ALGEBRAIC PURE TONE COMPOSITIONS CONSTRUCTED VIA SIMILARITY

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Abstract. We describe a family of musical compositions constructed by algebraic techniques, based on the notion of similarity between musical passages.

1. Similarity in music.

If we hear a piece of music played twice we are in a position to recognise the similarity of the second musical experience to the first. Some pieces of music operate on memory by introducing similar passages within a single piece [1] [4]. Similarity operates on the mind more generally: people form sentimental attachments to things, and language is built from words, which are representations of similar entities. Musical traditions can carry the use of similar techniques over centuries. An excess of similarity can be felt to be bad: monotony and cliché are negative words associated with such an excess. The response of a listener to hearing similarity in a given piece of music will vary, depending on previous experience.

In this article we introduce a class of musical compositions built around the notion of similarity. As well as carrying internal similarities, our pieces are designed to be similar to music previously written, and we record the way in which they are so. We insist on a certain sort of progression within our pieces, in an attempt to avoid monotony. Likewise, the introduction of new techniques implies our pieces are a little different from previous musical compositions.

Algebra carries a formal notion representing a self-transformation: an endomorphism. We use certain endomorphisms to generate our compositions. Roughly our idea is the following: Suppose we have a set of musical phrases that is closed under concatenation, and we have a set of transformations of this set that carries phrases to similar phrases. Take an initial note, apply a sequence of transformations to obtain a sequence of notes, and concatenate to obtain a phrase, each of whose notes is similar to the previous one. Apply a sequence of transformations to this phrase to obtain a sequence of phrases, and concatenate to obtain a longer phrase. Iterate this process a few times. The longest resulting phrase obtained is a musical composition, constructed by similarity. To avoid direct monotony in the composition we use nonidentity transformations. In actuality our compositions will involve an accompaniment, constructed via a slight variation on this strategy. Figure 1 is a diagram of transformations involved in such a composition. The transformation of the initial note corresponding to a given leaf of the tree is obtained by tracing the unique path from the root to the leaf, multiplying all the transformations along the way.

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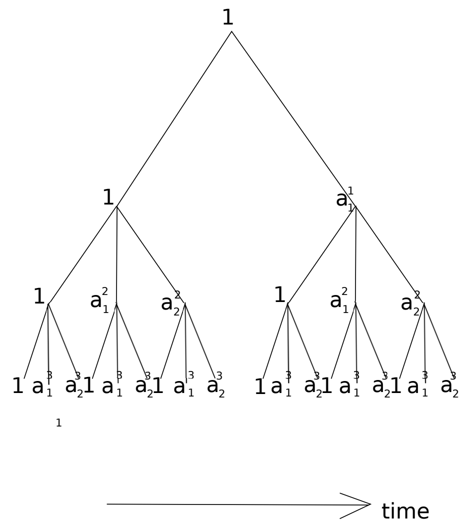

Figure 1. Transformations

Sharing music is a form of communication. This communication may occur between people in the same location at the same time, but also between people in different locations at different times. The music may have human players, in which case an additional sensual and intellectual dimension is added to the communication for those players. The music given here is to be played by a computer. However, by recording the associations we use to construct our compositions, we potentially add an intellectual dimension to the communication given by sharing the music.
Musical pieces commonly have passages designed to encourage associations in the mind of listeners. However there is potentially a great difference between the associations to a piece of music experienced by a listener, and those written into the piece. For example, the associations written into the music may be of a formal nature, and difficult to detect with the ear. Furthermore, the listener may form all kinds of additional associations that are not part of the formal design of the piece. Passage C may be designed to be similar to passage B, and passage B designed to be similar to passage A, but there may be an additional resemblance between passages C and A that was not part of the formal design. Passage E may be obtained from passage D by a similarity transformation, by design, but some of the notes of passage E may coincide with notes in passage D giving additional associations between passages E and D. Rhythmic or melodic motifs may remind a listener of some other music, unknown to the composer of the piece. Via memory, a listener may form associations between the piece and a place where they hear it. Sharing the experience of a piece of music is a form of communication, and may have social associations. Further examples of musical associations beyond sounds are found when listeners take an interest in musicians' personal lives, and beyond: in Austria Mozart's name is used to sell spherical chocolates (Mozartkugeln).
More ornate applications of the techniques introduced here are given elsewhere [7].

2. Similarity transformations.

In a piece of piano music, we have a set of notes on the stave. This set can naturally be ordered by pitch, and its elements are therefore indexed by elements of $\mathbb{Z}$, with middle C corresponding to zero. We call the integer corresponding to a given note on the stave its stave point. A piano note has a fundamental frequency f, and a set of overtones, whose frequencies are $2f$, $3f$, $4f$, etc. [2] The first five of these overtones are approximately $12, 19, 24, 28, 31$ semitones above the original frequency f. If S is a subset of $\mathbb{Z}$, and $m \in \mathbb{Z}$ then we can transform one phrase written on the stave with notes in S to a second by writing the elements of S in ascending order, and forming the second phrase by substituting the $i+m^{th}$ element of S for the i^{th} element of S wherever it appears in the phrase. We denote this transformation t_m. In case $S = \mathbb{Z}$ such a transformation is called transposition by m semitones. A phrase and its transposition share common overtones of low degree in the case of transposition by 5, 7, and 12 semitones since $5 = 24 - 19$, $7 = 19 - 12$, $12 = 12 - 0 = 24 - 12$. Formally at least, transpositions by these numbers carry special significance. Transposition is commonly used in music written on the stave, but the more general type of transformation we alluded to before is also used to generate similarity between phrases. For example, take S to correspond to the notes of the C major scale, and consider the treble part of Invention No. 1 in two parts by J.S. Bach [1]. Excepting the first note, the second bar is obtained from the first by applying t_4. Since up to octave equivalence the notes of the major scale are related by a sequence of perfect fifths, the second bar is also close to being a transposition of the first by 7 semitones. However, bars 3 and 4 contain similar phrases that are not related by transpositions. Indeed bar 3 contains a pair of similar two crotchet motifs that are related by the transformation t_2, and the two crotchet motif at the beginning of bar 4 is related to the second of these two by t_2 as well.

The building blocks of the compositions introduced here are sounds we call *quasi-notes*, which are chords of pure tones that we treat like notes. A note on a stringed instrument has a fundamental frequency f, and a set of overtones, whose frequencies are $2f$, $3f$, $4f$, etc.. The first five of these overtones are approximately $12, 19, 24, 28, 31$ semitones above the original frequency f. In a previous set of compositions [5], we introduced a set of positive integers ≤ 32 called *quasi-note generators*. For every point on the stave we took a quasi-note consisting of a *fundamental*, sounding together with a number of *overtones*. The fundamental is a pure tone whose frequency is given by the relevant point on the stave, whilst each overtone is a pure tone whose frequency is given by the fundamental frequency, raised by a quasi-note generator number of semitones. The fundamental, and the overtones are called the *partials* of the quasi-note. As similarity transformations in our compositions we used transposition by intervals given by the difference between a pair of quasi-note generators.

Here we use a more general sort of quasi-note, and a more general sort of similarity transformation. The motivation for generalising is the following: since quasi-notes are synthesised, we do not need to use a fixed set of quasi-note generators as we move up the stave. We expect to be able to allow the intervals between partials to vary, and obtain a notion of quasi-note we can use.

Take a set of strictly increasing maps $s_1, s_2, ..., s_p : \mathbb{Z} \to \mathbb{Z}$. Our i^{th} quasi-note consists of the p pure tones given by the stave points $s_1(i), s_2(i)..., s_p(i)$, for $i \in \mathbb{Z}$. We call the pure tone given by $s_1(i)$ the fundamental of the i^{th} quasi-note, and the pure tone given by $s_j(i)$ the j^{th} partial of the i^{th} quasi-note. We call the j^{th} partial, for $j > 1$, an overtone of the corresponding quasi-note. We thus have a set Q of quasi-notes, indexed by elements of $\mathbb{Z}$.

Our collection of quasi-notes is given equivalently by a set of subsets $\Omega_1, \Omega_2, ..., \Omega_p \subseteq \mathbb{Z}$ that are unbounded from above and below, with elements $\omega_1 \in \Omega_1, ..., \omega_p \in \Omega_p$. Indeed, such data emerges when we write $\Omega_i = s_i(\mathbb{Z})$ and $\omega_i = s_i(0)$, for $i = 1, 2, ..., p$.

Suppose $\Omega_u \supseteq \Omega_v$. Then the v^{th} partial of a quasi-note q is equal to the u^{th} partial of a second quasi-note q'. We write $t_{u,v}(q) = q'$, and thus define a transformation $t_{u,v}$ of Q. We denote by Φ the collection of such transformations. These transformations $t_{u,v}$ are the transformations we use as similarity transformations to generate our compositions. In our examples, we control our choices of $s_1, ..., s_p$ and our choices of the transformations $t_{u,v}$ we use, so that these transformations do indeed transform phrases to phrases that sound similar, beyond having common partials.

Our quasi-notes have partials corresponding to stave points, to create a similarity between our music and the large body of music that can be notated on a stave.

3. Accompaniment.

Here we work under the assumption that $\Omega_1 \supseteq \Omega_2, ..., \Omega_p$. We discuss the construction of a minimising consonant accompaniment to a sequence of quasi-notes, avoiding parallel motion.

Suppose we are given a sequence s of quasi-notes, and a subsequence t of s that dictates which elements of this sequence are to be accompanied.

Suppose we have a *consonant accompaniment* to s, which is a sequence of quasi-notes indexed by the elements of t whose fundamentals are given by overtones of the corresponding elements of t. We call an overtone of an element of t that forms a fundamental in the accompaniment a *harmonised overtone*. We say our accompaniment *avoids parallel motion* if the harmonised overtones of consecutive quasi-notes in t are indexed by distinct elements of $\{2, ..., p\}$ (cf. Fux's rules concerning contrapuntal motion [3]).

Suppose we fix a harmonised overtone for the first element of t. Suppose the i^{th} note of a consonant accompaniment of s is taken to avoid parallel motion, and is taken to minimise the number of semitones separating its fundamental from the fundamental of the $i-1^{th}$ note of our accompaniment for each i; then we say our accompaniment is a *minimising consonant accompaniment avoiding parallel motion*.

4. The compositions.

Our compositions have a *principal part*, and an accompaniment.

For n a natural number, let $\underline{n} = \{1, 2, ..., n\}$.

Let r be a natural number, and let $c_1, ..., c_r$ be natural numbers. For $i = 1, ..., r$ we take maps $\lambda_i : \underline{c_i} \to \mathbb{R}$ and $t_i : \underline{c_i} \to \Phi$, and for $i = 1, ..., r-1$ we take maps $f_i : \underline{c_i} \to \mathbb{F}_2^{c_r}$. We insist that $\lambda_i(1)$ and $t_i(1)$ are all the identity, for $i = 1, ..., r$, but that $\lambda_i(x)$ and $t_i(x)$ are different from the identity, for $x = 2, ..., c_i$ and $i = 1, ..., r$. We insist that $f_i(1) = 0$ for $i = 1, ..., r-1$.

For $i = 1, ..., r$ we define maps $\mu_i : \underline{c_i} \to \mathbb{R}$ and $u_i : \underline{c_i} \to End(Q)$, by

$$\mu_i(x) = \lambda_i(x)\lambda_i(x-1)...\lambda_i(1),$$

$$u_i(x) = t_i(x)t_i(x-1)...t_i(1),$$

and for $i = 1, ..., r-1$ we define maps $g_i : \underline{c_i} \to \mathbb{F}_2^{c_r}$ by
$$g_i(x) = f_i(x) + f_i(x-1) + ... + f_i(1).$$
Let us fix a duration $d \in \mathbb{R}$, a quasi-note q with fundamental in Ω_1, and an element $b \in \mathbb{F}_2^{c_r}$. The quasi-notes of the principal part of our composition correspond to elements $x = (x_1, ..., x_r)$ of $\underline{c_1} \times \underline{c_2} \times ... \times \underline{c_r}$, ordered lexicographically. They are given by $u_1(x_1)u_2(x_2)...u_r(x_r)q$. They have duration $\mu_1(x_1)\mu_2(x_2)...\mu_r(x_r)d$.
To form the accompaniment, consider the x_r^{th} coordinate of
$$g_1(x_1) + g_2(x_2) + ... + g_{r-1}(x_{r-1}) + b.$$
If this coordinate is 1 then we accompany our quasi-note, if it is 0 then we do not accompany our quasi-note. We fix an overtone in the first quasi-note that has an accompaniment. We then take a minimum consonant accompaniment avoiding parallel motion.

5. Examples.

Here we record the data sets for two examples for which recordings are available [6].

Example 1 Let $p = 4$. Let
$$\Omega_1 = \mathbb{Z},\ \Omega_2 = \{0, 1, 2, 3, 4, 6, 7, 8, 9, 10\} + 12\mathbb{Z},$$
$$\Omega_3 = \{0, 1, 3, 4, 6, 7, 9, 10\} + 12\mathbb{Z},\ \Omega_4 = \{0, 1, 3, 4, 5, 6, 7, 9, 10, 11\} + 12\mathbb{Z}.$$
Thus $\Omega_1 \supset \Omega_2, \Omega_4 \supset \Omega_3$. To specify a complete set of quasi-notes it is enough to specify one: $\omega_1 = -12, \omega_2 = -9, \omega_3 = -8, \omega_4 = 1$.
We take $r = 4$, $c_1 = c_2 = c_3 = 3$, and $c_4 = 7$. We take $\lambda_i(2) = \lambda_i(3) = 2^{\frac{1}{6}}$, for $1 \leq i \leq 3$, and take $\lambda_4(i)$ to be $1, 1, 1, 2, \frac{1}{2}, 1$ for $i = 2, 3, 4, 5, 6, 7$ respectively. We take $t_i(2) = t_{2,3}$ and $t_i(3) = t_{1,2}$ for $1 \leq i \leq 3$ and take $t_4(i)$ to be $t_{1,2}, t_{2,3}, t_{2,3}, t_{4,3}, t_{2,3}, t_{2,3}$ for $i = 2, 3, 4, 5, 6, 7$ respectively.
We choose an empty accompaniment in this case, thus $f_i = g_i = 0$ for all i and $b = 0$. We take $d = \frac{1}{2}$ and q to be the quasi-note whose fundamental is the C an octave below middle C.

Example 2 Let $p = 4$. Let
$$\Omega_1 = \{7i | -1 \leq i \leq 5\} + 12\mathbb{Z},\ \Omega_2 = \{7i | -1 \leq i \leq 4\} + 12\mathbb{Z},$$
$$\Omega_3 = \{7i | 0 \leq i \leq 4\} + 12\mathbb{Z},\ \Omega_4 = \{7i | 0 \leq i \leq 5\} + 12\mathbb{Z},$$
Thus $\Omega_1 \supset \Omega_2, \Omega_4 \supset \Omega_3$. To specify a complete set of quasi-notes it is enough to specify one: $\omega_1 = 17, \omega_2 = 9, \omega_3 = 0, \omega_4 = 4$. The partials of our quasi-notes lie in a major scale, creating a similarity between our piece and many others.
We take $r = 6$, $c_1 = c_2 = c_3 = c_4 = c_5 = 2$, and $c_6 = 7$. We take $\lambda_i(2) = 2^{-\frac{1}{5}}$, for $1 \leq i \leq 5$, and take $\lambda_6(i)$ to be $2, 2, \frac{1}{2}, 2, \frac{1}{2}, \frac{1}{2}$ for $i = 2, 3, 4, 5, 6, 7$ respectively. We take $t_i(2) = t_{1,2}$ for $1 \leq i \leq 5$, and take $t_6(i)$ to be $t_{1,2}, t_{1,2}, t_{1,2}, t_{4,3}, t_{4,3}, t_{1,2}$ for $i = 2, 3, 4, 5, 6, 7$ respectively. For $i = 1, ..., 5$ we take $f_i(2)$ to have 0s in all seven coordinates excepting the $7 - i^{th}$ which is 1.
We take $d = \frac{1}{8}$ and q to be the quasi-note whose fundamental is the F four octaves and a perfect fourth above middle C. We take $b = (1, 0, 0, 0, 0, 0, 0)$. We use the second partial of q to generate our minimum consonant accompaniment avoiding parallel motion.

References

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