EIGHT SHORT MATHEMATICAL COMPOSITIONS CONSTRUCTED BY SIMILARITY

WILL TURNER

Abstract. Similar sounds are a formal feature of many musical compositions, for example in pairs of consonant notes, in translated passages, and in continuous musical lines. The aim of this article is to introduce techniques derived from those of existing tonal music via the notion of similarity, and demonstrate interactions with existing techniques, also derived from the notion of similarity.

We describe eight short musical compositions that have been constructed mathematically using these techniques. The building blocks of our compositions are entities we call quasi-notes, which correspond to points on the stave, and are certain composites of pure tones. We explore various ways to combine these quasi-notes, pursuing the principle that similar sounds should combine and follow each other.

1. Introduction.

A musical score is a sort of combinatorial construction. It may be built via a combination of formal techniques and experiments made on instruments that sound similar to those on which the composition is to be performed. The formal techniques may have a mathematical flavour, meaning the composition resembles a mathematical construction. Here we present a handful of short compositions that have been constructed mathematically, using techniques derived from the notion of similarity. The principle behind our techniques is that similar sounds should combine and follow each other.

The techniques are derived from techniques for tonal music written on a stave by generalisation and abstraction.

The eight compositions are titled qn1-qn8. Sound files recording these compositions are available [11]. The letters 'qn' are an abbreviation of *quasi-note*, which is the name we have given to the composite sounds our compositions are built from.

In section 2 we describe the techniques we use, and how they are derived from the notion of similarity. In section 3 we outline the relation of our techniques to past music and past literature, with the hope of illuminating their origin, as well as clarifying what is new about them. In section 4 we describe the compositions qn1-qn8 as mathematical constructions, built using the techniques of section 2.

We have formulated our constructions in the language of set theory. Such abstraction raises the possibility of generalisation and development (for examples, see [13], [12]).

Date: April 5, 2018.

2. Techniques.

To appeal to the memory of the listener, our compositions are similar to much music already in existence. For example, the sounds of our pieces are combinations of tones whose frequencies correspond to points on the stave.

The building blocks of our compositions are sounds we call *quasi-notes*, which are chords of pure tones that we treat like notes (cf. mixture stops). A note on a stringed instrument has a fundamental frequency f, and a set of overtones, whose frequencies are $2f$, $3f$, $4f$, etc. [3] The first five of these overtones are approximately $12, 19, 24, 28, 31$ semitones above the original frequency f. For each of our compositions we fix a set of positive integers ≤ 32 called *quasi-note generators.* For every point on the stave we have a *quasi-note* consisting of a *fundamental*, sounding together with a number of *overtones* of identical amplitude and duration. The fundamental is a pure tone whose frequency is given by the relevant point on the stave, whilst each overtone is a pure tone whose frequency is given by the fundamental frequency, raised by a quasi-note generator number of semitones. The fundamental, and the overtones are called the *partials* of the quasi-note. The amplitudes of the partials are evenly distributed.

Formally speaking, let $a, d, f \in \mathbb{R}_+$. Let χ_I denote the indicator function of an interval $I \subset \mathbb{R}$. For a function $\eta : \mathbb{R} \to \mathbb{R}$ of bounded support, we call the infimum of the support of η the *start* of η, and we call the supremum of the support of η the *end* of η. We define a *pure tone* of amplitude a, duration d, and frequency f, to be given by a function from $\mathbb{R}$ to $\mathbb{R}$ sending t to $a\chi_{(o,o+d)} \sin(2\pi ft + \phi)$, for some $o, \phi \in \mathbb{R}$.

Points on the stave are given by frequencies, with the A above middle C corresponding to 440 Hz, and the operation of raising by a semitone corresponding to multiplication by $2^{\frac{1}{12}}$. Points on the stave are also given by integers, with middle C corresponding to 0, and the operation of raising by a semitone corresponding to addition of 1.

Fix $a \in \mathbb{R}_+$. We define a *chromatic combination* of duration d to be given by a function from $\mathbb{R}$ to $\mathbb{R}$ sending t to $a\chi_{(o,o+d)} \sum_{j=1}^{p} \sin(2\pi f_j t + \phi_j)$, for some $o, \phi_j \in \mathbb{R}$, and frequencies f_j given by points on the stave, $j = 1, ..., p$.

For $j = 2, ..., p$, we fix *quasi-note generators* $\rho_2, ..., \rho_p \in \{1, 2, 3, ..., 32\}$. We define $\rho_1 = 0$. Let $\zeta_j, j = 1, ..., p$ be maps from $\mathbb{Z}$ to the set of pure tones of duration d, amplitude a, and start o, such that $\zeta_j(x)$ has frequency determined by the point on the stave given by $x + \rho_j$, for $x \in \mathbb{Z}$. Let ζ be the map from $\mathbb{Z}$ to the set of chromatic combinations that sends x to $\sum_{j=1}^{p} \zeta_j(x)$. We call $\zeta(x)$ the x^{th} *quasi-note of duration d and start o.* We call $\zeta_1(x)$ the *fundamental*, we call $\zeta_2(x), ..., \zeta_p(x)$ the *overtones*, and we call $\zeta_1(x), ..., \zeta_p(x)$ the *partials* of the x^{th} quasi-note of duration d and start o, for $x \in \mathbb{Z}$.

For fixed $d \in \mathbb{R}_+$, and $x \in \mathbb{Z}$, we abuse terminology and call the class of x^{th} quasi-notes of duration d and start o, as o runs through elements of $\mathbb{R}$, the x^{th} *quasi-note of duration d*; we call the class of x^{th} quasi-notes of duration d, as d runs through elements of $\mathbb{R}_+$, the x^{th} *quasi-note*, etc.. This abuse of terminology is consistent with standard musical terminology for notes, which may or may not have a well defined start, duration, timbre, etc..

We thus have a set of quasi-notes, indexed by elements $x \in \mathbb{Z}$.

Let us remark here that the software we have used to turn our compositions into audio files manipulates our chromatic combinations somewhat.

A note on a stringed instrument has overtones at roughly $12, 19, 24 = 12 + 12, 31 = 12 + 19$ semitones, meaning that it shares more than one partial with a note 12 semitones above. In compositions qn1, qn6, and qn7, we include quasi-note generators $\{a, b, a + b\}$ for some positive integers a and b, so that distinct quasi-notes can share more than one partial.

A standard technique in music for stringed instruments is sounding distinct notes with shared partials, or close partials, simultaneously. Notes with shared partials may be said to be consonant, whilst partials that differ by a semitone may be said to be dissonant. We say a pair of distinct quasi-notes whose fundamentals differ by d semitones are *consonant* if d is a quasi-note generator. We say a pair of distinct quasi-notes whose fundamentals differ by d semitones are *upper dissonant* (respectively *lower dissonant*) if $d-1$ (respectively $d+1$) is a quasi-note generator. We use consonant quasi-notes in qn4-qn8, and upper dissonant quasi-notes in qn7. The quasi-notes themselves have partials that differ by a semitone in qn5 and qn8.

Many of our techniques involve introducing musical material that is similar to previous musical material in the same composition. This notion relates our compositions to music already in existence, where it is very common to hear similar sounds in adjacent parts of a piece of music. For example, adjacent notes in a musical line are often separated by small numbers of semitones. To give a more specific example, in Bach prelude no. 1 [1], the minims of the successive bars of this piece differ by 0,-1,1,0,0,-1,0,-2,-7,5,0,-2,0,-1,0,-2,-7,5,0,-7,1,2,-1,0,0,0,0,0,0,0,-7,0,0 semitones respectively. Transposition and reversal of musical phrases are also standard techniques.

Our compositions are built up from a *principal motif* which occurs at the beginning of the piece and is repeated, with variation and accompaniment.

Given a set of quasi-note generators, we define a corresponding set of *harmonic leaps*, to be given by those integers g, where g is a quasi-note generator, or $|g - g'|$ where g and g' are distinct quasi-note generators, which are less than or equal to 12. In all our compositions, within our principal motifs we assume consecutive quasi-notes differ by a number of semitones given by a harmonic leap. This means such notes have a partial in common, and are therefore similar.

Given a list of frequencies, there are precisely two ways to order these frequencies as $f_1, f_2, ..., f_n$ to minimise $\sum_i |f_{i+1} - f_i|$: we either take them in increasing order or in decreasing order. For this reason a *monotonic* sequence of quasi-notes, whose fundamentals increase as the sequence goes on, or decrease as the sequence goes on, maximises the similarity between consecutive notes of the sequence. In each of our compositions, when we choose a principal motif, we concatenate a small number of monotonic sequences.

Translation (often called ‘transposition’ in the musical literature) is a standard technique in music written on a stave to generate similar material. In our compositions we frequently follow a phrase with the same phrase translated by a harmonic leap, so that such phrases have a sequence of partials in common (see qn4, qn5, qn6). In qn2 and qn8 we use the technique of following a phrase with the same phrase reversed and translated by a harmonic leap. In this case the difference between the end of the first phrase and the beginning of the second phrase is a harmonic leap. We can take an n note phrase, swap the first r quasi-notes with last $n - r$ quasi-notes, and translate it to generate a new phrase. We use this technique in qn1.

Another method of generating similar adjacent material is to follow a phrase with the same phrase, only with two quasi-notes permuted. This method is used in qn7. A further technique for generating similar adjacent material is adding quasi-notes to, or deleting quasi-notes from, a given phrase or rhythm. This technique is used in qn4, qn5, and qn6.

Suppose we are given a sequence s of quasi-notes, a rhythm r to accompany that sequence, and a single quasi-note n whose fundamental differs from the fundamental of the first in our sequence by a number of semitones given by a quasi-note generator. Suppose we have an accompaniment to s, which is a sequence of quasi-notes whose first note is n, whose rhythm is r. We say our accompaniment *avoids parallel motion* if the difference between fundamentals of two successive quasi-notes in the accompaniment is not equal to the difference between fundamentals of the two quasi-notes of s they accompany (cf. Fux's rules concerning contrapuntal motion [6]). We say our accompaniment is *consonant* if the fundamental of each quasi-note of the accompaniment differs from the fundamental of the first quasi-note it accompanies by a number of semitones given by a quasi-note generator. Suppose the ith quasi-note of our accompaniment is taken to be consonant with the first note it accompanies, is taken to avoid parallel motion, and is taken to minimise the number of semitones separating its fundamental from the fundamental of the $i-1$th note of our accompaniment for each i; then we say our accompaniment is a *minimising consonant accompaniment avoiding parallel motion*. Minimising consonant accompaniments avoiding parallel motion are used to give similarities between consecutive quasi-notes of our accompaniments, in qn4-qn6 and qn8.

We say an accompaniment is *upper dissonant* if the fundamental of each quasi-note of the accompaniment differs from the fundamental of the first quasi-note it accompanies by a number of semitones given by a quasi-note generator plus 1. Suppose the ith quasi-note of our accompaniment is taken to be upper dissonant with the first quasi-note it accompanies, is taken to avoid parallel motion, and is taken to minimise the number of semitones separating its fundamental from the fundamental of the $i-1$th quasi-note of our accompaniment for each i; then we say our accompaniment is a *minimising upper dissonant accompaniment avoiding parallel motion*. A minimising upper dissonant accompaniment avoiding parallel motion is used in qn7.

Suppose we are given a scale $n_1, n_2, ..., n_l$ of quasi-notes, where n_i is n_{i-1} raised by a number of semitones given by a harmonic leap. Suppose we have a sequence $\alpha_1, ..., \alpha_t$ of integers $\leq l-u$, and a sequence of durations $d_1, ..., d_t$. Then we have a phrase given by sounding the quasi-notes n_{α_i} successively with durations d_i. We have another phrase given by sounding the quasi-notes n_{α_i+u} successively with durations d_i. These phrases may be similar sounding, in having common notes, or in having similar phrase structure. We use this technique for generating similar material in qn3.

We can generate similarities in musical compositions by structuring the material in sections, which are constructed according to a common method. We exploit this technique in qn4 and qn6.

3. Relation of techniques to works of the past.

Our techniques are built using the occurrence of similar sounds, either simultaneously (as in consonance) or one after another.

Sounds occurring simultaneously can appear to merge together to form a single sound. This merging effect can be exaggerated if the sounds themselves are similar, for example if we hear a number of people speaking at similar pitches in a crowded room.

Similar sounds occurring one after another play a role in the building of language; spoken words are shared by their repetition. For example a parent may show an apple to a child learning to talk, say the word 'apple', and encourage the child to repeat the same word.

Our compositions have harmonics that are given by positions on the stave. In this respect they are similar to a large body of music written on a stave. If a listener has experience of such stave music, their memory is referred to when they listen to these pieces. The definition of a quasi-note is designed so that we can generalise tonal musical techniques for stringed instruments to pure tone compositions, and create music that is like tonal music for stringed instruments, but a little different. Quasi-notes are different from notes played on stringed instruments, in that their harmonics are not necessarily integer multiples of a fundamental frequency.

Another body of music to which our compositions are similar is the collection composed with sine waves. For example there is the music of the Cologne school [9], I.3, [10]. Our quasi-notes differ from the note mixtures of the Cologne School, because they have harmonics that are written on a stave.

Quasi-notes are less general than notes played on a synthesiser whose partials have harmonic frequencies given by $\lambda_1 f, ..., \lambda_r f$, for some $\lambda_1, ..., \lambda_r$, as f runs through the frequencies corresponding to notes on a stave (for more advanced methods of synthesis, see [5]). We need to restrict ourselves to quasi-notes to use the techniques of consonance, and successive sounds with common harmonics.

The technique of combining sounds with shared harmonics (consonance), and of using similar passages one after another is standard in music for instruments with strings. Examples are to be found in Bach's Inventions [2]. We apply these ideas to reveal new techniques for quasi-notes. Successive quasi-notes of our motifs share a common harmonic, giving a sense of continuity to the motifs. Successive passages of our pieces are also constructed to have common harmonic material, again creating a sense of continuity. Let us note that the practice of only using successive motivic quasi-notes with a common harmonic is not commonly used in tonal music for stringed instruments: a scale is commonly used as a restrictive mechanism instead.

The notion of a minimising consonant accompaniment avoiding parallel notion is inspired by Fux's rules on counterpoint [6], although for quasi-notes it is new. Altering it slightly we obtain the notion of a minimising upper dissonant accompaniment avoiding parallel motion (partials separated by a semitone are often registered by the ear as dissonant [3]).

Translating along a scale to create similar passages by is a technique to be found in bars 3-4 of Bach's Invention No. 1 [2], where the C major scale is used.

Implicit in our minimising accompaniments is a geometry, or notion of distance: we look to minimise the number of semitones between successive quasi-notes in an accompaniment. The geometry of n-chord spaces $\mathbb{Z}^n/S_n$ and their relatives has been explored by Tymozcko [14] and by Callender, Quinn and Tymoczko [4]. The metric in our case corresponds to the trivial case $n = 1$, where quasi-notes are indexed by $\mathbb{Z}$. If $n - 1$ is the number of quasi-note generators, then quasi-notes consist of n partials and thus also correspond naturally to elements of $\mathbb{Z}^n/S_n$. The quasi-notes indexed by $\mathbb{Z}$ correspond to an orbit of $\mathbb{Z}$ in its diagonal action on $\mathbb{Z}^n/S_n$.

Our pieces naturally fall into hierarchies of sequences of similar sections. The grouping of sounds into hierarchies of sections of increasing size is analysed in some generality in A generative theory of tonal music [8].
The psychology of the experience of similarity in music is explored in Sweet Anticipation [7].

4. The compositions.

qn1. The quasi-note generators are $\{2, 4 = 2 + 2, 12\}$. The harmonic leaps are $\{2, 4, 8, 10\}$. The time signature is $\frac{4}{4}$. There is a principal motif, consisting of a crotchet followed by two quavers four times consecutively. The first quasi-note of the principal motif has fundamental given by middle C. The consecutive leaps in the principal motif are $2, 2, 8, -4, -2, -4, 2, 4, 8, -4, -4$ semitones. The leap from the last to the first quasi-note in the principal motif is -8 semitones.
The principal motif is repeated with variation 13 times. To obtain the $i + 1$th iteration from the ith iteration, make the first seven notes of the ith iteration the last seven notes of the $i + 1$th iteration, translated, and the last five notes of the ith iteration the first five notes of the $i + 1$th iteration, translated. The successive translations of the motif are by $8, -2, -2, -2, 8, -2, -2, -2, 8, -2, -2, -2$ semitones.

qn2. The quasi-note generators are $\{2, 7, 16\}$. The harmonic leaps are $\{2, 5, 7, 9\}$. The time signature is $\frac{4}{4}$. We have 2^7 quasi-notes, all of which are consecutive crotchets. The fundamental of the first quasi-note is middle C. The first 2^i quasi-notes are repeated, translated and taken in reverse order to form the passage from the $2^i + 1$th to the 2^{i+1}th quasi-note, for $0 \leq i \leq 6$. The successive translations are by 2, 5, 2, 5, 2, 5, 2 semitones.

qn3. The quasi-note generators are $\{14, 16, 17\}$. The harmonic leaps are $\{1, 2, 3\}$. The time signature is $\frac{4}{4}$. We form a 27 quasi-note scale s whose first quasi-note has fundamental given by the C below middle C. The successive quasi-notes in the scale are separated by $1, 2, 1, 3, 1, 1, 2, 1, 3, 1, 1, 1, 2, 1, 3, 1, 1, 1, 1, 2, 1, 3, 1, 1, 1, 1$ semitones respectively. The piece consists of 12 two bar phrases. Each of these consists of a sequence of seven quasi-notes, $3, 1, 1, 2, 5, 2, 2$ quavers in duration. In the ith 2 bar phrase these seven quasi-notes are the $2i - 1$th, $2i$th, $2i + 1$th, $2i + 2$th, $2i + 3$th, $2i + 2$th, and $2i$th quasi-notes of s.

qn4. The quasi-note generators are $\{10, 12, 16\}$. The harmonic leaps are given by $\{2, 4, 6, 10, 12\}$. The time signature is $\frac{4}{4}$. The piece has four sections, of 10 bars each, divided into 5 two bar phrases. There is a two bar principal motif, consisting of sixteen consecutive quavers. The first quasi-note of the principal motif has fundamental given by middle C. The consecutive leaps in the principal motif are $2, 2, 2, 2, 2, 2, 2, -6, -6, 4, 4, 4, -4, -4, -4, 2$ semitones. The principal motif recurs in the first two bars of each section, translated by $0, 12, 10, 16$ semitones respectively. In each section the motif is repeated in the following two bars, this time with an accompaniment of eight consecutive crotchets. This phrase of eight crotchets is then repeated with an accompaniment of four consecutive minims. This phrase of four minims is then repeated with an accompaniment of two semibreves.

This phrase of two semibreves is then repeated. In a given section, the pairs of quasi-notes that begin each two bar phrase are the same. They are the first quasi-note of the translated principal motif and the quasi-note whose fundamental is $10, -10, -12, -16$ semitones above that, in the four respective sections. After fixing their rhythms and first quasi-notes in this way, the two bar accompaniments are taken to be minimising consonant accompaniments, avoiding parallel motion.

qn5. The quasi-note generators are $\{18, 19, 23\}$. The harmonic leaps are $\{1, 4, 5\}$. The time signature is $\frac{4}{4}$. We have a sixteen term sequence t given by the list $1, 9, 5, 13, 3, 7, 11, 15, 2, 4, 6, 8, 10, 12, 14, 16$. It begins with 1, is followed by $1, 9$, then $1, 5, 9, 13$, then $1, 3, 5, ..., 15$, then $1, 2, 3, .., 16$, with repetitions deleted. There is a two bar principal motif, consisting of sixteen consecutive quavers. The first quasi-note of the principal motif has middle C as fundamental. The consecutive leaps in the principal motif are $1, 4, 1, 4, 1, 1, 1, -4, -4, -5, 5, 4, -4, -5, -1$ semitones. The piece consists of sixteen two bar phrases. The principal motif recurs in each of these, successively translated by $5, -1, -1, 5, -1, -1, 5, -1, -1, 5, -1, -1, 5, -1, -1$ semitones. Our motifs come with an accompaniment. The ith recurrence of the principal motif is accompanied by i quavers, at positions t_1,t_2,...,t_i in the two bar phrase. If $i \leq 8$, the first quasi-note of the ith accompaniment has a fundamental 18, (respectively 19, 23) semitones above the first quasi-note of the ith recurrence of the principal motif, if $i = 1 \mod 3$ (respectively $i = 2, 0 \mod 3$). If $i \geq 9$, the first quasi-note of the ith accompaniment has a fundamental 18, (respectively 19, 23) semitones below the first quasi-note of the ith recurrence of the principal motif, if $i = 1 \mod 3$ (respectively $i = 2, 0 \mod 3$). After fixing their rhythms and first quasi-notes in this way, the two bar accompaniments are taken to be minimising consonant accompaniments, avoiding parallel motion.

qn6. The quasi-note generators are $\{7, 9, 16 = 7 + 9\}$. The harmonic leaps are $\{2, 7, 9\}$. The time signature is $\frac{4}{4}$. The piece has four sections, of 10 bars each, divided into 5 two bar phrases. The four sections are labelled $s2, s1, f1, f2$ consecutively. There is a two bar principal motif, consisting of eight notes whose durations are $3, 1, 2, 2, 2, 2, 2, 2$ quavers respectively. The first quasi-note of the principal motif has fundamental given by the G below middle C. The consecutive leaps in the principal motif are $7, 2, 2, -9, 7, 2, 2$ semitones. The first two bar phrases of the four sections contain the principal motif translated by $0, 7, 9, 16$ semitones respectively. The next two bar phrases contain that motif with the odd/even numbered notes of the first/second bar deleted. The next two bar phrases contain that motif with the odd/even numbered notes of both bars deleted. The next two bars contain that motif with the beginning/final three notes of the first/second bar deleted, and the odd/even notes of the other bar deleted. The next two bars contain that motif with the beginning/final three notes of both bars deleted. Here we take odd or even, depending on whether our section labelling is 1 or 2. We take beginning or final, depending on whether our labelling is 1 or 2. We take first or second, depending on whether labelling is f or s. In each section, the deteriorating motif has an accompaniment, which is silent for the last two bars of the section. The rhythm of the accompaniment for the ith two bar phrase is the rhythm of the deteriorated motif for the $i + 1$th two bar phrase. In each section, the first quasi-note of the accompaniment has fundamental that is 16 semitones above the fundamental of the quasi-note it accompanies. After fixing their rhythms and first quasi-notes in this way, the accompaniments are taken to be minimising consonant accompaniments, avoiding parallel motion.

qn7. The quasi-note generators are $\{8, 10, 18 = 8 + 10\}$. The harmonic leaps are $\{2, 8, 10\}$. The time signature is $\frac{4}{4}$. There is a two bar principal motif, consisting of two minims followed by a semibreve. The first quasi-note of the principal motif has fundamental the C below middle C. The consecutive leaps in the principal motif are $8, 2$ semitones. The piece has seven sections of 6 bars, each of which contains the principal motif, its three notes permuted and repeated three times. The seven successive permutations, as elements of S_3, are $1, a, ba, aba, baba, ababa, bababa = 1$, where $a = (1\ 2)$ and $b = (2\ 3)$. The occurrences of the motif in the first six sections are accompanied, with the rhythm of the accompaniment identical to that of the motif. We have three quasi-notes accompanying the ith iteration of the motif in a given section. The fundamental of the first of these is 18 (respectively $10, 8$) semitones above the fundamental of the note it accompanies for $i = 1$ (respectively $i = 2, 3$). The second is a nearest consonant quasi-note to the first, avoiding parallel motion. The third is a nearest upper dissonant quasi-note to the second, avoiding parallel motion.

qn8. The quasi-note generators are $\{11, 12, 17\}$. The harmonic leaps are given by $\{1, 5, 6, 11, 12\}$. The time signature is $\frac{3}{4}$. The piece is in two sections of 9 bars. The second section is the first section, reversed and translated by 6 semitones. The piece is in three parts. The first part of the first section consists of three quasi-notes, each three bars long. The first quasi-note of the first part has fundamental given by the C below middle C. The consecutive leaps in the first part of the first section are $1, 5$ semitones. The second part consists of 9 quasi-notes, each a bar long. The ith bar of the second part consists of a quasi-note whose fundamental is 11 (respectively $12, 17$) semitones above the fundamental of the quasi-note of the first part it accompanies, for $i = 1 \mod 3$ (respectively $i = 2, 0 \mod 3$). The third part consists of 27 quasi-notes, each a crotchet long. The ith quasi-note of the second part consists of a quasi-note whose fundamental is 11 (respectively $12, 17$) semitones above the fundamental of the quasi-note of the second part it accompanies, for $i = 1 \mod 3$ (respectively $i = 2, 0 \mod 3$).

5. Concluding remarks.

To conclude we make some remarks on the sound of our compositions. They sound like music written on a stave, as indeed they are. They sound different from tonal music for stringed instruments, although our techniques were obtained by generalising and abstracting techniques of tonal music for stringed instruments. The composition that sounds closest to tonal music for stringed instruments is perhaps qn2, unsurprising since the quasi-note generators include a perfect fifth and an octave plus a major third.

Consonant quasi-notes merge into a single sound. Successive quasi-notes of our motifs share a common harmonic, giving a sense of continuity to the motifs. Successive passages of our pieces are also constructed to have common harmonic material, creating a sense of additional continuity. To observe these phenomena more clearly, for comparison we have stripped the fundamentals of their upper partials and recorded the results for qn6 and qn7 as qn6 and qn7stripped [11]. Scores for qn6 and qn7 are also available [11].

The contrapuntal effects of the minimising accompaniments of qn4-qn6 are somewhat similar to contrapuntal effects found in tonal music (eg. [2]), although the technique we use concerns quasi-notes, and is new. The minimising upper dissonant accompaniment of qn7 is slightly different from these: the contrapuntal feeling is maintained, but it does indeed sound more dissonant, at least to the author's ear.

References

[1] J. S. Bach, *Prelude in C major*, BWV 846.
[2] J. S. Bach, *Two Part Inventions*, BWV 772-786.
[3] D. J. Benson, *Music: A Mathematical Offering*, Cambridge University Press, November 2006.
[4] C. Clifton, I. Quinn, and D. Tymoczko, *Generalized Voice Leading Spaces*, Science 320, 346-348, 2008.
[5] A. Farnell, *Designing Sound*, Applied Scientific Press, 2006.
[6] J. J. Fux, *Gradus ad Parnassum*, 1725.
[7] D. Huron, *Sweet Anticipation. Music and the Psychology of Expectation*, Bradford, 2008.
[8] F. Lerdahl and R. Jackendoff, *A generative theory of tonal music*, The MIT Press, 1983.
[9] P. Manning, *Electronic and Computer Music*, Oxford University Press, 2013.
[10] K. Stockhausen, *Studie II*, 1954.
[11] W. Turner, *qn1-8*, `http://homepages.abdn.ac.uk/w.turner/pages/`.
[12] W. Turner, *Some scales that are similar to the chromatic scale*, `http://homepages.abdn.ac.uk/w.turner/pages/`.
[13] W. Turner, *Algebraic pure tone compositions constructed via similarity* `http://homepages.abdn.ac.uk/w.turner/pages/`.
[14] D. Tymoczko, *The geometry of Musical Chords*, Science 313, 72-74, 2006.