SOME SCALES THAT ARE SIMILAR TO THE CHROMATIC SCALE

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Abstract. We construct musical compositions by similarity, by constructing scales that are formally similar to the chromatic scale, and defining transformations of such scales that take sounds to similar sounds.

1. Scales and transformations.

We are interested in using mathematical structures to define musical compositions that appeal to the memory, by similarity. We have previously observed the following: suppose we are given a set of musical phrases that contains a phrase of a single note, is closed under concatenation, and is closed under the action of a set of similarity transformations, that send phrases to similar phrases; we can then form such a composition by starting with the note, applying a sequence of similarity transformations to obtain a sequence of notes, concatenating these to form a phrase, applying a sequence of similarity transformations to obtain a sequence of phrases, concatenating these to form a longer phrase, etc. [6]. We call a composition generated in this way a *similarity concatenation composition.*

One way to construct a set of phrases to which the above formalism may apply is to define a scale of sounds, together with a set of similarity transformations of the scale, and consider the set of musical phrases written in that scale. Such a set is obviously closed under concatenation, and consequently, assuming the scale is nonempty, it adheres to our formalism. Again in the hope of appealing to the memory of the listener, we may wish to use a scale that is similar to a scale that the listener may be familiar with. To effect a progression from previous compositions we may wish to define a scale that is different from scales used in the past. This was the strategy we used previously [6]: we used scales of so called quasi-notes, indexed by elements of $\mathbb{Z}$. These scales bore a formal resemblance to the chromatic scale, but were nonstandard, with varying intervals separating the partials of the quasi-notes.

Here we introduce further scales which bear a formal resemblance to the chromatic scale, and introduce similarity transformations on those scales. Our constructions are more intricate mathematically than those we used previously [6]. Inspired by fugal writing, we first introduce scales whose partials are given by motifs. Inspired by work in representation theory [4], we then introduce cubist scales of quasi-notes of dimension $r \geq 1$ (the dimension 2 case is the one we are most interested in, since it is closest to dimension 1, which we have used previously, but the formalism works more generally). Finally, also inspired by representation theory, we introduce some scales which possess symmetries that lift to the music they generate. In each case

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we give an explicit example of a composition constructed using the scale, for which an audio file is available [7].

Here, by a *scale*, we mean a poset S of sounds. In the construction of the compositions given here, the partial orderings on our scales will not be invoked, but like traditional scales, our scales do come with natural orderings.

2. Chromatic combinations and quasi-notes.

A note on a stringed instrument has a fundamental frequency f, and a set of overtones, whose frequencies are $2f$, $3f$, $4f$, etc. [3] The first five of these overtones are approximately $12, 19, 24, 28, 31$ semitones above the original frequency f. In imitation of this situation we defined further scales indexed by $\mathbb{Z}$ [6]. In each scale, for every point on the stave we defined a *quasi-note*, consisting of a *fundamental*, sounding together with a number of *overtones* of the same amplitude and duration. The fundamental was a pure tone whose frequency was given by the relevant point on the stave, whilst each overtone was a pure tone whose frequency was given by the fundamental frequency, raised by a certain number of semitones. The fundamental, and the overtones were called the *partials* of the quasi-note.

Formally speaking, let $a, d, f \in \mathbb{R}_+$. Let χ_I denote the indicator function of an interval $I \subset \mathbb{R}$. For a function $\eta : \mathbb{R} \to \mathbb{R}$ of bounded support, we call the infimum of the support of η the *start* of η, and we call the supremum of the support of η the *end* of η. If $\eta, \eta' : \mathbb{R} \to \mathbb{R}$ and the end of η is equal to the start of η', we call $\eta + \eta'$ the *concatenation* of η and η'. We define a *pure tone* of amplitude a, duration d, and frequency f, to be given by a function from $\mathbb{R}$ to $\mathbb{R}$ sending t to $a\chi_{(o,o+d)} \sin(2\pi f t + \phi)$, for some $o, \phi \in \mathbb{R}$.

Points on the stave are given by frequencies, with the A above middle C corresponding to 440 Hz, and the operation of raising by a semitone corresponding to multiplication by $2^{\frac{1}{12}}$. Points on the stave are also given by integers, with middle C corresponding to 0, and the operation of raising by a semitone corresponding to addition of 1.

Fix $a \in \mathbb{R}_+$. We define a *chromatic combination* of duration d to be given by a function from $\mathbb{R}$ to $\mathbb{R}$ sending t to $a\chi_{(o,o+d)} \sum_{j=1}^{p} \sin(2\pi f_j t + \phi_j)$, for some $o, \phi_j \in \mathbb{R}$, and frequencies f_j given by points on the stave, $j = 1, ..., p$.

Take a set of strictly increasing maps $s_1, s_2, ..., s_p : \mathbb{Z} \to \mathbb{Z}$. Let $\zeta_j, j = 1, ..., p$ be maps from $\mathbb{Z}$ to the set of pure tones of duration d, amplitude a, and start o, such that $\zeta_j(x)$ has frequency determined by the point on the stave given by $s_j(x)$, for $x \in \mathbb{Z}$. Let ζ be the map from $\mathbb{Z}$ to the set of chromatic combinations that sends x to $\sum_{j=1}^{p} \zeta_j(x)$. We call $\zeta(x)$ the x^{th} *quasi-note of duration d and start o*. We call $\zeta_1(x)$ the x^{th} *fundamental*, we call $\zeta_2(x), ..., \zeta_p(x)$ the *overtones*, and we call $\zeta_1(x), ..., \zeta_p(x)$ the *partials* of the x^{th} quasi-note of duration d and start o, for $x \in \mathbb{Z}$.

For fixed $d \in \mathbb{R}_+$, and $x \in \mathbb{Z}$, we abuse terminology and call the class of x^{th} quasi-notes of duration d and start o, as o runs through elements of $\mathbb{R}$, the x^{th} *quasi-note of duration d*; we call the class of x^{th} quasi-notes of duration d, as d runs through elements of $\mathbb{R}_+$, the x^{th} *quasi-note*, etc.. This abuse of terminology is consistent with standard musical terminology for notes, which may or may not have a well defined start, duration, timbre, etc..

In this way have a set Q of quasi-notes, indexed by elements x of $\mathbb{Z}$. Our collection Q is given equivalently by a set of subsets $\Omega_1, \Omega_2, ..., \Omega_p \subseteq \mathbb{Z}$ that are unbounded

from above and below, with elements $\omega_1 \in \Omega_1, ..., \omega_p \in \Omega_p$. Indeed, such data emerges when we write $\Omega_j = s_j(\mathbb{Z})$ and $\omega_j = s_j(0)$, for $i = 1, 2, ..., p$.
Suppose $\Omega_u \supseteq \Omega_v$. Then the v^{th} partial of a quasi-note q is equal to the u^{th} partial of a second quasi-note q'. We write $t_{u,v}(q) = q'$, and thus define a transformation $t_{u,v}$ of Q. We denote by Φ the collection of such transformations.
Let us remark here that the software we have used to turn our compositions into audio files [7] manipulates our chromatic combinations somewhat.

3. Scales of motifs.

In a fugue, we hear motifs sounding repeatedly in different combinations [2]. This suggests the possibility of treating motifs as the partials of notes in a scale. In this section we pursue this possibility, by ornamenting quasi-notes with motifs.
Suppose we have fixed a set Q of quasi-notes, as in Section 2. Let us assume that $\Omega_1 \supseteq \Omega_2, ..., \Omega_p$.
Let n be a natural number. Let $\underline{n} = \{1, ..., n\}$. We define an n-motif m to be a map $\alpha : \underline{n} \to \mathbb{Z} \cup \{\emptyset\}$ and a map $\beta : \underline{n} \to \mathbb{Q}$ such that $\sum_{i=1}^{d} \beta(i) = 1$. We have an action $\mathbb{Z}/n\mathbb{Z}$ on $\underline{n}$ where $y \in \mathbb{Z}/n\mathbb{Z}$ acts as multiplication by $(1\ 2\ ...\ n)^y$. Composition of the action of an element $y \in \mathbb{Z}/n\mathbb{Z}$ with α, and composition with β, define a new n-motif $y * m$. The operation $*$ defines an action of $\mathbb{Z}/n\mathbb{Z}$ on the set of n-motifs.
Let l be a natural number with $p \leq l$. Let $n_1, ..., n_l$ be natural numbers. Let $A = \mathbb{Z}/n_1\mathbb{Z} \times ... \times \mathbb{Z}/n_l\mathbb{Z}$, let $B = S_l$, let $C = \mathbb{Z}$, and let $D = \mathbb{R}_+$. We have an action of $\mathcal{E}_A = \mathbb{Z}/n_1\mathbb{Z} \times ... \times \mathbb{Z}/n_l\mathbb{Z}$ on A by left multiplication. We have an action of $\mathcal{E}_B = S_l$ on B by left multiplication. We have an action of the monoid $\mathcal{E}_C$ generated by Φ on C. We have an action of $\mathcal{E}_D = \mathbb{R}_+$ on D by left multiplication.
Let m_j be an n_j-motif, for $j = 1, ..., l$. We describe a scale S indexed by elements (a, b, c, d) of $A \times B \times C \times D$. Let c correspond to the p-tuple of pure tones of duration d indexed by $(s_1(c), ..., s_p(c))$. For $j = 1, ..., p$, we have $s_j(c) = s_1(c_j)$ for some c_j since $\Omega_1 \supseteq \Omega_2, ..., \Omega_p$. For $j = 1, ..., p$ we have a motif given by $a_{b(j)} * m_{b(j)}$. Let $\alpha_j : \underline{n_{b(j)}} \to \mathbb{Z} \cup \{\emptyset\}$ and $\beta_j : \underline{n_{b(j)}} \to \mathbb{Q}$ be the data determining this motif. Consider the sequence σ_j of $n_{b(j)}$ pure tones given by points on the stave $s_1(c_j + \alpha_j(i))$, played consecutively, with duration $d\beta_j(i)$, for $i = 1, ..., n_{b(j)}$. Here, if $\alpha_j(i) = \emptyset$ we interpret the pure tone given by $s_1(c_j + \alpha_j(i))$ as silence. The element of S indexed by (a, b, c, d) consists of the sequences $\sigma_1, ..., \sigma_p$ played simultaneously, with the same start. The duration of the scale element is d.
We order S by $(a, b, c, d) \leq (a', b', c', d')$ if $a = a'$, $b = b'$, $c \leq c'$, and $d = d'$.
We describe similarity concatenation compositions with scale S.
Let w be a natural number, and let $\gamma_1, ..., \gamma_w$ be natural numbers. For $i = 1, ..., w$ we take maps $t_i : \underline{\gamma_i} \to \mathcal{E}_A \times \mathcal{E}_B \times \Phi \times \mathcal{E}_D$. We insist that $t_i(1)$ is equal to the identity, for $i = 1, ..., w$, but that $t_i(\xi)$ is different from the identity, for $\xi = 2, ..., \gamma_i$ and $i = 1, ..., w$.
For $i = 1, ..., w$ we define maps $u_i : \underline{\gamma_i} \to \mathcal{E}_A \times \mathcal{E}_B \times \mathcal{E}_C \times \mathcal{E}_D$, by

$$u_i(\xi) = t_i(\xi)t_i(\xi - 1)...t_i(1).$$

Let us fix an element $q = (a, b, c, d) \in A \times B \times C \times D$ where $c \in \Omega_1$. Our composition is obtained by concatenating the elements of S corresponding to the elements $(\xi_1, ..., \xi_w)$ of $\underline{\gamma_1} \times \underline{\gamma_2} \times ... \times \underline{\gamma_w}$, ordered lexicographically. The scale element corresponding to $(\xi_1, ..., \xi_w)$ is given by $u_1(\xi_1)u_2(\xi_2)...u_w(\xi_w)q$.

Example 1 Let $p = 4$. Let

$$\Omega_1 = \{7i| -3 \leq i \leq 5\} + 12\mathbb{Z},\ \Omega_2 = \{7i| -2 \leq i \leq 5\} + 12\mathbb{Z},$$
$$\Omega_3 = \{7i| -3 \leq i \leq 4\} + 12\mathbb{Z},\ \Omega_4 = \{7i| -2 \leq i \leq 4\} + 12\mathbb{Z},$$

Thus $\Omega_1 \supset \Omega_2, \Omega_4 \supset \Omega_3$. To specify a set of quasi-notes it is enough to specify one: $\omega_1 = -19, \omega_2 = -12, \omega_3 = -8, \omega_4 = -2$.
We take $w = 3$, and $\gamma_1 = \gamma_2 = \gamma_3 = 3$. We write $h_1 = (1\ 2\ 3\ 4)$, $h_2 = (1\ 2\ 3\ 4\ 5)$, $h_3 = (1\ 2\ 3)$. We take $t_3(2) = ((0,0,0,0,0), h_3, 1, 1)$, $t_3(3) = ((0,0,0,0,0), h_3, 1, 1)$. We take $t_2(2) = ((0,0,0,0,0), h_2, t_{1,2}, 1)$, $t_2(3) = ((1,2,3,0,0), h_2, t_{4,3}, 1)$. We take $t_1(2) = ((0,0,0,0,0), h_1, t_{1,2}, 1)$, $t_1(3) = ((0,0,1,2,3), h_1, t_{4,3}, 1)$. We fix $q = ((0,0,0,0,0), 1, 0, \frac{7}{2})$
We take $l = 5$, $n_1 = 3$, $n_2 = 4$, $n_3 = 5$, $n_4 = 6$, and $n_5 = 7$. We define β_1 to take $1,2,3$ to $\frac{3}{7}, \frac{3}{7}, \frac{1}{7}$ respectively. We define β_2 to take $1,2,3,4$ to $\frac{2}{7}, \frac{1}{7}, \frac{3}{7}, \frac{1}{7}$ respectively. We define β_3 to take $1,2,3,4,5$ to $\frac{2}{7}, \frac{1}{7}\frac{2}{7}, \frac{1}{7}, \frac{1}{7}$ respectively. We define β_4 to take $1,2,3,4,5,6$ to $\frac{2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}$ respectively. We define β_5 to take $1,2,3,4,5,6,7$ to $\frac{2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}$ respectively. We define α_1 to take $1,2,3$ to $0,2,4$ respectively. We define α_2 to take $1,2,3,4$ to $0,\emptyset,1,\emptyset$ respectively. We define α_3 to take $1,2,3,4,5$ to $0,0,1,2,4$ respectively. We define α_4 to take $1,2,3,4,5,6$ to $0,3,\emptyset,3,\emptyset,3$ respectively. We define α_5 to take $1,2,3,4,5,6,7$ to $0,1,2,3,4,5,6$ respectively. The motifs m_i have been chosen to be similar in various ways: for example, the rhythm of m_i is a refinement of the rhythm of m_{i-1} for $i = 2, ..., 5$, and each α_i defines an ascending sequence starting from zero.

4. Cubist scales.

In a previous article [5] we restricted our attention to a specific type of quasi-note, which we now describe. Fix a sequence of *quasi-note generators* $\rho_2, ..., \rho_p \in \{1, 2, 3, ..., 32\}$. Define $\rho_1 = 0$. The maps $s_j : \mathbb{Z} \to \mathbb{Z}$ given by $s_j(x) = x + \rho_j$, for $j = 1, ..., p$ define a set Q of quasi-notes. Quasi-notes obtained this way are formally similar to notes played on a stringed instrument, particularly in the special case where $\{\rho_j | j = 1, ..., p\}$ is some subset of $\{0, 12, 19, 24, 28, 31\}$.
Let $D = \mathbb{R}_+$. Implicitly, we thus worked [5] with a scale S of quasi-notes indexed by $(x, d) \in \mathbb{Z} \times D$, and defined by a set of maps $s_1, ..., s_p : \mathbb{Z} \to \mathbb{Z}$ such that $s_j(x+1) = s_j(x) + 1$ for $j = 1, ..., p$, so that $s_j(x)$ defined the position on the stave of the j^{th} partial corresponding to $x \in \mathbb{Z}$, and d denoted the duration of the quasi-note in our scale. The compositional techniques in our quasi-note compositions were built on a set of transformations, given by translations of $\mathbb{Z}$, and an accompaniment strategy based on the metric on $\mathbb{Z}$ given by $d(x, y) = |x - y|$.
Here we wish to generalise the above scales, to obtain scales indexed by $\mathbb{Z}^r \times D$, for r a natural number. We wish to generalise the transformations, and to generalise the metric.
The metric will merely be given by the metric on $\mathbb{Z}^r$ given by

$$d((x_1, ..., x_r), (y_1, ..., y_r)) = |x_1 - y_1| + ... + |x_r - y_r|.$$

We are looking for scales S that are mild generalisations of the above scales indexed by $\mathbb{Z} \times D$, in the hope that their sonic effect will be similar to music played on a stringed instrument on a stave: we hope for some awareness of uniformity in the mind of the listener as they listen to music written in our scales. To begin with, we wish to assume our scale is defined by a set of maps $s_1, ..., s_p : \mathbb{Z}^r \to \mathbb{Z}$ such that $s_j(x+(1,1,...,1)) = s_j(x)+1$ for $j = 1,...,p$. This generalises the $r = 1$ case discussed above in a natural way. To specify the map s_j we need only specify the set $\mathcal{X} \subset \mathbb{Z}^r$ on which s_j takes value zero. This set should form a set of representatives for the set of cosets $\mathbb{Z}(1,1,..,1)\backslash\mathbb{Z}^r$. However, we wish to assume further that $\mathcal{X}$ is a *cubist* set, and thus extends to a special kind of cell complex in $\mathbb{R}^r$. Such sets arise in representation theory [4], and give us scales which are mild generalisations as required.

Let $E = \mathbb{R}^{\oplus \underline{r}}$ denote Euclidean space, of dimension r. Let ϵ_i denote the standard basis element of E, for $i \in \underline{r}$.

Suppose $\Sigma \subset \underline{r}$. Let $F_\Sigma = F_1 \times ... \times F_r \subset E$, where $F_i = [0,1]$, if $i \in \Sigma$, and $F_i = \{0\}$, if $i \notin \Sigma$.

Let $\mathcal{Z} = \mathcal{Z}_r$ denote the polytopal complex, homeomorphic to E, whose i-dimensional cells are i-cubes in E of the form $x + F_\Sigma, x \in \mathbb{Z}^r, |\Sigma| = i$.

Let $H = (1,1,..,1)^\perp \subset \mathbb{R}^r$. Let $p : E \twoheadrightarrow H$ denote orthogonal projection.

We say a polytopal subcomplex $\mathcal{C} \subset \mathcal{Z}$ is *cubist* if the projection $p : \mathcal{C} \to H$ is a homeomorphism.

If $\mathcal{C}$ is a cubist complex, we write $\mathcal{X} = \mathcal{C} \cap \mathbb{Z}^r$, and call $\mathcal{X}$ a cubist set in $\mathbb{Z}^r$. We denote by $s_\mathcal{X}$ the map from $\mathbb{Z}^r$ to $\mathbb{Z}$ given by $s_\mathcal{X}(x + (k,k,....,k)) = k$ for $x \in \mathcal{X}, k \in \mathbb{Z}$.

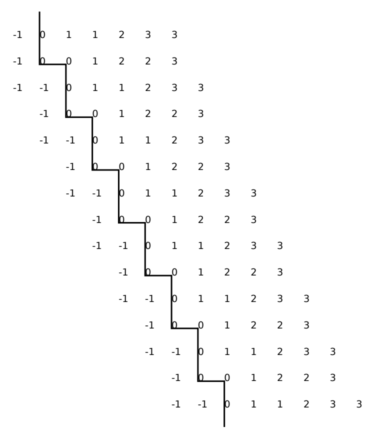

FIGURE 1. A cubist subset $\mathcal{X}$ of $\mathbb{Z}^2$ and the corresponding map $s_\mathcal{X}$ from $\mathbb{Z}^2$ to $\mathbb{Z}$.

Let $\mathcal{X}_1, ..., \mathcal{X}_p \subset \mathbb{Z}^r$ be cubist subsets. Let $s_1 = s_{\mathcal{X}_1}, ..., s_p = s_{\mathcal{X}_p}$. These maps define our scale S indexed by $\mathbb{Z}^r \times D$, where $(x,d) \in \mathbb{Z}^r \times D$ corresponds to a chromatic combination of duration d whose partials are given by positions $s_1(x), ..., s_p(x)$ on the stave.

We order S by $(x,d) \le (x',d')$ if $x \le x'$ and $d = d'$, where the ordering on $\mathbb{Z}^r$ is lexicographic.

Translation by $z \in \mathbb{Z}^r$ defines a transformation of our scale, as in the $r = 1$ case. For $z \in \mathbb{Z}^r$ we define τ_z to be translation by z.

We wish to describe some alternative similarity transformations on our scale. Such transformations take a note to a second note with a common partial. Suppose $1 \le u, v \le p$. Let $x \in \mathbb{Z}^r$. Suppose $s_v(x) = y$. The set of all elements z in $\mathbb{Z}^r$ with $s_u(z) = y$ is nonempty, and thus contains a subset M consisting of elements a

minimal distance from x in $\mathbb{Z}^r$. We define $t_{u,v}(x)$ to be the greatest element of M, taken with respect to the lexicographic ordering on $\mathbb{Z}^r$.
We denote by Ψ the set of transformations of $\mathbb{Z}^r$ consisting of translations τ_z and transformations $t_{u,v}$. We denote by $\mathcal{E}_{\mathbb{Z}^r}$ the monoid of transformations of $\mathbb{Z}^r$ generated by Ψ. We thus have an action of $\mathcal{E}_{\mathbb{Z}^r}$ on $\mathbb{Z}^r$.
We write $\mathcal{E}_D = \mathbb{R}_+$. Thus $\mathcal{E}_D$ acts on D by multiplication.
We describe generalised similarity concatenation compositions with scale S. We use the word 'generalised' due to the use of the permutation σ (see below).
Let w be a natural number, and let $\gamma_1, ..., \gamma_w$ be natural numbers. For $i = 1, ..., w$ we take maps $t_i : \underline{\gamma_i} \to \Psi \times \mathcal{E}_D$. We insist that $t_i(1)$ is equal to the identity, for $i = 1, ..., w$, but that $t_i(\xi)$ is different from the identity, for $\xi = 2, ..., \gamma_i$ and $i = 1, ..., w$.
For $i = 1, ..., w$ we define maps $u_i : \underline{\gamma_i} \to \mathcal{E}_{\mathbb{Z}^r} \times \mathcal{E}_D$, by

$$u_i(\xi) = t_i(\xi)t_i(\xi - 1)...t_i(1).$$

Let us fix an element $q = (x, d) \in \mathbb{Z}^r \times D$. Let us fix $\sigma \in S_w$. Our composition is obtained by concatenating scale elements corresponding to elements $(\xi_{\sigma 1}, ..., \xi_{\sigma w})$ of $\underline{\gamma_{\sigma 1}} \times \underline{\gamma_{\sigma 2}} \times ... \times \underline{\gamma_{\sigma w}}$, ordered lexicographically. The element of S corresponding to $(\xi_{\sigma 1}, ..., \xi_{\sigma w})$ is given by $u_1(\xi_1)u_2(\xi_2)...u_w(\xi_w)q$.

Example 2 We take $\mathcal{X}_1 = 0 \times \mathbb{Z}$, $\mathcal{X}_2 = \{(0,0), (-1,0), (-1,1), (-1,2)\} + \mathbb{Z}(-1,3)$, $\mathcal{X}_3 = \{(0,0), (-1,0), (-1,1), (-2,1), (-2,2)\} + \mathbb{Z}(-2,3)$, $\mathcal{X}_4 = \{(0,0), (-1,0)\} + \mathbb{Z}(-1,1)$.
We take $q = ((0,29),1)$, $w = 4$, and $\sigma = (4\ 3\ 2\ 1)$.
We take $\gamma_1 = 3$ and $t_1(2) = (\tau_{(0,-3)}, 1), t_1(3) = (\tau_{(0,-3)}, 1)$. We take $\gamma_2 = 3$ and $t_2(2) = (\tau_{(3,0)}, 1), t_2(3) = (\tau_{(3,0)}, 1)$. We take $\gamma_3 = 8$ and $t_3(2) = (t_{2,1}, 1), t_3(3) = (t_{3,2}, 1), t_3(4) = (t_{4,3}, 1), t_3(5) = (t_{1,4}, \frac{3}{2}), t_3(6) = (t_{2,1}, \frac{1}{3}), t_3(7) = (t_{3,2}, 2), t_3(8) = (t_{4,3}, 1)$. We take $\gamma_4 = 2$ and $t_4(2) = (\tau_{(0,6)}, \frac{1}{3})$.

5. Symmetric scales.

This section concerns scales constructed via modular group representations. Although we do not use it, an extensive theory has been developed for the study of such representations (see eg. [1]).
Suppose we are given a sequence s of r points on the stave, and $d \in \mathbb{R}_+$. Each subset Ω of $\{1, ..., r\}$ then corresponds to a chromatic combination of duration d whose frequencies are given by the elements of s indexed by elements of Ω. The collection of subsets of $\{1, ..., r\}$ is in bijection with the elements of $\mathbb{F}_2^r$ via the correspondence that sends a set Ω to the vector with a 0 in the i^{th} position if $i \notin \Omega$ and a 1 in the i^{th} position if $i \in \Omega$. We thus have a scale S of chromatic combinations indexed by elements of $\mathbb{F}_2^r$. We order S lexicographically.
Suppose we are given a set of vectors $v_1, ..., v_n \in \mathbb{F}_2^r$. We have a linear map from $\mathbb{F}_2^n$ to $\mathbb{F}_2^r$ sending $(a_1, ..., a_n)$ to $\sum_i a_i v_i$. This gives us a map from $\mathbb{F}_2^n$ to our scale S. We order the elements of $\mathbb{F}_2^n$ lexicographically. We have a similarity concatenation composition given by scale elements corresponding to the images of elements of $\mathbb{F}_2^n$, concatenated in this order. The similarity transformations of our composition correspond to symmetries of $\mathbb{F}_2^r$ given by $w \mapsto v + w$, for some $v \in \mathbb{F}_2^n$.

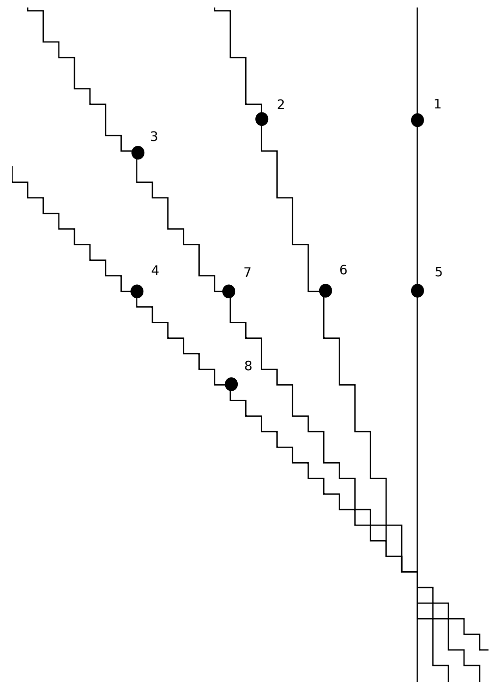

FIGURE 2. The cubist sets and first eight notes of Example 2.

The above compositions have been formally constructed via an action of $\mathbb{F}_2^n$ on S. However, the chromatic combinations of the compositions described above may be quite heterogeneous; for example chromatic combinations corresponding to $v_1, ..., v_n$ may have diverse numbers of partials. It seems reasonable to place further restrictions on the structure of our sets to create further homogeneity within the compositions. One way to do this is using group actions. We consider the case where the vector spaces $\mathbb{F}_2^r$ and $\mathbb{F}_2^n$ above are permutation $\mathbb{F}_2G$-modules, generated by transitive G-sets.

Let G be a finite group, and H a subgroup of G. Let G/H be the set of left cosets of H in G, and $\mathbb{F}_2[G/H]$ the permutation $\mathbb{F}_2G$-module generated by this set. Let $r = |G/H|$. Suppose we are given a map c from G/H to the set of points on the stave. Each subset Ω of G/H thus corresponds to a chromatic combination of duration d whose partials are given by the images under c of the elements of G/H indexed by elements of Ω. The collection of subsets of G/H is in bijection with the elements of $\mathbb{F}_2[G/H]$ via the correspondence that sends a set Ω to $\sum_{\omega \in \Omega} \omega$. We thus have a scale S of chromatic combinations indexed by elements of $\mathbb{F}_2[G/H]$.

Suppose K is a subgroup of G, and x is a double coset in $K\backslash G/H$. The canonical isomorphism $Hom_G(\mathbb{F}_2[G/K], \mathbb{F}_2[G/H]) \cong \mathbb{F}_2[K\backslash G/H]$ means that x determines a G-homomorphism ϕ from $\mathbb{F}_2[G/K]$ to $\mathbb{F}_2[G/H]$. Composing ϕ with the indexing bijection between $\mathbb{F}_2[G/H]$ and S, we obtain a map $\tilde{\phi}$ from $\mathbb{F}_2[G/K]$ to S.

Suppose we are given a linear ordering of the set G/K, and $|G/K| = n$. Such an ordering determines an identification of $\mathbb{F}_2[G/K]$ with $\mathbb{F}_2^n$. The lexicographic ordering

on $\mathbb{F}_2^n$ thus determines a linear ordering on $\mathbb{F}_2[G/K]$. We have a similarity concatenation composition with chromatic combinations of duration d corresponding to the images of elements of $\mathbb{F}_2[G/K]$ under $\tilde{\phi}$, concatenated in this order.
The compositions thus constructed have symmetries given by the elements of G. The chromatic combinations indexed by the elements of G/K all have the same number of partials, given by the number of elements of x/H. We have a set of distinguished transformations of our composition corresponding to the endomorphisms of $\mathbb{F}_2[G/K]$ sending w to $v + w$, when v belongs to the kernel of ϕ; such transformations send a given chromatic combination to itself.

A composition as described above depends on a choice of map c from G/H to the set of points on the stave, as well as a choice of linear ordering of the set G/K. We wish to describe some systematic methods for determining such data. Here is the idea: The group $\mathbb{Z}$ acts on the set of points on the stave, with $i \in \mathbb{Z}$ acting by raising by i semitones. Since G is finite, it is impossible to construct a nontrivial homomorphism from G to $\mathbb{Z}$. However, it is possible to exploit weaker correspondences between the group structure on G, and the group structure on $\mathbb{Z}$ to define the data we require.
Suppose $\{g_1, ..., g_f\}$ is a set of generators of G.
Let W_K be a set of words in the letters $g_1, ..., g_f$ such that the map $\pi_K : W_K \to G/K$ sending w to wK is bijective. The lexicographic ordering on W_K gives us a linear ordering of G/K, as required.
Let W_H be a set of words in the letters $g_1, ..., g_f$ such that the map $\pi_H : W_H \to G/H$ sending w to wH is bijective. Suppose we are given a set of intervals $i_1, ..., i_f \in \mathbb{Z}$ and a point e on the stave. Let $\mathcal{F}$ denote the free group on letters $g_1, ..., g_f$. We have a homomorphism from $\mathcal{F}$ to $\mathbb{Z}$ sending $g_1, ..., g_f$ to $i_1, ..., i_f$ respectively; we have a bijection from $\mathbb{Z}$ to the set of points on the stave sending $i \in \mathbb{Z}$ to the pitch e, raised by i semitones; composing these maps gives us a map from $\mathcal{F}$ to the set of points on the stave. Restricting to W_H gives us a map from W_H to the set of points on the stave. Composing with π_H^{-1} defines our map c from G/H to the set of points on the stave.
When c is injective, an alternative bijection c' from G/H to the set of points on the stave, with image $c(G/H)$, is obtained by taking the lexicographic ordering on W_H given by $g_f < g_{f-1} < ... < g_1$, and taking the bijection π_H to define a linear ordering of G/H. Under c' we identify $c(G/H)$ with G/H in such a way that the linear ordering of $c(G/H)$ by pitch corresponds to our linear ordering of G/H.

Example 3 Suppose $G = S_7$, and H is the Young subgroup of S_7 given by $Sym\{1,2\} \times Sym\{3,4,5,6,7\}$, whilst K is the Young subgroup of S_7 given by $Sym\{2,3,4,5,6,7\}$. Then $r = \frac{7!}{2!5!} = 21$ and $n = \frac{7!}{1!6!} = 7$. Let $x = KH$. Take $f = 2$, $d = \frac{1}{2}$, and

$$g_1 = (1\ 2\ 3\ 4\ 5\ 6\ 7),$$

$$g_2 = (2\ 3\ 4\ 5\ 6\ 7).$$

We take $W_H = \{{g_1}^{a_1}{g_2}^{a_2} | 0 \leq a_1, 0 \leq a_2, 0 \leq a_1 + a_2 \leq 5,\}$ and $W_K = \{{g_1}_1^{b} | 0 \leq b_1 \leq 6\}$. Let $i_1 = 8, i_2 = 5$ and e to be the pitch 19 semitones below middle C. We linearly order G/K as

$$K < g_1K < g_1^2K < ... < g_1^6K.$$

Associated to our data, we have two similarity concatenation compositions according to the recipes defined above: one for c, and one for c'. Each of these compositions has $2^7 = 128$ chromatic combinations.
Since the two compositions defined above have the same partials, and are constructed according to a similar formalism, it makes sense to play them one after another, to create a larger composition with two similar parts. This is what we have done in our audio file [7].

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