ON REPRESENTING CONSONANCE STRUCTURES

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Abstract. We consider the representation of consonance structures in music, and consider two families of examples. We thus derive pieces of music which are similar to existing ones, via generalised tunings.

In the first family of examples, certain representations define interpretations of two part compositions, written on a stave. The relevant consonance structures are quivers, built from consonances between notes that are close in the score, as determined by a certain algorithm. We present some interpretations of Bach's Invention No. 9 in F minor. Special cases of our interpretations are certain tunings to equal temperament and just intonation.

In the second family of examples, we represent consonance structures found in one part compositions written on the stave with contrapuntal pieces, with harmonics of notes in the one part composition corresponding to motifs in the contrapuntal pieces. We present an example where the one part composition is the first twelve crotchets of the folk song 'The False Bride'.

1. Introduction.

Here, by a representation of a mathematical object, we mean merely a realisation of that object in a second context. In our examples, this context will be musical. Often in the mathematical literature, a representation has a more specific definition, namely a functor from some category to a category of vector spaces. Some such representations appear in section 2.

Consider the piece of music consisting of three consecutive piano notes, the first of which is middle C, the second of which is the C above middle C, and the third of which is the C two octaves above middle C. We can think of this as a representation of the labelled quiver

$$\circ \xrightarrow{\alpha} \circ \xrightarrow{\alpha} \circ$$

where the vertices of the quiver are represented by the notes of the piece, and the arrows are represented by certain consonances between the notes, namely octaves. We call the quiver a consonance structure. Thus the piece is a representation of a consonance structure.

We can represent the above quiver in different ways, with vertices represented as musical elements and arrows represented by consonances. We can reorder the three notes of our piece, by playing them in reverse, for example. Or we can represent the arrows by perfect fifths instead of octaves: eg. we can take the piece whose three consecutive notes are middle C, the G above middle C, and the D above that. To give another example, we can represent the vertices of the quiver with motifs and the arrows as octaves: we can take a piece that consists of three consecutive motifs, the first of which begins on middle C, the second of which is the same motif

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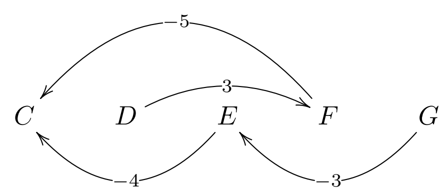
transposed up an octave, and the third of which is that motif transposed up an octave.

In this article, we present two families of examples of musical pieces obtained by representing consonance structures in this way. Our motivation is to derive pieces of music which are similar to existing ones, in a specified way. There is a long tradition of deriving music from existing music by similarity, for example by transposition, by retuning, by arrangement, by giving variations, cover versions, interpretations, pieces derived by machine learning (e.g. [5]), etc.. Our approach is formal, and each example we give associates music to a single source piece, rather than to a large body of work.

In the first family of examples, we represent consonance structures of two part compositions written on the stave. We associate quivers to musical scores of such two part compositions, where the quivers' arrows correspond to certain consonances between notes that are close in the score, in a certain sense. We then represent these quivers in various ways, to create interpretations of the musical scores that may differ substantially from conventional interpretations played on a piano. Arrows in our quiver also correspond to consonances in the interpretation. We demonstrate this where the score is a certain Two Part Invention by Bach [1].

To give a flavour of the analysis required, let us describe a quiver associated to the ascending five note sequence in the C major scale, beginning at middle C and ending at the G above (for more details, see Example 4):



Note that there is one vertex of our quiver for every note of our piece, and when an arrow pointing from vertex x to vertex x' is labelled by an integer i, it takes a rise of i semitones to get from x to x'. The intervals indexing arrows of our quiver are consonant intervals, in the sense that two notes which differ by such an interval share a common harmonic. For example C is 4 semitones below E so the interval relating them, in just intonation, has a frequency ratio of $\frac{4}{5}$; this means the fifth harmonic of C has the same frequency as the fourth harmonic of E. It is not the case that every consonant interval corresponds to an arrow in our quiver. For example the interval from C to G is a perfect fifth, which is consonant, being given by a frequency ratio of $\frac{3}{2}$ in just intonation; however, there is no arrow connecting C and G in our quiver.

The underlying graph of the quiver is a tree: there is a unique path between any two vertices in this graph.

In our representation, we retune the harmonics of all our notes, as well as their frequencies, in a consistent way. The resulting notes have fundamental frequencies

$$F_0 \qquad \zeta_2\zeta_3^{-2}\zeta_5F_0 \qquad \zeta_2^{-2}\zeta_5F_0 \qquad \zeta_2^2\zeta_3^{-1}F_0 \qquad \zeta_2^{-1}\zeta_3F_0$$

for some positive real numbers $F_0, \zeta_2, \zeta_3, \zeta_5$. We recover equal temperament as one of our tunings ($F_0 = 2^{\frac{1}{4}} \cdot 220$, $\zeta_2 = 2$, $\zeta_3 = 2^{\frac{19}{12}}$, $\zeta_5 = 2^{\frac{28}{12}}$), and a certain form of just intonation as another example ($F_0 = 264$, $\zeta_2 = 2$, $\zeta_3 = 3$, $\zeta_5 = 5$).

In the second family of examples, we represent consonance structures found in one part compositions written on the stave as contrapuntal interpretations of the one part compositions, with harmonics of notes in the one part composition corresponding to motifs in the interpretation. We demonstrate this where the original one part composition is the first twelve crotchets of the folk song 'The False Bride' [10]. We consider a graph whose vertices are the notes in the C major scale between two Ds an octave apart, and whose edges are various consonances between these (see Figure 1; here we forget the orientation of the relevant quiver, and we relabel the edges from 0 to 11). This defines a consonance structure. We represent this structure with its 12 edges represented by 12 motifs, and each vertex v represented in various ways as a contrapuntal piece with motifs taken from those corresponding to edges attached to v. The first 12 crotchets of the folk song 'The False Bride' belong to the C major scale between two Ds an octave apart. We associate to this a composition so that notes in the original correspond to short contrapuntal pieces in the interpretation, and harmony is given by the representation of the 12 motifs as above. These 12 motifs are taken from 'The False Bride' as well. The durations of the short contrapuntal pieces are a multiple of the durations of the corresponding notes of the original.

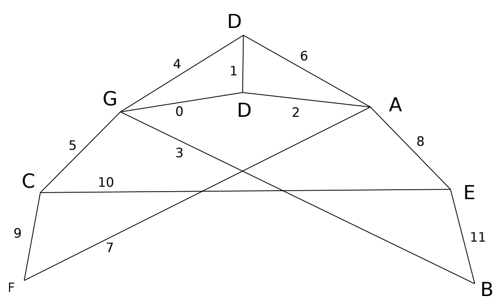


FIGURE 1. The notes of the C major scale running from D to D, together with consonances labelled from 0 to 11.

Above and below, by a consonance, we mean sonic material common to a pair of sounds, for example a common harmonic. In our representations, a consonance represents an edge of a graph, whose two vertices are represented by the sounds with common material. It is sometimes convenient to work instead with an orientation of such a graph, which is a quiver.

2. Representing consonance structures in two part compositions.

Suppose for each frequency $f \in \mathbb{R}$ we have a note modelled on the sound of a string vibrating with frequency f, given by the function from $\mathbb{R}$ to $\mathbb{R}$ sending t to $\sum_{i=1}^{5} \sin(2\pi f i t)/\sqrt{i}$. If we are given two such notes, whose fundamental frequencies f differ by a ratio of 5/4, then these notes will be consonant, with the frequency of the fourth harmonic of one coinciding with the frequency of the fifth harmonic of the other. Let us tweak our fifth harmonic, so that our function from $\mathbb{R}$ to $\mathbb{R}$ indexed by f sends t to $\sum_{i=1}^{5} \sin(2\pi f \zeta(i) t)/\sqrt{i}$, where $\zeta(i) = i$ for $i = 1, 2, 3, 4$ and $\zeta(5) = \frac{24}{5}$. If we are given two notes whose fundamental frequencies f differ by a ratio of 5/4, then these notes will no longer be consonant in the above sense. Consequently, if we play a piece of music with a given set of fundamental frequencies with these tweaked functions, its consonances will not necessarily correspond to those associated to the same piece of music played with the original functions.

Here we introduce a technique for adjusting harmonics away from their respective multiples of the fundamental, whilst preserving a significant portion of the consonance structure of a piece of music. Except in certain special cases, in which no pitch drift occurs (see Remark 1), we do not preserve all of the consonances of the piece, but only consonances that are close in the score, as specified by a certain algorithm. Preserving only these consonances allows for more exotic interpretations of our piece than restricting ourselves to the case of no pitch drift would.

More precisely, we retune the second, third, fifth, and seventh harmonics of our notes, from frequencies that are approximately 12, 19, 28, 34 semitones above the fundamental, to frequencies that are ζ_2, ζ_3, ζ_5, ζ_7 times the fundamental. We retune the fourth, sixth, eighth, ninth and tenth harmonics to frequencies that are ζ_2^2, $\zeta_2\zeta_3$, ζ_2^3, ζ_3^2, $\zeta_2\zeta_5$ times the fundamental. In case $\zeta_2 = 2$, $\zeta_3 = 3$, $\zeta_5 = 5$, $\zeta_7 = 7$ we recover our original piece, played in various forms of just intonation.

So that the sounds we create are relatively simple, we restrict our attention to pieces of music in two parts.

Let $\mathcal{M}$ denote the subgroup of $\mathbb{Q}^\times$ generated by $2, 3, 5, 7$. Consider the group homomorphism $m : \mathcal{M} \to \mathbb{Z}$ sending $2, 3, 5, 7$ to $12, 19, 28, 34$ respectively. Suppose we have a fixed natural number n with $5 \leq n \leq 10$. We define g to be the restriction of m to $\{1, 2, ..., n\} \cdot \{1, 2, ..., n\}^{-1}$. Let $\mathcal{G}$ denote the image of g. Suppose we have a fixed section s of g.

Suppose we are given a two part composition on the stave, such as a Bach Two-Part Invention. We denote one of the parts 1 and the other part 2. We have a linear order of the notes of our composition, where notes are ordered by start time, and given two notes starting at the same time we precede the note in part 2 by the note in part 1. We denote by N the number of notes of our composition.

For $1 \leq x \leq y \leq N$ define the sequence $S(x, y)$ of elements of $\{1, 2, ..., N\}$ to be

$$(x + 1, x + 2, x + 3, ..., y, x - 1, x - 2, x - 3, ..., 1).$$

We denote by $i(x, y) \in \mathbb{Z}$ the number of semitones required to ascend from the x^{th} note to the y^{th} note.

We define a quiver Q whose vertices are given by the notes of our composition, and whose arrows are labelled with elements of $\mathcal{G}$. This is the consonance structure which, when represented, defines an interpretation of our two part composition.

Our algorithm to define Q begins with a quiver with a single vertex, corresponding to the first note of the composition, and no arrows; it adds vertices and arrows successively. We run through elements y of $\{1, 2, ..., N\}$ consecutively, in standard order. For a fixed y we run through the elements x with $1 \leq x \leq y$ in reverse order. For a fixed x and y we search through $S(x, y)$ for vertices in our quiver to connect to x. If x already belongs to our quiver, we abandon our search through $S(x, y)$ straightaway. Otherwise we run through the elements z of $S(x, y)$ in sequence. If $i(x, z) \in \mathcal{G}$ and z belongs to our quiver, we add x to our quiver, draw an arrow from x to z, labelled with $i(x, z)$, and discontinue the search through $S(x, y)$.

The underlying graph of Q is a tree, since our algorithm involves adding leaves successively. We will assume that the vertex set of Q is the set of all notes of our composition, although there do exist examples where this is not the case.

Choose positive real numbers ζ_2, ζ_3, ζ_5, ζ_7. These determine a homomorphism ζ from $\mathcal{M}$ to $\mathbb{R}$ sending p to ζ_p, for $p = 2, 3, 5, 7$. Consider the double of Q, which is the quiver obtained from Q by adjoining a single reverse arrow from v' to v for every arrow from v to v' in Q. We label the arrows of the double of Q as follows: given an arrow in our quiver labelled by i, we label the corresponding arrow in our double quiver with $\zeta(s(i))$ and the corresponding reverse arrow with $\zeta(s(i))^{-1}$.
A path in the underlying graph of Q determines a path in the double of Q, and thus a real number, via the above representation: this real number is the product of the real numbers labelling the arrows in the path. Choose an initial frequency $F_0 \in \mathbb{R}$. Every vertex v of our quiver is connected by a unique path in the underlying graph of Q from the first note of the composition, and thus multiplying the real number given by this path by F_0 determines a frequency, which gives the frequency of v.
To a frequency $f \in \mathbb{R}$ we assign the function from $\mathbb{R}$ to $\mathbb{R}$ sending t to the sum $\sum_{i=1}^{n} \sin(2\pi f \zeta(i) t)/\sqrt{i}$. We call $f\zeta(i)$ the i^{th} harmonic of this function. To a vertex v of our quiver we have associated a frequency, and to a frequency we have assigned a function. We call the resulting function 'the function assigned to v'.
By construction, an arrow in Q directed from v_1 to v_2 corresponds to at least one common harmonic of the functions assigned to v_1 and v_2. Indeed, if $s(i(v_1, v_2)) = \alpha/\beta$, for $1 \leq \alpha, \beta \leq n$, then the frequencies of v_1 and v_2 differ by the factor $\zeta(\alpha/\beta)$, and the fact that ζ is a group homomorphism implies the α^{th} harmonic of the function assigned to v_1 is equal to the β^{th} harmonic of the function assigned to v_2. We obtain a piece by playing, for every vertex v, the function assigned to v, for the duration of the note associated to v in our score. This piece is a representation of the consonance structure Q, and an interpretation of the original two part composition.

Remark 1 The kernel of m is generated by $2^{-4}{\cdot}3^4{\cdot}5^{-1}$, $2^7{\cdot}5^{-3}$, and $2^2{\cdot}3^2{\cdot}5^{-1}{\cdot}7^{-1}$. We denote the images of these elements under ζ as σ_1, σ_2, and σ_3. In case two notes of our score occupy the same position on the stave, their pitches in our interpretation differ by a frequency ratio r in the multiplicative group generated by σ_1, σ_2, and σ_3. In this case we we call r the *pitch drift* between the two notes of our interpretation.
The cases where σ_1, σ_2 and σ_3 are all equal to 1 are special. In such a case there is no pitch drift (i.e. each pitch drift ratio is equal to 1), and we have $\zeta_2 = 2^a$, $\zeta_3 = 2^{\frac{19a}{12}}$, $\zeta_5 = 2^{\frac{28a}{12}}$, $\zeta_7 = 2^{\frac{34a}{12}}$ for some $a \in \mathbb{R}$. The case $a = 1$ corresponds to the tuning of our piece to equal temperament.

Remark 2 Strictly speaking, we do not need the double quiver of Q for our construction, since every path in the underlying graph of Q from a vertex to the first vertex is in fact a path in Q, as is visible by induction on the number of vertices. However, it makes sense to introduce the double of the quiver so that we have the following property: Suppose v and v' are vertices of Q. Then the ratio of the frequency of v' to the frequency of v is given by the real number associated to any path in the double of Q from v to v'.

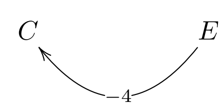
Remark 3 A quiver is a set of generators for a free category: the objects of the free category are given by the vertices of the quiver, and the generating arrows of the free category are given by the arrows of the quiver.

Let us associate to our quiver Q the category $\mathcal{C}_Q$, which is the free category associated to the double of Q, modulo the relations $\alpha\beta = id$, for α and β opposing arrows in our double quiver. Since the underlying graph of Q is a tree, the category $\mathcal{C}_Q$ has a unique morphism from v to v' for any pair of vertices v and v' [6]. It is the free groupoid on our quiver Q.
The data associated to our two part composition above encodes a linear representation of $\mathcal{C}_Q$, which means a functor from that category to a category of vector spaces. This functor sends a vertex of Q to the real vector space $\mathbb{R}$; it sends an arrow from v to w in Q, labelled by γ, to the scalar in $Hom_{\mathbb{R}}(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}$ given by $\zeta(s(\gamma))$; and it sends the corresponding reverse arrow to the corresponding inverse element of $Hom_{\mathbb{R}}(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}$.
Actions of groupoids, or even groups, on musical spaces have been studied by other authors in different contexts, see [3] and the references therein, eg. [7]. Quivers feature in an exploration of the relation between gestures and music by Mazzola and Andreatta [8]. Quiver representations have been extensively studied in algebra, eg. see [4].

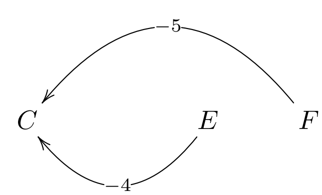
Example 4 Let $n = 6$. We run through our procedure to represent the ascending five note sequence in the C major scale, beginning at middle C and ending at the G above.
By definition, $\mathcal{G}$ is the set of differences between elements of $\{0, 12, 19, 24, 28, 31\}$. We see $4 = 28 - 24$ belongs to $\mathcal{G}$, as does $3 = 31 - 28$ and $5 = 24 - 19$. However 1 and 2 do not belong to $\mathcal{G}$. We have $N = 5$.
We first associate a labelled quiver to the five note sequence. Our algorithm begins with the first note of the ascending five note sequence, which belongs to our quiver:

$$C$$

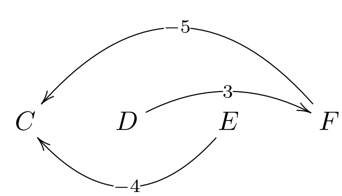
When $y = 1, 2$ the algorithm produces no new vertices or arrows. When $y = 3$ and $x = 3$ the algorithm gives a new vertex and a new arrow, which is labelled by $-4 \in \mathcal{G}$:



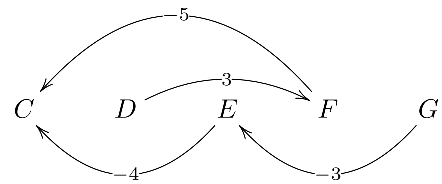
When $y = 4$ and $x = 4$ the algorithm gives a new vertex and a new arrow, which is labelled -5:



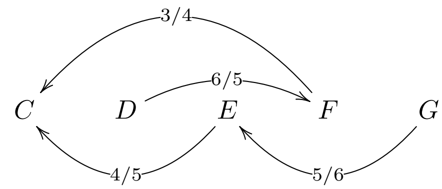
When $y = 4$ and $x = 2$ the algorithm gives a new vertex and a new arrow, which is labelled 3:



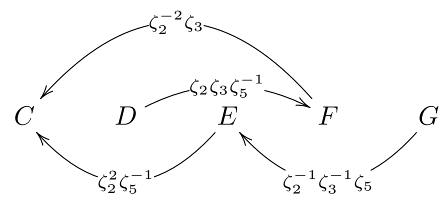
When $y = 5$ and $x = 5$ the algorithm gives a new vertex and a new arrow, which is labelled -3:



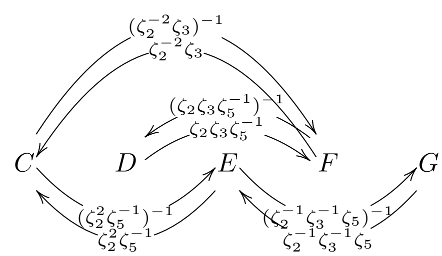
We next lift our intervals in $\mathcal{G}$ to their corresponding frequency ratios in the set $\{1, 2, ..., 6\}.\{1, 2, ..., 6\}^{-1}$:



We then represent our arrows via the group homomorphism ζ:



The corresponding labelled double quiver is:



By tracing paths in our double quiver from the vertex C, with its corresponding base frequency F_0, we associate the following five frequencies to the five vertices of our quiver:

$$F_0, (\zeta_2\zeta_3\zeta_5^{-1})^{-1}(\zeta_2^{-2}\zeta_3)^{-1}F_0, (\zeta_2^2\zeta_5^{-1})^{-1}F_0, (\zeta_2^{-2}\zeta_3)^{-1}F_0, (\zeta_2^{-1}\zeta_3^{-1}\zeta_5)^{-1}(\zeta_2^2\zeta_5^{-1})^{-1}F_0$$

or in other words:

$$F_0 \qquad \zeta_2\zeta_3^{-2}\zeta_5F_0 \qquad \zeta_2^{-2}\zeta_5F_0 \qquad \zeta_2^2\zeta_3^{-1}F_0 \qquad \zeta_2^{-1}\zeta_3F_0$$

This sequence of frequencies defines our interpretation of the ascending five note sequence in the C major scale, beginning at C and ending at G. If we set $F_0 = 264$, $\zeta_2 = 2$, $\zeta_3 = 3$, $\zeta_5 = 5$, then our sequence of five frequencies is

$$264 \qquad \tfrac{10}{9}\cdot 264 \qquad \tfrac{5}{4}\cdot 264 \qquad \tfrac{4}{3}\cdot 264 \qquad \tfrac{3}{2}\cdot 264$$

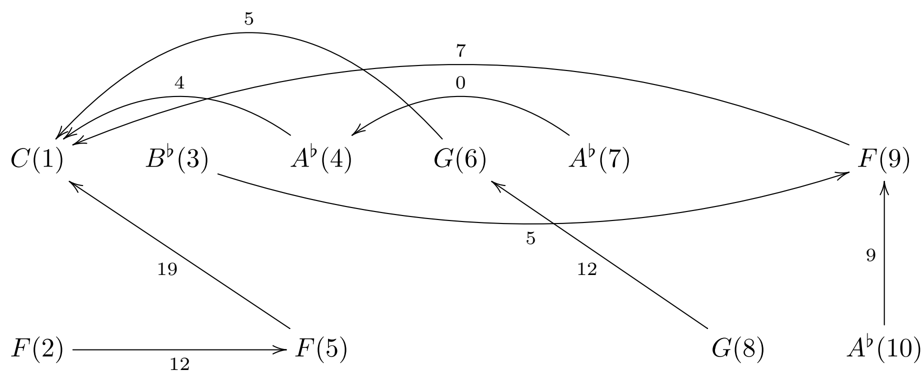
If we play notes of these frequencies in sequence, we recover our original five note sequence, played in a form of just intonation.

If we set $F_0 = 2^{\frac{1}{4}} \cdot 220$, $\zeta_2 = 2$, $\zeta_3 = 2^{\frac{19}{12}}$, $\zeta_5 = 2^{\frac{28}{12}}$, then our sequence of five frequencies is

$$2^{\frac{3}{12}} \cdot 220 \qquad 2^{\frac{5}{12}} \cdot 220 \qquad 2^{\frac{7}{12}} \cdot 220 \qquad 2^{\frac{8}{12}} \cdot 220 \qquad 2^{\frac{10}{12}} \cdot 220$$

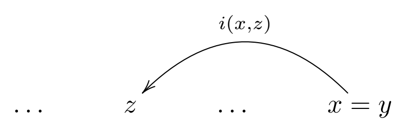
If we play notes of these frequencies in sequence, we recover our original five note sequence, played in equal temperament.

Example 5 Let $n = 5$. Consider Bach's Two Part Invention No. 9. The associated quiver has 559 vertices. Let us describe the arrows between the first ten vertices:



Here, the top line corresponds to the first part of the piece, and the bottom line corresponds to the second part of the piece. Time moves from left to right. The letter indexing a vertex indicates its pitch class whilst the corresponding number in brackets indicates the position in the set $\{1, 2, 3..., 559\}$ that it occupies.

Beyond the first ten notes, the quiver is simpler, in the sense that every vertex is consonant with some vertex preceding it. Let us say two notes v and v' are $\mathcal{G}$-differing if their pitches differ by an element of $\mathcal{G}$ when measured in semitones. Then the y^{th} step in our algorithm merely involves connecting $x = y$ to the closest preceding $\mathcal{G}$-differing element z with an arrow; since $z < x$ this arrow points in the same direction as $<$.



An example of this pattern is already visible in the ten vertex quiver above, in case $y = 10$. The tenth note of the piece, an $A^\flat$, has closest preceding $\mathcal{G}$-differing element given by the ninth note of the piece, an F. The $y = 10$ step of our algorithm consists of attaching the tenth vertex, and a labelled arrow from that vertex to the ninth vertex.

The 49^{th} note of the piece is a C, like the first note of the piece. If we follow our procedure, we find the frequency ratio of the 49^{th} note to the first is $\sigma_1 = \zeta_2^{-4}\zeta_3^4\zeta_5^{-1}$. The ratio σ_1 determines the pitch drift from the first note to the 49^{th} note of our interpretation. Note that $\sigma_1 = 1$ implies a consonance between the first note and the 49^{th} note of our interpretation.

More specifically, in case there is no pitch drift throughout the piece, we have $\sigma_1 = \sigma_2 = 1$ and so $\zeta_2 = 2^a$, $\zeta_3 = 2^{\frac{19a}{12}}$, $\zeta_5 = 2^{\frac{28a}{12}}$, for some $a \in \mathbb{R}$. In this case,

every interval of i semitones in our score, $i \in \mathcal{G}$, is represented by the frequency ratio $2^{\frac{ai}{12}}$, and corresponds to a consonance in our interpretation.
If we specify $\zeta_2 = 2$, $\zeta_3 = 3$, $\zeta_5 = 5$ then we have a form of just intonation. Then pitch drift occurs through the piece so that, in some cases, two notes occupy the same point on the stave but are represented by different frequencies. Some other tunings with this feature are outlined in an article by Stange, Wick, and Hinrichsen [9]. Restricting ourselves to tunings such that two notes occupying the same point on the stave are represented by the same frequency, a number of forms of just intonation are described in Benson's book [2].

Our remaining examples are all interpretations of Bach's Invention No. 9. In the sound files that accompany the paper [12], the durations of semiquavers are 0.1s, 0.3s, 0.3s, 0.2s, 0.1s, 0.7s, 0.4s, 0.5s, 0.3s, and 0.3s respectively.

Example 6 Let $n = 6$. Let $F_0 = 528$. Only the first six harmonics are involved here, so the real numbers ζ_7 and σ_3 are not relevant. We take $\zeta_2 = 2^a$, $\zeta_3 = 2^{\frac{19a}{12}}$, $\zeta_5 = 2^{\frac{28a}{12}}$, where $a = 1.2$. Then $\sigma_1 = \sigma_2 = 1$ and we have no pitch drift.

Example 7 Let $n = 5, 8$ and $F_0 = 528$. We take $(\zeta_2, \zeta_3, \zeta_5, \zeta_7) = (2, 3, 5, 7)$. These are forms of just intonation, and we have $\sigma_1 = \frac{81}{80}$, $\sigma_2 = \frac{128}{125}$, $\sigma_3 = \frac{36}{35}$.

Example 8 Let $n = 5$. Only the first five harmonics are involved here, so the real numbers ζ_7 and σ_3 are not relevant. The gradual pitch drift from the C at the beginning of the piece to the C in the second bar from the end is given by the factor $\sigma_1^8\sigma_2^2$, so in choosing ζ_2, ζ_3 and ζ_5, steps are taken to make this factor fairly close to 1, to avoid the composition becoming inaudible to a large extent. Here we choose natural numbers for the ζ_ps, to create similarities with the harmonic series. Our two examples are $(\zeta_2, \zeta_3, \zeta_5) = (3, 5, 10), (2, 3, 4)$. We take $F_0 = 528, 132$.

Example 9 Let $n = 5$. Our three examples are $(\zeta_2, \zeta_3, \zeta_5) = (2, 31/10, 47/10)$, $(\frac{9}{4}, \frac{27}{8}, 5)$, $(\frac{3}{2}, 2, 3)$. We take $F_0 = 168$, 264, 528.
Note that in these examples, we select some natural numbers for ζ_2, ζ_3 and ζ_5 to create partial similarities with the harmonic series. Approximate relations between short words in the ζ_ps and rational numbers with small numerator and denominator also occur, such as $(\frac{31}{10})^2 \cdot 2^{-3} \approx 6/5$, $(\frac{9}{4})^3 \cdot (\frac{27}{8})^{-2} = 1$, $(\frac{9}{4})^2 \cdot 5^{-1} \approx 1$. These are audible as recognisable intervals in the music. For example, if we look at the interval between the first two notes in the first part of our piece, from $C(1)$ to $B^\flat(3)$, we see in our interpretation it is given by a frequency ratio of $\zeta_2^3\zeta_3^{-2}$. In case $(\zeta_2, \zeta_3, \zeta_5) = (2, 31/10, 47/10)$ this frequency ratio is $2^3 \cdot (\frac{31}{10})^{-2} \approx 5/6$, and therefore approximates the familiar interval of a minor third.

Example 10 We end this section with two examples where none of the ζ_ps are natural numbers. Let $n = 5$. Let $(\zeta_2, \zeta_3, \zeta_5) = (\frac{63}{16}, \frac{15}{2}, \frac{31}{2}), (\frac{42}{10}, \frac{63}{10}, \frac{105}{10})$. Let $F_0 = 264$, 1056.

3. Some background.

Our second family of representations of consonance structures have components which are contrapuntal pieces constructed from a family of motifs.

Our constructions are based on the following observation, made previously: suppose we are given a set of musical phrases that contains a phrase of a single note, is closed under concatenation, and is closed under the action of a set of similarity transformations, that send phrases to similar phrases; we can then form such a composition by starting with the note, applying a sequence of similarity transformations to obtain a sequence of notes, concatenating these to form a phrase, applying a sequence of similarity transformations to obtain a sequence of phrases, concatenating these to form a longer phrase, etc. [11]. We call a composition generated in this way a *similarity concatenation composition.*

The scales of motifs we use are generalisations of scales of so-called quasi-notes. We recall some of the relevant definitions [13].

A note on a stringed instrument has a fundamental frequency f, and a set of overtones, whose frequencies are $2f$, $3f$, $4f$, etc. [2] The first five of these overtones are approximately $12, 19, 24, 28, 31$ semitones above the original frequency f. In imitation of this situation we define further scales indexed by $\mathbb{Z}$ [11]. In each scale, for every point on the stave we define a *quasi-note*, consisting of a *fundamental*, sounding together with a number of *overtones* of the same amplitude and duration. The fundamental is a pure tone whose frequency is given by the relevant point on the stave, whilst each overtone is a pure tone whose frequency is given by the fundamental frequency, raised by a certain number of semitones. The fundamental, and the overtones are called the *partials* of the quasi-note.

Formally speaking, let $a, d, f \in \mathbb{R}_+$. Let χ_I denote the indicator function of an interval $I \subset \mathbb{R}$. For a function $\eta : \mathbb{R} \to \mathbb{R}$ of bounded support, we call the infimum of the support of η the *start* of η, and we call the supremum of the support of η the *end* of η. If $\eta, \eta' : \mathbb{R} \to \mathbb{R}$ and the end of η is equal to the start of η', we call $\eta + \eta'$ the *concatenation* of η and η'. We define a *pure tone* of amplitude a, duration d, and frequency f, to be given by a function from $\mathbb{R}$ to $\mathbb{R}$ sending t to $a\chi_{(o,o+d)} \sin(2\pi f t + \phi)$, for some $o, \phi \in \mathbb{R}$.

Points on the stave are given by frequencies, with the A above middle C corresponding to 440 Hz, and the operation of raising by a semitone corresponding to multiplication by $2^{\frac{1}{12}}$. Points on the stave are also given by integers, with middle C corresponding to 0, and the operation of raising by a semitone corresponding to addition of 1.

Fix $a \in \mathbb{R}_+$. We define a *chromatic combination* of duration d to be given by a function from $\mathbb{R}$ to $\mathbb{R}$ sending t to $a\chi_{(o,o+d)} \sum_{j=1}^{p} \sin(2\pi f_j t + \phi_j)$, for some $o, \phi_j \in \mathbb{R}$, and frequencies f_j given by points on the stave, $j = 1, ..., p$.

Take a set of strictly increasing maps $s_1, s_2, ..., s_p : \mathbb{Z} \to \mathbb{Z}$. Let $\zeta_j, j = 1, ..., p$ be maps from $\mathbb{Z}$ to the set of pure tones of duration d, amplitude a, and start o, such that $\zeta_j(x)$ has frequency determined by the point on the stave given by $s_j(x)$, for $x \in \mathbb{Z}$. Let ζ be the map from $\mathbb{Z}$ to the set of chromatic combinations that sends x to $\sum_{j=1}^{p} \zeta_j(x)$. We call $\zeta(x)$ the x^{th} *quasi-note of duration d and start o*. We call $\zeta_1(x)$ the x^{th} *fundamental*, we call $\zeta_2(x), ..., \zeta_p(x)$ the *overtones*, and we call $\zeta_1(x), ..., \zeta_p(x)$ the *partials* of the x^{th} quasi-note of duration d and start o, for $x \in \mathbb{Z}$. For fixed $d \in \mathbb{R}_+$, and $x \in \mathbb{Z}$, we abuse terminology and call the class of x^{th} quasi-notes of duration d and start o, as o runs through elements of $\mathbb{R}$, the x^{th} *quasi-note* *of duration* d; we call the class of x^{th} quasi-notes of duration d, as d runs through elements of $\mathbb{R}_+$, the x^{th} *quasi-note*, etc.. This abuse of terminology is consistent with standard musical terminology for notes, which may or may not have a well defined start, duration, timbre, etc..

In this way, we have a set Q of quasi-notes, indexed by elements x of $\mathbb{Z}$. Our collection Q is given equivalently by a set of subsets $\Omega_1, \Omega_2, ..., \Omega_p \subseteq \mathbb{Z}$ that are unbounded from above and below, with elements $\omega_1 \in \Omega_1, ..., \omega_p \in \Omega_p$. Indeed, such data emerges when we write $\Omega_j = s_j(\mathbb{Z})$ and $\omega_j = s_j(0)$, for $i = 1, 2, ..., p$.

Suppose $\Omega_u \supseteq \Omega_v$. Then the v^{th} partial of a quasi-note q is equal to the u^{th} partial of a second quasi-note q'. We write $t_{u,v}(q) = q'$, and thus define a transformation $t_{u,v}$ of Q. We denote by Φ the collection of such transformations.

4. A family of contrapuntal compositions.

In this section we construct contrapuntal similarity concatenation compositions from a family of motifs. This is similar to a previous approach [13].

Suppose we have fixed a set Q of quasi-notes, as in section 3. Let us assume that $\Omega_1 \supseteq \Omega_2, ..., \Omega_p$.

Let n be a natural number. Let $\underline{n} = \{1, ..., n\}$. We define an n-motif m to be a map $\alpha : \underline{n} \to \mathbb{Z} \cup \{\emptyset\}$ and a map $\beta : \underline{n} \to \mathbb{Q}_{\geq 0}$ such that $\sum_{i=1}^{n} \beta(i) = 1$.

We have a correspondence between motifs and sequences of notes defined as follows: Let $d \in \mathbb{R}$ with $d > 0$. Let m be an n-motif as defined above. Consider the sequence given by $\zeta_1(\alpha(i))$, $i = 1, ..., n$, played consecutively with duration $d\beta(i)$, where $\zeta(\emptyset)$ is taken to be silence. The resulting sequence of notes has total duration d.

We define a cyclic motif to be a finite subset of $\mathbb{Q}/\mathbb{Z}$, together with a map ζ from that subset to $\mathbb{Z} \cup \{\emptyset\}$. If we write the elements of our subset as $\gamma_1 < \gamma_2 < ... < \gamma_f$, with $0 \leq \gamma_1$ and $\gamma_f < 1$ then we have an associated $f+1$-motif m defined as follows: the corresponding map α sends i to $\zeta(\gamma_{i-1})$ for $2 \leq i \leq f+1$, and sends 1 to $\zeta(\gamma_f)$, whilst the corresponding map β sends i to $\gamma_i - \gamma_{i-1}$ for $2 \leq i \leq f$, sends 1 to γ_1, and sends $f+1$ to $1 - \gamma_f$. We have an action $*$ of $\mathbb{Z}/e\mathbb{Z}$ on $\mathbb{Q}/\mathbb{Z}$ where $y \in \mathbb{Z}/e\mathbb{Z}$ acts as addition of $\frac{y}{e}$. Thus, in our notation y sends $q \in \mathbb{Q}/\mathbb{Z}$ to $y * q \in \mathbb{Q}/\mathbb{Z}$. This action extends to an action $*$ of $\mathbb{Z}/e\mathbb{Z}$ on subsets of $\mathbb{Q}/\mathbb{Z}$. This action extends to an action of $\mathbb{Z}/e\mathbb{Z}$ on cyclic motifs, where ζ_{y*C} sends $y * s$ to $\zeta_C(s)$, for ζ_C the map defining the cyclic motif C and ζ_{y*C} the map defining the cyclic motif $y * C$.

Let l be a natural number with $p \leq l$. Let $n_1, ..., n_l$ be natural numbers. Let $A = \mathbb{Z}/e_1\mathbb{Z} \times ... \times \mathbb{Z}/e_l\mathbb{Z}$, let $B = S_l$, let $C = \mathbb{Z}$, and let $D = \mathbb{R}_+$. We have an action of $\mathcal{E}_A = \mathbb{Z}/e_1\mathbb{Z} \times ... \times \mathbb{Z}/e_l\mathbb{Z}$ on A by left multiplication. We have an action of $\mathcal{E}_B = S_l$ on B by left multiplication. We have an action of the monoid $\mathcal{E}_C$ generated by Φ on C. We have an action of $\mathcal{E}_D = \mathbb{R}_+$ on D by left multiplication.

Let m_j be a cyclic motif, for $j = 1, ..., l$. We describe a scale S indexed by elements (a, b, c, d) of $A \times B \times C \times D$. Let c correspond to the p-tuple of pure tones of duration d indexed by $(s_1(c), ..., s_p(c))$. For $j = 1, ..., p$, we have $s_j(c) = s_1(c_j)$ for some c_j since $\Omega_1 \supseteq \Omega_2, ..., \Omega_p$. For $j = 1, ..., p$ we have a cyclic motif given by $a_{b(j)} * m_{b(j)}$. Let $\alpha_j : \underline{n_{b(j)}} \to \mathbb{Z} \cup \{\emptyset\}$ and $\beta_j : \underline{n_{b(j)}} \to \mathbb{Q}$ be the data determining the associated n-motif. Consider the sequence σ_j of $n_{b(j)}$ pure tones given by points on the stave $s_1(c_j + \alpha_j(i))$, played consecutively, with duration $d\beta_j(i)$, for $i = 1, ..., n_{b(j)}$. Here, if $\alpha_j(i) = \emptyset$ we interpret the pure tone given by $s_1(c_j + \alpha_j(i))$ as silence. The element of S indexed by (a, b, c, d) consists of the sequences $\sigma_1, ..., \sigma_p$ played simultaneously, with the same start. The duration of the scale element is d.

Here, by a *scale* we mean a set of sounds, together with a partial ordering. We order S by $(a,b,c,d) \leq (a',b',c',d')$ if $a = a'$, $b = b'$, $c \leq c'$, and $d = d'$.
We describe similarity concatenation compositions with scale S.
Let w be a natural number, and let $\gamma_1, ..., \gamma_w$ be natural numbers. For $i = 1, ..., w$ we take maps $t_i : \underline{\gamma_i} \to \mathcal{E}_A \times \mathcal{E}_B \times \Phi \times \mathcal{E}_D$. We insist that $t_i(1)$ is equal to the identity, for $i = 1, ..., w$, but that $t_i(\xi)$ is different from the identity, for $\xi = 2, ..., \gamma_i$ and $i = 1, ..., w$.
For $i = 1, ..., w$ we define maps $u_i : \underline{\gamma_i} \to \mathcal{E}_A \times \mathcal{E}_B \times \mathcal{E}_C \times \mathcal{E}_D$, by

$$u_i(\xi) = t_i(\xi)t_i(\xi - 1)...t_i(1).$$

Let us fix an element $q = (a,b,c,d) \in A \times B \times C \times D$ where $c \in \Omega_1$. Our contrapuntal similarity concatenation composition is obtained by concatenating the elements of S corresponding to the elements $(\xi_1, ..., \xi_w)$ of $\underline{\gamma_1} \times \underline{\gamma_2} \times ... \times \underline{\gamma_w}$, ordered lexicographically. The scale element corresponding to $(\xi_1, ..., \xi_w)$ is given by $u_1(\xi_1)u_2(\xi_2)...u_w(\xi_w)q$.

Example 11 We describe a short example, based on the ascending five note sequence in C major, from middle C to the G above. Let $d = 1$, let $l = p = 2$, let $e_1 = e_2 = 4$, and let $\Omega_1 = \Omega_2$ denote the C major scale. Let ω_1 be given by the F below middle C, and ω_2 be given by middle C. Let $q = (0, id, 0, 1)$.
Let m_1 denote the cyclic motif given by the subset $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ of $\mathbb{Q}/\mathbb{Z}$, and the map which sends 0 to 0, $\frac{1}{4}$ to 4, $\frac{1}{2}$ to 2, and $\frac{3}{4}$ to 4. Let m_2 denote the cyclic motif given by the subset $\{0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}\}$ of $\mathbb{Q}/\mathbb{Z}$, and the map which sends 0 to 0, $\frac{1}{8}$ to 1, $\frac{1}{4}$ to 2, $\frac{3}{8}$ to 3, and $\frac{1}{2}$ to 4.
Let $w = 1$ and $\gamma_1 = 2$. Let $t_1(1) = id$ and let $t_1(2) = ((2,3), (1\ 2), t_{2,1}, 1)$. The short contrapuntal piece consists of two consecutive scale elements, namely the two bars depicted below:

We see the motif m_1 first in the lower part, and then repeated, with a cycle, in the upper part. We see the motif m_2 first in the upper part, and then repeated, at the same pitch, with a cycle, in the lower part. The first notes of the first bar are ω_1 and ω_2.

5. Interpretation of 'The False Bride'

Suppose we have a one part composition, whose notes are taken from a scale V. Form a graph whose vertices are the elements of V, and whose edges are certain consonances between elements of V. For example, for our piece we take the first twelve crotchets of 'The False Bride' [10], and we consider consonances as in Figure 1. We represent this consonance structure with a motif for each edge, and a family of contrapuntal similarity concatenation compositions built from such motifs for each vertex, one for each note of the one part composition representing that vertex. The motifs used to build similarity concatenation compositions at a given vertex are those associated to the edges attached to that vertex.

We proceed to use such a representation to create a contrapuntal interpretation of the first twelve crotchets of 'The False Bride'.
We associate the twelve pitch classes with $\mathbb{Z}/12\mathbb{Z}$, so that 0 corresponds to C, and addition of 1 corresponds to a rise in a semitone. In this way the C major scale corresponds to the elements $\{7i+12\mathbb{Z}|-1 \leq i \leq 5\}$. To each pitch class $p = 7i+12\mathbb{Z}$ in the major scale we associate a subset $\omega(p) = \{7(i+k) + 12\mathbb{Z}| - 4 \leq k \leq 4\}$ of $\mathbb{Z}/12\mathbb{Z}$, whose preimage under the quotient map $\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$ is a subset $\Omega(p) \subset \mathbb{Z}$. For every pitch class p of 'The False Bride', the subset $\omega(p)$ contains the pitch class G, and the set $\Omega(p)$ is the image of an increasing map $s(p) : \mathbb{Z} \to \mathbb{Z}$ that sends 0 to the G below the note an octave below middle C.
The elements of the scale of C major all belong to $\Omega(2+12\mathbb{Z})$. We consider the first twelve crotchets of 'The False Bride', which we identify with elements of $\mathbb{Z}/12\mathbb{Z}$ so that the first crotchet corresponds to 0, the second crotchet to 1, etc. We associate a cyclic motif $m(i)$ to $i \in \mathbb{Z}/12\mathbb{Z}$ as follows: Consider the notes to be found in the crotchets corresponding to $i, i+1, i+2$ as defining a sequence of elements of $\Omega(2+12\mathbb{Z})$ together with durations, and thus via $s(2+12\mathbb{Z})^{-1}$ a sequence of elements of $\mathbb{Z}$ together with durations, and if we scale the durations by $\frac{1}{3}$ we obtain a corresponding cyclic motif $n(i)$. We define $m(i)$ to be the cyclic motif given by $\zeta_{m(i)} = \zeta_{n(i)} - \zeta_{n(i)}(0)$. Upon labelling the edges of our consonance structure with elements of $\mathbb{Z}/12\mathbb{Z}$ as in Figure 1, we have a motif associated to each edge of our consonance structure.
We next proceed to associate contrapuntal similarity concatenation compositions to the notes of the first twelve crotchets of 'The False Bride'. We call a similarity concatenation composition corresponding to a given note a section of the piece. The quasi-note that begins a section corresponding to a vertex v in our consonance structure has harmonics with pitch classes that are given by the pitch classes of the harmonics associated to the edges attached to v. Notes of 'The False Bride' of duration d are interpreted as sections of duration $6d$.
Throughout $q = (0, id, 0, \frac{3}{2})$.

Example 12 Section 1 (D). We take $\Omega_i = \Omega(2+12\mathbb{Z})$ for $i = 1, 2, 3$. We take

$$(s_1(0), s_2(0), s_3(0)) = (2, -3, -10).$$

We take $(m_1, m_2, m_3) = (m(2), m(1), m(0))$.
We take $w = 2$ and $(\gamma_1, \gamma_2) = (2, 3)$. We take $t_2(2) = ((0, -1, 0), (2\ 3), t_{2,3}, 1)$, $t_2(3) = ((0, 0, -1), (1\ 2), t_{2,1}, 1)$, $t_1(2) = ((-1, -1, -1), (1\ 2\ 3), t_{1,2}, 1)$.
Section 2 (G). We take $\Omega_i = \Omega(7+12\mathbb{Z})$ for $i = 1, 2, 3, 4$. We take

$$(s_1(0), s_2(0), s_3(0), s_4(0)) =$$
$$(7, 2, -1, -10).$$

We take $(m_1, m_2, m_3, m_4) = (m(5), m(4), m(3), m(0))$.
We take $w = 2$ and $(\gamma_1, \gamma_2) = (3, 3)$. We take $t_2(2) = ((-1, -1, 0, 0), (3\ 4), t_{1,2}, 1)$, $t_2(3) = ((0, 0, -1, -1), (2\ 3), t_{2,3}, 1)$, $t_1(2) = ((-1, -1, -1, -1), (1\ 2\ 3\ 4), t_{3,4}, 1)$, $t_1(3) = ((-1, -1, -1, -1), (1\ 2\ 3\ 4), t_{2,1}, 1)$.
Section 3 (A). We take $\Omega_i = \Omega(9+12\mathbb{Z})$ for $i = 1, 2, 3, 4$. We take

$$(s_1(0), s_2(0), s_3(0), s_4(0)) =$$
$$(9, -3, -8, -15).$$

We take $(m_1, m_2, m_3, m_4) = (m(7), m(6), m(8), m(2))$.

We take $w = 1$ and $\gamma_1 = 3$. We take $t_1(2) = ((-1,-1,0,0), Id, t_{3,4}, 1)$, $t_1(3) = ((0,-1,-1,0), Id, t_{3,4}, 1)$.
Section 4 (G). We take $\Omega_i = \Omega(7+12\mathbb{Z})$ for $i = 1,2,3,4$. We take

$$(s_1(0), s_2(0), s_3(0), s_4(0)) =$$

$$(2,-1,-5,-10).$$

We take $(m_1, m_2, m_3, m_4) = (m(0), m(3), m(5), m(4))$.
We take $w = 2$ and $(\gamma_1, \gamma_2) = (2,3)$. We take $t_2(2) = ((0,0,-1,-1), (1\ 2\ 3\ 4), t_{2,3}, 1)$, $t_2(3) = ((-1,-1,0,0), (1\ 2\ 3\ 4), t_{4,3}, 1)$, $t_1(2) = ((-1,0,-1,0), (1\ 2)(3\ 4), t_{1,2}, 1)$.
Section 5 (F). We take $\Omega_i = \Omega(5+12\mathbb{Z})$ for $i = 1,2$. We take

$$(s_1(0), s_2(0)) = (-12,-15).$$

We take $(m_1, m_2) = (m(9), m(7))$.
We take $w = 2$ and $(\gamma_1, \gamma_2) = (2,3)$. We take $t_2(2) = ((-1,0), (1\ 2), t_{1,2}, 1)$, $t_2(3) = ((0,-1), (1\ 2), t_{1,2}, 1)$, $t_1(2) = ((-1,-1), Id, t_{2,1}, 1)$.
Section 6 (E). We take $\Omega_i = \Omega(4+12\mathbb{Z})$ for $i = 1,2,3$. We take

$$(s_1(0), s_2(0), s_3(0)) = (16, 11, 4).$$

We take $(m_1, m_2, m_3) = (m(8), m(11), m(10))$.
We take $w = 2$ and $(\gamma_1, \gamma_2) = (2,3)$. We take $t_2(2) = ((-1,0,0), (1\ 2\ 3), t_{2,1}, 1)$, $t_2(3) = ((0,-1,0), (1\ 2\ 3), t_{2,3}, 1)$, $t_1(2) = ((-1,-1,0), Id, t_{2,3}, 1)$.
Section 7 (D). We take $\Omega_i = \Omega(2+12\mathbb{Z})$ for $i = 1,2,3$. We take

$$(s_1(0), s_2(0), s_3(0)) = (2,-10,-15).$$

We take $(m_1, m_2, m_3) = (m(0), m(2), m(1))$.
We take $w = 2$ and $(\gamma_1, \gamma_2) = (2,3)$. We take $t_2(2) = ((-1,0,0), (2\ 3), t_{1,2}, 1)$, $t_2(3) = ((0,-1,0), (1\ 2), t_{3,2}, 1)$, $t_1(2) = ((-1,-1,0), (1\ 2\ 3), t_{2,3}, 1)$.
Section 8 (G). We take $\Omega_i = \Omega(7+12\mathbb{Z})$ for $i = 1,2,3,4$. We take

$$(s_1(0), s_2(0), s_3(0), s_4(0)) =$$

$$(2,-5,-10,-13).$$

We take $(m_1, m_2, m_3, m_4) = (m(4), m(5), m(0), m(3))$.
We take $w = 2$ and $(\gamma_1, \gamma_2) = (3,4)$. We take $t_2(2) = ((-1,0,0,0), (1\ 2), t_{3,4}, 1)$, $t_2(3) = ((0,-1,0,0), (3\ 4), t_{3,4}, 1)$, $t_2(4) = ((0,0,-1,-1), (1\ 2)(3\ 4), t_{3,2}, 1)$, $t_1(2) = ((-1,0,0,0), (1\ 3)(2\ 4), t_{1,2}, 1)$, $t_1(3) = ((0,-1,0,0), (1\ 3)(2\ 4), t_{3,2}, 1)$.
Section 9 (A). We take $\Omega_i = \Omega(9+12\mathbb{Z})$ for $i = 1,2,3,4$. We take

$$(s_1(0), s_2(0), s_3(0), s_4(0)) =$$

$$(9,4,-3,-15).$$

We take $(m_1, m_2, m_3, m_4) = (m(7), m(8), m(2), m(6))$.
We take $w = 1$ and $\gamma_1 = 2$. We take $t_1(2) = ((-1,-1,0,0), (1\ 2), t_{2,3}, 1)$.
Section 10 (B). We take $\Omega_i = \Omega(11+12\mathbb{Z})$ for $i = 1,2$. We take

$$(s_1(0), s_2(0)) = (-1,-13).$$

We take $(m_1, m_2) = (m(11), m(3))$.
We take $w = 1$ and $\gamma_1 = 2$. We take $t_1(2) = ((0,0,0,0), Id, t_{1,2})$.
Section 11 (C). We take $\Omega_i = \Omega(0+12\mathbb{Z})$ for $i = 1,2,3$. We take

$$(s_1(0), s_2(0), s_3(0)) = (7,4,0).$$

We take $(m_1, m_2, m_3) = (m(5), m(10), m(9))$.

We take $w = 1$ and $\gamma_1 = 2$. We take $t_1(2) = ((0, -1, -1), (2\ 3), t_{2,3}, 1)$.
Section 12 (D). We take $\Omega_i = \Omega(2 + 12\mathbb{Z})$ for $i = 1, 2, 3$. We take

$$(s_1(0), s_2(0), s_3(0)) = (2, -3, -10).$$

We take $(m_1, m_2, m_3) = (m(1), m(6), m(4))$.
We take $w = 2$ and $(\gamma_1, \gamma_2) = (3, 4)$. We take $t_2(2) = ((-1, 0, 0), Id, t_{2,3}, 1)$, $t_2(3) = ((0, -1, 0), Id, t_{1,2}, 1)$, $t_2(4) = ((0, 0, -1), Id, t_{3,1}, 1)$, $t_1(2) = ((-1, 0, 0), (1\ 2\ 3), t_{2,3}, 1)$, $t_1(3) = ((0, -1, 0), (1\ 2\ 3), t_{2,3}, 1)$

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