HOMOLOGICAL STABILITY FOR IWAHORI-HECKE ALGEBRAS

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Abstract. We show that the Iwahori-Hecke algebras $\mathcal{H}_n$ of type A_{n-1} satisfy homological stability, where homology is interpreted as an appropriate Tor group. Our result precisely recovers Nakaoka's homological stability result for the symmetric groups in the case that the defining parameter is equal to 1. We believe that this paper, and our joint work with Boyd on Temperley-Lieb algebras, are the first time that the techniques of homological stability have been applied to algebras that are not group algebras.

1. Introduction

1.1. Homological stability.

1.1. Homological stability. A family of discrete groups

$$G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \cdots$$

satisfies *homological stability* if the maps

$$H_d(G_{n-1}) \longrightarrow H_d(G_n)$$

are isomorphisms when n is sufficiently large compared to d. Homological stability can similarly be formulated for sequences of topological groups, and for families of spaces that are not necessarily classifying spaces of groups. Examples of families for which homological stability holds include symmetric groups [Nak60], general linear groups [Qui73, Cha80, vdK80], mapping class groups of surfaces and 3-manifolds [Har85, RW16, Wah13, HW10], diffeomorphism groups of highly connected manifolds [GRW18], automorphism groups of free groups [HV04, HV98], families of Coxeter groups [Hep16] and Artin monoids [Boy20], configuration spaces of manifolds [Chu12], [RW13], and a great many others besides.

The homology $H_*(G;R)$ of a discrete group G with coefficients in a ring R can be written as the Tor group

$$\mathrm{Tor}_*^{RG}(\mathbb{1},\mathbb{1})$$

over the group algebra RG, where $\mathbb{1}$ denotes the trivial representation. This formulation shows that the homology of a group depends only on the group algebra. We can therefore say that a family of algebras

$$\mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \cdots,$$

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equipped with a consistent choice of 'trivial representation' $\mathbb{1}$ satisfies *homological stability* if the maps

$$\mathrm{Tor}_d^{\mathcal{A}_{n-1}}(\mathbb{1},\mathbb{1}) \longrightarrow \mathrm{Tor}_d^{\mathcal{A}_n}(\mathbb{1},\mathbb{1})$$

are isomorphisms when n is sufficiently large compared to d. Here the algebras need not be group algebras, and the only requirement on $\mathbb{1}$ is that it is a module for each $\mathcal{A}_n$ and that the module structures are compatible with the maps $\mathcal{A}_{n-1} \to \mathcal{A}_n$.

The purpose of this paper is to demonstrate that homological stability holds in this sense for Iwahori-Hecke algebras of type A_{n-1}, and moreover that it can be proved by adapting the suite of techniques used to study families of groups to the setting of algebras. In [BH20], Boyd and the author prove homological stability for the *Temperley-Lieb algebras*. There we again use the techniques of homological stability, but encounter — and resolve — novel obstructions that are not present in the setting of groups or of Iwahori-Hecke algebras.

To the best of our knowledge, the present paper and [BH20] are the first homological stability results of their kind for algebras that are not group algebras, and our hope is that they will serve as a proof of concept for the export of homological stability techniques into new algebraic contexts.

1.2. **Iwahori-Hecke algebras.** The symmetric group $\mathfrak{S}_n$ has presentation with generators

$$s_1, \ldots, s_{n-1},$$

and with relations

$$\begin{aligned} s_i s_j &= s_j s_i && \text{for } |i-j|>1,\\ s_i s_j s_i &= s_j s_i s_j && \text{for } |i-j|=1,\\ s_i^2 &= e && \text{for all } i. \end{aligned}$$

where s_i is the adjacent transposition $s_i = (i\ i+1)$. This is the presentation of $\mathfrak{S}_n$ as the *Coxeter group of type A_{n-1}*.

Now let R be a commutative ring and let $q \in R^\times$ be a unit. The *Iwahori-Hecke algebra of type A_{n-1}*, denoted $\mathcal{H}_n$, is the R-algebra with generators

$$T_1, \ldots, T_{n-1}$$

and with relations

$$\begin{aligned} T_i T_j &= T_j T_i && \text{for } |i-j|>1,\\ T_i T_j T_i &= T_j T_i T_j && \text{for } |i-j|=1,\\ (T_i+1)(T_i-q) &= 0 && \text{for all } i. \end{aligned}$$

When $q=1$, the final relation can be rewritten as $T_i^2 = 1$, so that $\mathcal{H}_n \cong R\mathfrak{S}_n$ by the isomorphism that sends T_i to s_i. Thus $\mathcal{H}_n$ is a 'deformation' of $R\mathfrak{S}_n$ depending on the parameter q. Taking $R = \mathbb{C}$, then $\mathcal{H}_n \cong \mathbb{C}\mathfrak{S}_n$ unless q is a d-th root of unity for $2 \leqslant d \leqslant n$ [Wen88, Theorem 2.2], in which case no such isomorphism exists.

The algebras $\mathcal{H}_n$ are important from several points of view, and we mention just a couple. In knot theory, the $\mathcal{H}_n$ are a crucial ingredient in certain definitions of the HOMFLY-PT polynomial [FYH+85, Jon87], and their categorifications via Soergel bimodules are used to define categorifications of this polynomial [Kho07]. In representation theory, if we take $R = \mathbb{C}$ and q a prime power, then $\mathcal{H}_n$ is isomorphic to the endomorphism algebra of a certain representation of $\mathrm{GL}_n(\mathbb{F}_q)$, and this allows the construction of an irreducible representation of $\mathrm{GL}_n(\mathbb{F}_q)$ from each irreducible of $\mathfrak{S}_n$, see [Mat99, pp.x-xi]. For general introductions to the $\mathcal{H}_n$ we suggest [Mat99, Chapter 1] and [KT08, Chapters 4-5].

In general, there is an Iwahori-Hecke algebra associated to any Coxeter system, and these more general Iwahori-Hecke algebras are important in many parts of representation theory, see for example [GP00], [Hum90, Chapter 7], [KL79] and [Lib19]. We will often refer to the $\mathcal{H}_n$ as *Iwahori-Hecke algebras* without explicitly mentioning their type.

1.3. **Homological stability for Iwahori-Hecke algebras.** The Iwahori-Hecke algebra $\mathcal{H}_n$ has two natural rank-1 modules, denoted $\mathbb{1}$ and ε, where each T_i acts on $\mathbb{1}$ as multiplication by q, and on ε as multiplication by (-1), see Corollary 1.14 of [Mat99]. When $q = 1$ the modules $\mathbb{1}$ and ε become the trivial representation and the sign representation respectively. We may therefore consider

$$\mathrm{Tor}_*^{\mathcal{H}_n}(\mathbb{1}, \mathbb{1}) \quad \text{and} \quad \mathrm{Ext}^*_{\mathcal{H}_n}(\mathbb{1}, \mathbb{1})$$

to be the *homology* and *cohomology* of $\mathcal{H}_n$, and indeed when $q = 1$ these become simply $H_*(\mathfrak{S}_n; R)$ and $H^*(\mathfrak{S}_n; R)$ respectively. We can now state our main result:

Theorem 1.1. *The maps*

$$\mathrm{Tor}_d^{\mathcal{H}_{n-1}}(\mathbb{1}, \mathbb{1}) \longrightarrow \mathrm{Tor}_d^{\mathcal{H}_n}(\mathbb{1}, \mathbb{1})$$

and

$$\mathrm{Ext}^d_{\mathcal{H}_n}(\mathbb{1}, \mathbb{1}) \longrightarrow \mathrm{Ext}^d_{\mathcal{H}_{n-1}}(\mathbb{1}, \mathbb{1})$$

are isomorphisms for $d \leqslant \frac{n-1}{2}$.

When $q = 1$ then $\mathcal{H}_n \cong R\mathfrak{S}_n$, and Theorem 1.1 gives exactly Nakaoka's stability result for the homology and cohomology of symmetric groups. See [Nak60, Corollary 6.7], [Ker05, Theorem 2] and [RW13, Theorem 5.1]. Nakaoka in in fact gave a complete computation of $H_*(\mathfrak{S}_n; \mathbb{F}_p)$ for any prime p, and this can be used to show that for $k \geqslant 1$ the map $H_k(\mathfrak{S}_{2k-1}; \mathbb{F}_2) \to H_k(\mathfrak{S}_{2k}; \mathbb{F}_2)$ is not surjective. Thus the range $d \leqslant \frac{n-1}{2}$ appearing in the theorem cannot be improved in general.

1.4. **Comparison with work of Benson-Erdmann-Mikaelian.** The cohomology ring $\mathrm{Ext}^*_{\mathcal{H}_n}(\mathbb{1}, \mathbb{1})$ of $\mathcal{H}_n$ was explicitly computed by Benson, Erdmann and Mikaelian [BEM10] in the case where $R = \mathbb{C}$ and q is a primitive ℓ-th root of unity with $\ell \geqslant 2$. In the case $\ell > n$, the result of Wenzl mentioned above shows that $\mathcal{H}_n \cong \mathbb{C}\mathfrak{S}_n$, so that $\mathrm{Ext}^d_{\mathcal{H}_n}(\mathbb{1}, \mathbb{1}) = H^*(\mathfrak{S}_n; \mathbb{C})$ is trivial, but when $2 \leqslant \ell \leqslant n$ then no such isomorphism holds, and indeed Benson-Erdmann-Mikaelian show that $\mathrm{Ext}^d_{\mathcal{H}_n}(\mathbb{1},\mathbb{1})$ is nontrivial. Furthermore, one can use their results to observe that in this case the stabilisation maps $\mathrm{Ext}^d_{\mathcal{H}_n}(\mathbb{1},\mathbb{1}) \to \mathrm{Ext}^d_{\mathcal{H}_{n-1}}(\mathbb{1},\mathbb{1})$ are isomorphisms up to and including (at least) degree $(n-2)$. So [BEM10] serves as an antecedent of the present work, but more interestingly, it demonstrates a much stronger stable range in this case, of slope 1 rather than slope $\frac{1}{2}$. This is reminiscent of the slope 1 rational homological stability results for configuration spaces of manifolds (see for example Corollary 3 of [Chu12] and Theorem B of [RW13]). It suggests that there may be a slope 1 stability result for the $\mathcal{H}_n$ in characteristic 0.

1.5. **Discussion: Homological stability for Coxeter groups and Artin monoids.** The present paper builds strongly on previous work of the author [Hep16], which proved homological stability for families of Coxeter groups, and of Boyd [Boy20], which proved homological stability for families of Artin monoids. These papers demonstrated that one can do all of the normal work of a homological stability proof purely in terms of a Coxeter or Artin-type presentation, rather than in terms of a concrete model of the group or monoid being studied. The defining presentation of the Iwahori-Hecke algebra $\mathcal{H}_n$ is of course very close to both of these, being a deformation of the Coxeter presentation of $\mathfrak{S}_n$, and a quotient of the Artin presentation of the braid group (or rather of their group rings).

In both [Hep16] and [Boy20], the results apply to families of groups or monoids obtained from sequences of Coxeter diagrams that 'grow a tail' of type A_{n-1} as n increases. These families are very general, but include as the basic case the families of type A, B and D. So one may ask whether Theorem 1.1 can be extended to apply to any of these more general families. This seems likely, but we were not able to prove Theorem 1.1 by generalising the method of [Hep16] from Coxeter groups to Iwahori-Hecke algebras; this is discussed further in section 1.8 below.

1.6. **Discussion: Stable homology.** Theorem 1.1 shows that, in a fixed degree d, then for n sufficiently large the groups $\mathrm{Tor}^{\mathcal{H}_n}_d(\mathbb{1},\mathbb{1})$ all agree and coincide with the *stable homology*

$$\mathrm{colim}_n\, \mathrm{Tor}^{\mathcal{H}_n}_d(\mathbb{1},\mathbb{1}) = \mathrm{Tor}^{\mathcal{H}_\infty}_d(\mathbb{1},\mathbb{1}),$$

where $\mathcal{H}_\infty = \mathrm{colim}_n\, \mathcal{H}_n$ is the 'infinite' Iwahori-Hecke algebra. When $q=1$, the stable homology $\mathrm{Tor}^{\mathcal{H}_\infty}_*(\mathbb{1},\mathbb{1})$ coincides with the homology of the infinite symmetric group, $H_*(\Sigma_\infty;R)$, which is computed by the Barratt-Priddy-Quillen theorem [BP72], [FM94]:

$$H_*(\Sigma_\infty;R) \cong H_*(\Omega^\infty_0 S^\infty;R).$$

Here $\Omega^\infty S^\infty = \mathrm{colim}_n\, \Omega^n S^n$ is the infinite loop space of the sphere spectrum, and $\Omega^\infty_0 S^\infty$ is the path component of its basepoint. It is therefore natural to ask what is the stable homology $\mathrm{Tor}^{\mathcal{H}_n}_*(\mathbb{1},\mathbb{1})$ in general? To put it another way, what is the Iwahori-Hecke analogue of $H_*(\Omega^\infty_0 S^\infty;R)$?

1.7. **Discussion: Homological stability for algebras.** As we said earlier, we believe that the work of the present paper on Iwahori-Hecke algebras and of [BH20] on Temperley-Lieb algebras are the first time the techniques of homological stability have been applied to families of algebras that are not group algebras, and we hope that they will serve as a starting point for new work in this area. We refer the reader to the introduction of [BH20], where several possible directions are discussed in some detail.

1.8. **Method of proof.** Proofs of homological stability for sequences of groups $(G_n)_{n\geqslant 0}$ can often be placed in the following broad framework:

- Find a complex (a simplicial complex, or semisimplicial set, or chain complex) upon which the n-th group G_n acts in such a way that the simplex stabilisers are of the form G_m for $m < n$. (Or at least, the simplex stabilisers must be associated to the previous groups in the sequence in some way).
- Prove that the complex is highly acyclic, i.e. that its homology vanishes up to a certain point.
- Use an algebraic method (often but not always a spectral sequence argument) based on the complex in order to prove stability by induction.

While many different proofs fit this framework when viewed from a distance, there are many choices to be made and many variations are possible. It may be possible to prove stability for the same family of groups by choosing different complexes to begin with. It may be possible to prove high-acyclicity of the same complex in multiple ways. And it may be possible to use the same complex in different algebraic arguments to prove stability.

Our approach to proving Theorem 1.1 fits into the framework outlined above. There is a well-known complex, called the *complex of injective words*, that is used in many proofs of homological stability for the symmetric group. For our complex, we construct an Iwahori-Hecke analogue of the complex of injective words. While the complex of injective words has an action of $\mathfrak{S}_n$, our new complex is a chain complex of $\mathcal{H}_n$-modules; and while the generators of the complex of injective words have stabilisers given by smaller symmetric groups, our new complex is built out of tensor products like $\mathcal{H}_n \otimes_{\mathcal{H}_m} \mathbb{1}$ for $m < n$. The proof that our complex is highly acyclic is closely modelled on, but far more involved than, a proof that the complex of injective words is highly acyclic, and requires us to make careful use of the theory of distinguished coset representatives in Coxeter groups and the basis theorem for Iwahori-Hecke algebras. Furthermore, new difficulties arise because q is no longer equal to 1, so that one must now account for many hitherto-invisible powers of q. (Surprisingly, the formula $T_k^2 = (q-1)T_k + q$ explicitly surfaces in only one place, and it quickly disappears again.) The final step of our argument is a spectral sequence argument closely related to ones in the literature.

A lot of the difficulty in the present paper boils down to the fact that we are operating under two significant constraints. First, we are not working with the symmetric group, but with its Iwahori-Hecke algebra, and while $\mathcal{H}_n$ is closely related to $\mathfrak{S}_n$ *thought of as a Coxeter group*, it is not useful to think of $\mathcal{H}_n$ in terms of permutations of the set $\{1,\dots,n\}$. This means that we can approach the complex of injective words only in terms of the Coxeter presentation of $\mathfrak{S}_n$. Second, the *linear* nature of Iwahori-Hecke algebras heavily restricts the suite of topological tools that we can apply. For example, the approach of [Hep16] to proving homological stability for Coxeter groups could not be adapted to this setting since it made use of simplicial complexes and barycentric subdivision, which do not seem to have analogues in the linear setting.

There are by now several systematic approaches to proving homological stability results, for example Randal-Williams and Wahl's approach [RWW17] via homogeneous categories, the author's approach via families of groups with multiplication [Hep20], and Kupers, Galatius and Randal-Williams' approach via cellular E_k-algebras [GKRW18]. One may ask whether the present results could be proved using any of these frameworks. In the first two cases the answer is no, since these are designed purely for the study of groups, though it is plausible that a 'linearised' version of [RWW17] would produce the same complex that we use. In the final case, it seems that the methods of [GKRW18] could possibly be applied in the present situation, but we have taken a significantly more elementary approach.

1.9. **Outline of the paper.**

- We begin in section 2 with some detailed background on Iwahori-Hecke algebras.
- In section 3 we give a short account of the complex of injective words and a proof that it is highly-acyclic. This will give us motivation and reference points for our Iwahori-Hecke analogue of the complex.
- In section 4 we define our analogue of the complex of injective words, $\mathcal{D}(n)$.
- In sections 5, 6, 7 and 8 we show that the homology of $\mathcal{D}(n)$ is zero up to degree $(n-2)$. Section 5 defines a filtration of $\mathcal{D}(n)$, while sections 6, 7 and 8 identify the filtration quotients in terms of the $\mathcal{D}(m)$ for $m < n$, allowing an inductive proof of high-acyclicity.
- In section 9 we obtain a spectral sequence from $\mathcal{D}(n)$ and identify its E_1 and E_∞ terms.
- In section 10 we use the spectral sequence to give an inductive proof of Theorem 1.1.

2. Background on Iwahori-Hecke algebras

This section is a rapid run through the theory of Coxeter groups and Iwahori-Hecke algebras that is necessary for the applications in this paper. The intention is to give the reader a flavour of the extent and depth of the theory required.

Everything that we recall here is basic in the theory of Coxeter groups and Iwahori-Hecke algebras, but it nevertheless amounts to a significant amount of nontrivial theory. However, none of this theory is strictly necessary until section 8, and some readers may wish to skim or skip the section until then. For reading we recommend chapters 1, 2 and 4 of [GP00], chapters 3 and 4 of [Dav08], or chapter 1 of [Mat99] in the $\mathcal{H}_n$-case.

2.1. **Coxeter systems and Coxeter groups.** A *Coxeter matrix* on a set S is a symmetric $S \times S$ matrix whose entries lie in $\{1, 2, 3, \dots, \infty\}$ and satisfy $m_{ss} = 1$ for all $s \in S$, $m_{st} \geqslant 2$ if $s \neq t$. A Coxeter matrix determines a *Coxeter group*

$$W = \left\langle S \;\middle|\; s^2 = e \text{ for } s \in S,\ \underbrace{sts\cdots}_{m_{st}\text{ terms}} = \underbrace{tst\cdots}_{m_{st}\text{ terms}} \text{ for } s, t \in S \right\rangle$$

When $m_{st} = \infty$ no relation is applied. The relations

$$\underbrace{sts\cdots}_{m_{st}\text{ terms}} = \underbrace{tst\cdots}_{m_{st}\text{ terms}}$$

are called the *braid relations* or *braid moves*. The pair (W, S) is called a *Coxeter system*.

Example 2.1 (The Coxeter system of type A_{n-1})**.** Let $W = \mathfrak{S}_n$ be the symmetric group on n letters, and let $S_n = \{s_1, \dots, s_{n-1}\}$ where s_i is the adjacent transposition $(i\ i+1)$. Then $(\mathfrak{S}_n, S_n)$ is a Coxeter system, called the *Coxeter system of type* A_{n-1}. Observe that:

$$m_{s_i s_j} = \begin{cases} 2 & \text{if } |i-j| > 1 \\ 3 & \text{if } |i-j| = 1 \end{cases}$$

Indeed, if $|i-j| > 1$ then s_i and s_j are disjoint transpositions, so that $s_i s_j$ has order 2, while if $|i-j| = 1$ then $s_i s_j$ is a 3-cycle, and so has order 3. (The observation really just shows that if W is the Coxeter group of this type, then the relevant relations hold in $\mathfrak{S}_n$ so that there is a surjection $W \to \mathfrak{S}_n$. To show that this is an isomorphism, one must show that the relations of the Coxeter group are sufficient to relate any two words representing the same element of $\mathfrak{S}_n$. That is a simple exercise.)

Let (W, S) be a Coxeter system. A *word* in S is a tuple $\mathbf{s} = (s_1, \dots, s_l)$ of elements of S. We say that $w = w(\mathbf{s}) = s_1 \dots s_l$ is the element *represented* by $\mathbf{s}$, and we say equivalently that $\mathbf{s}$ is an *expression* for $w = w(\mathbf{s})$. The *length* of an element $w \in W$, denoted $\ell(w)$, is the minimum length of a word representing w. We say that $\mathbf{s}$ is a *reduced expression* for w if it is a word of minimum length representing w.

We will often blur the difference between words and their expressions, writing $w = s_1 \cdots s_l$ for an element of W, and referring to $s_1 \cdots s_l$ as a *word* or *expression* for w, hoping that it will be clear from what is written that the expression $(s_1, \ldots, s_l)$ is to be understood.

Given an element $w \in W$, there are two possibilities for $\ell(sw)$:

- $\ell(sw) = \ell(w) + 1$. In this case one can obtain a reduced expression for sw by putting s in front of a reduced expression for w.
- $\ell(sw) = \ell(w) - 1$. In this case w has a reduced expression beginning with s.

(See [Dav08, pp.35-36].)

Here are two important results on reduced words in Coxeter groups.

Theorem 2.2 (Matsumoto's theorem [Mat64], [GP00, section 1.2])**.** *Let (W, S) be a Coxeter system. Then any reduced expression for an element of W can be transformed into any other by repeatedly replacing subwords of the form $sts\cdots$ (with m_{st} terms) with $tst\cdots$ (again with m_{st} terms).*

Theorem 2.3 (The word problem, Tits [Tit69], [Dav08, 3.4.2])**.** *Let (W, S) be a Coxeter system. Then a word in S is a reduced expression if and only if it cannot be shortened by applying a sequence of the following* M-operations*:*

- *Delete a subword of the form ss.*
- *Replace a subword of the form $\underbrace{sts\cdots}_{m_{st}\text{ terms}}$ with $\underbrace{tst\cdots}_{m_{st}\text{ terms}}$.*

Any two reduced expressions for the same element differ only by a sequence of moves of the second kind.

The final part of the theorem is called

Let (W, S) be a Coxeter system. Let $T \subseteq S$. The associated *special subgroup* is the subgroup of W generated by T, and is denoted W_T. The pair (W_T, T) is then a Coxeter system, which is to say, W_T is precisely the Coxeter group with generators T and with Coxeter matrix obtained from the Coxeter matrix of (W, S) in the evident way [Dav08, 4.1.6].

2.2. **Cosets in Coxeter groups.** For the material in this subsection we refer to section 2.1 of [GP00] and section 4.3 of [Dav08].

Let (W, S) be a Coxeter system, and let $J \subseteq S$. The cosets $W_J\backslash W$ are the subject of the following theory, which will be extremely useful to us. Define

$$X_J = \{w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in J\}.$$

Thus X_J consists of all elements of W that have no reduced expressions beginning with an element of J. The elements of X_J are called $(J, \emptyset)$*-reduced*, and referred to as the *distinguished right coset representatives* for W_J, for reasons that the next theorem will make clear. If $J \subseteq K \subseteq S$, then we write X_J^K for the set of distinguished right-coset representatives for W_J in W_K.

Theorem 2.4. (1) *$x \in X_J$ if and only if $\ell(vx) = \ell(v) + \ell(x)$ for all $v \in W_J$.*
(2) *For each $w \in W$ there exist unique $x \in X_J$ and $v \in W_J$ such that $w = vx$.*
(3) *X_J forms a complete set of representatives for $W_J \backslash W$.*
(4) *If $x \in X_J$ then x is the unique shortest element in $W_J x$.*

There is a similar theory for the cosets W/W_J, in which the role of X_J is now played by X_J^{-1}, elements of which are called $(\emptyset, J)$*-reduced*.

Moreover, if $J, K \subseteq S$ then there is also a theory for the double cosets $W_J \backslash W / W_K$. We define $X_{JK} = X_J \cap X_K^{-1}$. Thus an element $x \in W$ lies in X_{JK} if and only if it has no reduced expressions beginning with a letter in J or ending with a letter in K. The elements of X_{JK} are called *distinguished coset double coset representatives of W_J and W_K in W*, and we also refer to them as (J,K)*-reduced*. They form a complete set of representatives for the double cosets $W_J \backslash W / W_K$, and each one is the unique shortest element in its double coset.

The *Mackey decomposition* states that for $J, K \subseteq S$,

$$X_J = \bigsqcup_{d \in X_{J,K}} d \cdot X^K_{J^d \cap K}$$

Inverting the Mackey decomposition gives us a version for the left cosets

$$X_J^{-1} = \bigsqcup_{d \in X_{K,J}} (X^K_{K \cap {}^d J})^{-1} \cdot d$$

Observe that in both Mackey decompositions the lengths add in products. For example, suppose that $d \in X_{J,K}$ and $y \in X^K_{J^d \cap K}$, so that $dy \in X_J$, then since d is (J,K)-reduced,

2.3. **Iwahori-Hecke algebras.** Let (W,S) be a Coxeter system, and let R be a commutative ring and let $q \in R^\times$ be a unit. The *Iwahori-Hecke algebra* associated to (W,S) is the algebra $\mathcal{H}_W$ with generators

$$T_s \text{ for } s \in S$$

and relations:

$$\underbrace{T_s T_t T_s \cdots}_{m_{st} \text{ terms}} = \underbrace{T_t T_s T_t \cdots}_{m_{st} \text{ terms}} \qquad \text{for } s, t \in S$$

$$(T_s + 1)(T_s - q) = 0 \qquad \text{for } s \in S$$

Example 2.5 (The Iwahori-Hecke algebra of type A_{n-1})**.** Take $W = \mathfrak{S}_n$ and $S_n = \{s_1, \ldots, s_{n-1}\}$, as in Example 2.1, so that $(W,S) = (\mathfrak{S}_n, S_n)$ is the Coxeter system of type A_{n-1}.

$$m_{s_i s_j} = \begin{cases} 2 & \text{if } |i-j| > 1 \\ 3 & \text{if } |i-j| = 1 \end{cases}$$

Then $m_{s_is_j} = 2$ if $|s_i - s_j| > 1$, and and $m_{s_is_j} = 3$ if $|s_i - s_j| = 1$, so that $\mathcal{H}_{\mathfrak{S}_n}$ has generators

$$T_{s_1}, \dots, T_{s_{n-1}}$$

and relations

$$\begin{aligned} T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} && \text{for } |i-j|>1,\\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j} && \text{for } |i-j|=1,\\ (T_{s_i}+1)(T_{s_i}-q) &= 0 && \text{for all } i. \end{aligned}$$

Thus, if we write $T_i = T_{s_i}$, then $\mathcal{H}_{\mathfrak{S}_n}$ becomes exactly the algebra $\mathcal{H}_n$ defined in the introduction.

Let $w \in W$, and let $w = s_1 \cdots s_r$ be any reduced expression for w. Then by Matsumoto's Theorem 2.2, the quantity

$$T_w = T_{s_1}T_{s_2}\cdots T_{s_r}$$

depends only on w and not on the reduced expression. Suppose that $u, v \in W$ satisfy $\ell(uv) = \ell(u) + \ell(v)$. Then one can obtain a reduced expression for uv by combining reduced expressions for u and v. We therefore obtain:

$$T_uT_v = T_{uv} \text{ for } u, v \in W \text{ such that } \ell(uv) = \ell(u) + \ell(v)$$

The significance of the elements T_w is the following central result, for which see chapter IV, section 2, exercise 23 of [Bou02], or Theorem 4.4.6 of [GP00], or Theorem 1.13 of [Mat99] for the case $\mathcal{H}_W = H_n$.

Theorem 2.6 (Basis theorem)**.** *The elements T_w for $w \in W$ form a basis for $\mathcal{H}_W$ as an R-module, called the* standard basis*.*

And we have the following consequence, which is extremely important for the present paper.

Proposition 2.7. *Let (W, S) be a Coxeter system and let $J \subseteq S$. Then $\mathcal{H}_W$ is free as a left $\mathcal{H}_{W_J}$-module with basis $\{T_x \mid x \in X_J\}$. In particular, $\mathbb{1} \otimes_{\mathcal{H}_{W_J}} \mathcal{H}_W\mathbb{1}$ is free with basis $\{1 \otimes T_x \mid x \in X_J\}$.*

Similarly, $\mathcal{H}_W$ is free as a right $\mathcal{H}_{W_J}$-module with basis $\{T_x \mid x \in X_J^{-1}\}$, and $\mathcal{H}_W \otimes_{\mathcal{H}_{W_J}} \mathbb{1}$ is free with basis $\{T_x \otimes 1 \mid x \in X_J^{-1}\}$.

This follows by combining Theorems 2.4 and 2.6. The point is that there is a bijection $W_J \times X_J \to W$, $(v, x) \mapsto vx$ satisfying $\ell(vx) = \ell(v) + \ell(x)$ for every $(v, x) \in W_J \times X_J$, so that $T_{vx} = T_vT_x$. See [GP00, 4.4.7].

3. Symmetric groups and the complex of injective words

In this section, we will recall the definition of the complex of injective words, and we will give a proof that it is highly acyclic. This result is originally due to Farmer [Far79], and has since been proved in different ways by many authors, including Björner-Wachs [BW83], Kerz [Ker05], and Randal-Williams [RW13]. The approach that we present here is closest to that of Kerz. Throughout the section we fix a commutative ring R.

If A is a set, then an *injective word* on A is an ordered tuple $(a_0,\dots,a_r)$ of elements of A such that no element appears more than once. We allow the empty word $()$.

Definition 3.1 (The complex of injective words)**.** Let $n \geqslant 0$. The *complex of injective words* $\mathcal{C}(n)$ is the chain complex, concentrated in degrees $-1 \leqslant r \leqslant n-1$, that in degree r is the R-module with basis consisting of the injective words $(a_0,\dots,a_r)$ of length $(r+1)$ on the set $\{1,\dots,n\}$. The differential $\partial^r\colon \mathcal{C}(n)_r \to \mathcal{C}(n)_{r-1}$ is defined to be given by the alternating sum

$$\partial^r(a_0,\dots,a_r) = \sum_{i=0}^{r}(-1)^i(a_0,\dots,\widehat{a_i},\dots,a_r).$$

We regard $\mathcal{C}(n)$ as a chain complex of $\mathfrak{S}_n$-modules by allowing $\mathfrak{S}_n$ to act on the letters of a word in the evident way. Note that $\mathcal{C}(n)_{-1}$ is a copy of R generated by the empty word $()$.

Remark 3.2. The complex of injective words appears in many forms, for example as the realisation of a poset in [Far79] and [BW83], a chain complex in [Ker05], or as a semisimplicial set in [RW13]. We are working in the linear setting of $R\mathfrak{S}_n$-modules, and so our complex is a chain complex of $R\mathfrak{S}_n$-modules.

Note 3.3. Throughout the paper we will use notation like ∂^r in Definition 3.1, where the superscript indicates the degree in which the differential originates. This causes visual clutter and is sometimes extraneous, but will be extremely helpful later on in keeping track of degrees.

Theorem 3.4 (Farmer [Far79])**.** *$H_d(\mathcal{C}(n)) = 0$ for $d \leqslant n-2$.*

In order to prove this theorem we will define a filtration of $\mathcal{C}(n)$ by looking at the position of the letter n. This is essentially the technique used by Kerz [Ker05].

Definition 3.5 (The filtration of $\mathcal{C}(n)$)**.** Let $0 \leqslant p \leqslant n-1$. Define $F_p \subseteq \mathcal{C}(n)$ to be the subcomplex of $\mathcal{C}(n)$ spanned by all words for which the letter n appears in the last $(r+1)$ places, or not at all. Thus we obtain a filtration

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} = \mathcal{C}(n).$$

Observe that the F_p are not submodules with respect to the $\mathfrak{S}_n$ action, since that can change the position of n, but that they are submodules with respect to the restricted action of $\mathfrak{S}_{n-1}$.

The following notation fixes our conventions for cones and suspensions of chain complexes. The conventions are chosen so as to make the subsequent parts of the proof as direct as possible.

Definition 3.6. Let X be a chain complex with differentials d^r_X. The *cone* on X, denoted CX, is the chain complex defined by

$$(CX)_r = X_r \oplus X_{r-1}$$

with

$$d^r_{CX}\colon (CX)_r \longrightarrow (CX)_{r-1}$$

defined by $d^r_{CX}(x,y) = (d^r_X(x) + (-1)^r y, d^{r-1}_X(y))$. The *suspension* ΣX is the chain complex defined by

$$(\Sigma X)_r = X_{r-1}$$

with

$$d^r_{\Sigma X}\colon (\Sigma X)_r \longrightarrow (\Sigma X)_{r-1}$$

defined by $d^r_{\Sigma X} = d^{r-1}_X$.

Lemma 3.7. *F_0 is isomorphic to the cone on on $\mathcal{C}(n-1)$.*

Proof. F_0 is the span of all words in which either n does not appear, or appears in the final position. Ignoring differentials temporarily, the submodule spanned by the words in which n does not appear is exactly $\mathcal{C}(n-1)$, while the submodule spanned by the words in which n appears in the final position is isomorphic to $\mathcal{C}(n-1)$ with degrees shifted up by one, the isomorphism being given by simply appending n to an injective word on $\{1,\ldots,n\}$. This discussion establishes an isomorphism between F_0 and $C(\mathcal{C}(n-1))$, and one can check directly that it is a chain map. □

Lemma 3.8. *Let $1 \leqslant p \leqslant n-1$. Then F_p/F_{p-1} is isomorphic to a direct sum of $n!/(n-p)!$ copies of $\Sigma^{p+1}\mathcal{C}(n-p-1)$.*

Proof. Recall that F_p is the span of words in which n appears in the last $p+1$ positions, or does not appear at all. Thus F_p/F_{p-1} has basis consisting of all words for which n appears in exactly the $(p+1)$-st position from the end. Let us work in degree $(p+1)+r$ and consider a word in this basis, written as $(y_0,\ldots,y_r,n,z_1,\ldots,z_p)$. The effect of the boundary map $\partial^{(p+1)+r}$ on this word is

$$(y_0,\ldots,y_r,n,z_1,\ldots,z_p) \longmapsto \sum_{j=0}^{r}(-1)^j(y_0,\ldots,\widehat{y_j},\ldots,y_r,n,z_1,\ldots,z_p).$$

Observe that the last $(p+1)$ summands of $\partial^{(p+1)+r}$, in which one of the last $(p+1)$ letters is deleted, are not present because in the resulting words n either does not appear, or appears in the last p places, and therefore lies in F_{p-1}.

The above discussion shows that F_p/F_{p-1} splits as a direct sum of subcomplexes, one for each choice of $\underline{z} = (z_1,\ldots,z_p)$, which can be any injective word on the set $\{1,\ldots,n-1\}$. Moreover, the summand corresponding to $\underline{z}$ is visibly isomorphic to the complex of injective words on the set $\{1,\ldots,n-1\}\setminus\{z_1,\ldots,z_p\}$, but with all degrees shifted up by $p+1$. For each $\underline{z}$ we may therefore choose an arbitrary identification of $\{1,\ldots,n-1\}\setminus\{z_1,\ldots,z_p\}$ with $\{1,\ldots,n-p-1\}$, and obtain an isomorphism of the corresponding summand with $\Sigma^{(p+1)}\mathcal{C}(n-p-1)$. Thus,

$$F_p/F_{p-1}\cong\Sigma^{p+1}\mathcal{C}(n-p-1)^{\oplus\frac{n!}{(n-p)!}}.$$

$\square$

Proof of Theorem 3.4. This is proved by induction on $n\geqslant 0$. In the case $n=0$, $\mathcal{C}(0)$ consists only of a copy of R in degree $-1=n-1$, and its homology has the same description. Now take $n>0$ and suppose that the claim holds for all smaller values of n. Then $\mathcal{C}(n)$ has filtration $F_0\subseteq\cdots\subseteq F_{n-1}$. The subcomplex $F_0\cong C(\mathcal{C}(n-1))$ is chain-contractible by Lemma 3.7. And the induction hypothesis tells us that each $\mathcal{C}(n-p-1)$ has zero homology in all degrees up to and including $(n-p-1)-2$, so that $\Sigma^{p+1}\mathcal{C}(n-p-1))^{\oplus\frac{n!}{(n-p)!}}$ has zero homology in degrees up to and including $(n-p-1)-2+(p+1)=n-2$, so that by Lemma 3.8 the same is true of F_p/F_{p-1}. Since F_0 and all F_p/F_{p-1} have vanishing homology in the stated range, the same follows for $\mathcal{C}(n)$ itself. $\square$

4. The chain complex $\mathcal{D}(n)$

Definition 4.1. Let $n\geqslant 0$. The complex $\mathcal{D}(n)$ is defined to be the chain complex of left $\mathcal{H}_n$-modules defined by

$$\begin{array}{rcl}\mathcal{D}(n)_{n-1} & = & \mathcal{H}_n\otimes_{\mathcal{H}_0}\mathbb{1}\\ & \vdots & \\ \mathcal{D}(n)_r & = & \mathcal{H}_n\otimes_{\mathcal{H}_{n-r-1}}\mathbb{1}\\ & \vdots & \\ \mathcal{D}(n)_{-1} & = & \mathcal{H}_n\otimes_{\mathcal{H}_n}\mathbb{1}\end{array}$$

for r in the range $-1\leqslant r\leqslant n-1$, and by $\mathcal{D}(n)_r=0$ for r outside that range. Observe that $\mathcal{D}(-1)=\mathcal{H}_n\otimes_{\mathcal{H}_n}\mathbb{1}\cong\mathbb{1}$. The differential

$$\partial^r\colon\mathcal{D}(n)_r\longrightarrow\mathcal{D}(n)_{r-1}$$

of $\mathcal{D}(n)$ is defined by

$$\partial^r=\sum_{i=0}^{r}(-1)^i q^{-i}\partial_i^r$$

where $\partial_i^r\colon\mathcal{D}(n)_r\to\mathcal{D}(n)_{r-1}$, $\partial_i^r(x\otimes y)=(x\cdot D_i^r)\otimes y$ and D_i^r is the element of $\mathcal{H}_n$ defined below.

Definition 4.2. For $r=0,\ldots,(n-1)$ and $i=0,\ldots,r$ define $D_i^r\in\mathcal{H}_n$ by

$$D_i^r=T_{n-r+i-1}\cdots T_{n-r+1}T_{n-r}.$$

Note that the subscripts decrease from left to right, and in particular if $i = 0$ then the product is empty and we have $D_0^r = 1$. Thus

$$\begin{array}{rcr} D_0^r &=& 1 \\ D_1^r &=& T_{n-r} \\ D_2^r &=& T_{n-r+1}T_{n-r} \\ &\vdots& \\ D_r^r &=& T_{n-1}\cdots T_{n-r+1}T_{n-r} \end{array}$$

We must verify that the differentials in $\mathcal{D}(n)$ are well defined, and that they satisfy $\partial^{r-1} \circ \partial^d = 0$. However before we do that, let us relate $\mathcal{D}(n)$ to the complex of injective words $\mathcal{C}(n)$ discussed in section 3. Compare with Example 35 of [Hep16].

Proposition 4.3. *Let $q = 1$ so that $\mathcal{H}_n = R\mathfrak{S}_n$. Then $\mathcal{D}(n)$ is isomorphic to the usual complex of injective words $\mathcal{C}(n)$.*

Proof. Define a chain map

$$\Theta\colon \mathcal{D}(n) \longrightarrow \mathcal{C}(n)$$

as follows. In degree $-1 \leqslant r \leqslant (n-1)$ the complex $\mathcal{D}(n)$ is $R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-r-1}} \mathbb{1}$ and $\mathcal{C}(n)$ is spanned by injective words of length $(r+1)$, and we define

$$\Theta_r(\sigma \otimes 1) = (\sigma(n-r), \sigma(n-r+1), \ldots, \sigma(n)).$$

for $\sigma \in \mathfrak{S}_n$. For this to be well-defined we must check that if $\tau \in \mathfrak{S}_{n-r-1}$ then $\sigma\tau \otimes 1$ and $\sigma \otimes (\tau \cdot 1) = \sigma \otimes 1$ have the same image under Θ_r, but that is clear because such τ fixes $n-r, n-r+1, \ldots, n$. The same reasoning shows that Θ_r restricts to a bijection between the basis of $\mathcal{D}(n)_r$ consisting of representatives for the cosets $\mathfrak{S}_n/\mathfrak{S}_{n-r-1}$ and the basis of $\mathcal{C}(n)_r$ given by the injective words of length $r+1$.

It remains to show that Θ is a chain map, and since in both domain and codomain the differential ∂^r is given by the alternating sum $\sum_{j=0}^r (-1)^j \partial_j^r$ (the powers of q^{-1} appearing in the definition of ∂^r for $\mathcal{D}(n)$ are now all 1) it is sufficient to check that

$$\Theta_{r-1} \circ \partial_j^r = \partial_j^r \circ \Theta_r.$$

In order to do this we observe that the element

$$D_j^r = T_{n-r+j-1}\cdots T_{n-r+1}T_{n-r} = s_{n-r+j-1}\cdots s_{n-r+1}s_{n-r}$$

is the permutation whose effect on the final r letters is

$$D_j^r(n-r+k) = \begin{cases} n-r+k-1 & 1 \leqslant k \leqslant j \\ n-r+k & j+1 \leqslant k \leqslant r \end{cases}$$

so that

$$(D_j^r(n-r+1), \ldots, D_j^r(n)) = (n-r, \ldots, \widehat{n-r+j}, \ldots, n).$$

Thus

$$\begin{aligned}(\Theta_{r-1}\circ\partial^r_j)(\sigma\otimes 1) &= \Theta_{r-1}(\sigma D^r_j\otimes 1)\\ &= (\sigma(D^r_j(n-r+1)),\ldots,\sigma(D^r_j(n)))\\ &= (\sigma(n-r),\ldots,\sigma(\widehat{n-r+j}),\ldots,\sigma(n))\\ &= \partial^r_j(\sigma(n-r),\ldots,\sigma(n))\\ &= \partial^r_j\Theta_r(\sigma\otimes 1)\end{aligned}$$

as required. □

We now verify that the differentials in $\mathcal{D}(n)$ are well defined, and that they satisfy $\partial^{r-1}\circ\partial^d = 0$. Compare with Example 35 of [Hep16] and Lemmas 7.2 and 7.3 of [Boy20, Lemma 7.3].

Lemma 4.4. *The maps ∂^r_i are well-defined.*

Proof. We must show that if $\lambda\in\mathcal{H}_{n-r-1}$, then $\partial^r_i(x\lambda\otimes y)$ and $\partial^r_i(x\otimes\lambda y)$ agree, or in other words that $(x\lambda D^r_i)\otimes y = (xD^r_i)\otimes(\lambda y)$. This amounts to showing that D^r_i commutes with the generators of $\mathcal{H}_{n-r-1}$. To see this, observe that D^r_i is a word in $T_{n-r},\ldots,T_{n-r+i-1}$, while $\mathcal{H}_{n-r-1}$ is generated by $T_1,\ldots,T_{n-r-2}$, and each of the former commutes with each of the latter. □

Lemma 4.5. *$\mathcal{D}(n)$ is a chain complex, i.e. $\partial^{r-1}\circ\partial^r = 0$ for all $r\geqslant 1$.*

Proof. We will show that for $0\leqslant i<j\leqslant r$ we have

$$\partial^{r-1}_i\partial^r_j = q\cdot\partial^{r-1}_{j-1}\partial^r_i,$$

for then

$$(q^{-i}\partial^{r-1}_i)\circ(q^{-j}\partial^r_j) = (q^{-(j-1)}\partial^{r-1}_{j-1})\circ(q^{-i}\partial^r_i)$$

and the result then follows quickly. (Indeed, the last relation shows that the $q^{-i}\partial^r_i$ make $\mathcal{D}(n)$ into a semi-simplicial object in $R[\mathfrak{S}_n]$-modules, and then the claim that $\partial^{r-1}\circ\partial^r = 0$ is standard.) By Lemma 4.6 below we have

$$D^r_j\cdot D^{r-1}_i = D^r_i\cdot D^{r-1}_{j-1}\cdot T_{n-r}$$

so that

$$\begin{aligned}\partial^{r-1}_i\partial^r_j(x\otimes y) &= xD^r_jD^{r-1}_i\otimes y\\ &= xD^r_iD^{r-1}_{j-1}T_{n-r}\otimes y\\ &= xD^r_iD^{r-1}_{j-1}\otimes T_{n-r}y\\ &= xD^r_iD^{r-1}_{j-1}\otimes qy\\ &= q\cdot(xD^r_iD^{r-1}_{j-1}\otimes y)\\ &= q\cdot\partial^{r-1}_{j-1}\partial^r_i(x\otimes y).\end{aligned}$$

Here the third equality comes from the fact that the elements lie in $\mathcal{D}(n)_{r-2} = \mathcal{H}_n\otimes_{\mathcal{H}_{n-r+1}}\mathbb{1}$, and $\mathcal{H}_{n-r+1}$ contains the element T_{n-r}. □

Lemma 4.6. *For $0 \leqslant i < j \leqslant r \leqslant (n-1)$ we have*

$$D^r_j D^{r-1}_i = D^r_i D^{r-1}_{j-1} T_{n-r}.$$

Proof. Suppose that T_k is a letter from the given expression for D^{r-1}_i, so that $n-r+1 \leqslant k \leqslant n-r+i$. Then in particular $n-r+1 \leqslant k \leqslant n-r+j-1$, so that

$$\begin{aligned}
D^r_j \cdot T_k &= (T_{n-r+j-1} \cdots T_{n-r}) \cdot T_k \\
&= (T_{n-r+j-1} \cdots T_{k+1}) \cdot (T_k T_{k-1}) \cdot (T_{k-2} \cdots T_{n-r}) \cdot T_k \\
&= (T_{n-r+j-1} \cdots T_{k+1}) \cdot (T_k T_{k-1} T_k) \cdot (T_{k-2} \cdots T_{n-r}) \\
&= (T_{n-r+j-1} \cdots T_{k+1}) \cdot (T_{k-1} T_k T_{k-1}) \cdot (T_{k-2} \cdots T_{n-r}) \\
&= T_{k-1} \cdot (T_{n-r+j-1} \cdots T_{k+1}) \cdot (T_k T_{k-1}) \cdot (T_{k-2} \cdots T_{n-r}) \\
&= T_{k-1} \cdot (T_{n-r+j-1} \cdots T_{n-r}) \\
&= T_{k-1} \cdot D^r_j
\end{aligned}$$

Thus

$$\begin{aligned}
D^r_j \cdot D^{r-1}_i &= D^r_j \cdot (T_{n-r+i} \cdots T_{n-r+1}) \\
&= (T_{n-r+i-1} \cdots T_{n-r}) \cdot D^r_j \\
&= D^r_i \cdot D^r_j \\
&= D^r_i \cdot D^{r-1}_{j-1} \cdot T_{n-r}
\end{aligned}$$

□

5. The filtration of $\mathcal{D}(n)$

Definition 5.1 (The generators of the filtration)**.** For $-1 \leqslant r \leqslant n-2$ we define an element of $\mathcal{H}_n$ as follows.

$$B^r \quad = \quad T_{n-1} T_{n-2} \cdots T_{n-r} T_{n-r-1}$$

Here the indices decrease from left to right, so that if $r = -1$ then the product is empty and we have $B^{-1} = 1$.

For $0 \leqslant p \leqslant (n-1)$ we define an element of $\mathcal{H}_n$ as follows.

$$C_p \quad = \quad T_{n-1} T_{n-2} \cdots T_{n-p}$$

Here the indices decrease from left to right, so that if $p = 0$ then the product is empty and $C_0 = 1$. Thus:

$$\begin{array}{lcl}
C_0 & = & 1 \\
C_1 & = & T_{n-1} \\
C_2 & = & T_{n-1} T_{n-2} \\
 & \vdots & \\
C_{n-1} & = & T_{n-1} T_{n-2} \cdots T_1
\end{array}$$

Definition 5.2 (The filtration of $\mathcal{D}(n)$)**.** Let $0 \leqslant p \leqslant (n-1)$. Define $F_p \subseteq \mathcal{D}(n)$ to be the subcomplex of $\mathcal{D}(n)$ that in degree r is generated as an $\mathcal{H}_{n-1}$-module by the element $B^r \otimes 1$ (for $-1 \leqslant r \leqslant (n-2)$), and the elements $C_0 \otimes 1, \ldots, C_{\min(p,r)} \otimes 1$ (for $0 \leqslant r \leqslant (n-1)$). Thus we obtain a filtration

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} = \mathcal{D}(n)$$

by $\mathcal{H}_{n-1}$-submodules. Note that F_p is not an $\mathcal{H}_n$-submodule of $\mathcal{D}(n)$.

At this stage we must verify that each F_p is indeed a subcomplex of $\mathcal{D}(n)$, but we leave that to the end of the section.

Right now we give a proposition that explains the relationship of the filtration of $\mathcal{D}(n)$ to the filtration

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} = \mathcal{C}(n)$$

of the complex of injective words $\mathcal{C}(n)$ given in Definition 3.5. In particular, the proof of the proposition 'explains' the definition of the elements C_p and B^r above.

Proposition 5.3. *Let $q = 1$ so that we have an isomorphism $\Theta\colon \mathcal{D}(n) \xrightarrow{\cong} \mathcal{C}(n)$ as in the proof of Proposition 4.3. Then the filtrations of $\mathcal{C}(n)$ and $\mathcal{D}(n)$ are identified by the isomorphism Θ, i.e. $\Theta(F_p) = F_p$ for all p.*

Proof. Recall that for $\mathcal{D}(n)$ we defined F_p in degree r to be the $\mathcal{H}_{n-1} = R\mathfrak{S}_{n-1}$ span of the elements $B^r \otimes 1$ (for $-1 \leqslant r \leqslant (n-2)$), and $C_0 \otimes 1, \ldots, C_{\min(p,r)} \otimes 1$ (for $0 \leqslant r \leqslant (n-1)$).

Recall that

$$B^r = T_{n-1}T_{n-2}\cdots T_{n-r}T_{n-r-1} = s_{n-1}s_{n-2}\cdots s_{n-r}s_{n-r-1}$$

so that

$$\begin{aligned}\Theta(B^r \otimes 1) &= (B^r(n-r), \ldots, B^r(n)) \\ &= (n-r-1, \ldots, n-1)\end{aligned}$$

with no appearance of the letter n. And for $t \leqslant \min(p,r)$,

$$C_t = T_{n-1}T_{n-2}\cdots T_{n-t} = s_{n-1}s_{n-2}\cdots s_{n-t}$$

so that

$$\begin{aligned}\Theta_r(C_t \otimes 1) &= (C_t(n-r), \ldots, C_t(n)) \\ &= (n-r, n-r+1, \ldots, n-t-1, n, n-t, n-t+1, \ldots, n-1)\end{aligned}$$

with the letter n appearing in the $(t+1)$-st position from the end. Thus in degree r, the generators of $F_p \subseteq \mathcal{D}(n)$ are sent to the prototypical words in which the letter n appears in the last $(p+1)$ places or not at all, and the $\mathcal{H}_{n-1} = R\mathfrak{S}_{n-1}$-submodule spanned by these elements is sent by Θ to the span of all words in which the letter n appears in the last $(p+1)$ places or not at all. In other words, $\Theta(F_p) = F_p$ as required. $\square$

Now we state the main result on the filtration.

Theorem 5.4. *There are isomorphisms*

$$F_0 \cong C(\mathcal{D}(n-1))$$

and, for $1 \leqslant p \leqslant (n-1)$,

$$F_p/F_{p-1} \cong \mathcal{H}_n \otimes_{\mathcal{H}_{n-p-1}} \Sigma^{p+1}\mathcal{D}(n-p-1).$$

In particular, since $\mathcal{H}_n$ *is free as a right-*$\mathcal{H}_{n-p-1}$*-module with rank* $n!/(n-p)!$, F_p/F_{p-1} *is isomorphic to a direct sum of* $n!/(n-p)!$ *copies of* $\Sigma^{p+1}\mathcal{D}(n-p-1)$.

Theorem 5.5. $H_d(\mathcal{D}(n)) = 0$ *for* $n \leqslant (n-2)$.

The proof that the Theorem 5.5 follows from Theorem 5.4 is entirely analogous to the proof of Theorem 3.4 given in section 3. The proof of Theorem 5.4 will occupy our attention throughout sections 6, 7 and 8. We conclude this section by verifying that F_p is indeed a filtration of chain complexes.

Lemma 5.6. F_p *is indeed a subcomplex of* $\mathcal{D}(n)$.

Proof. Fix $0 \leqslant p \leqslant (n-1)$ and $-1 \leqslant r \leqslant (n-1)$. Definition 5.2 lists the generators of F_p, as an $\mathcal{H}_{n-1}$-module, in each degree r. So it is enough to check that if we apply ∂^r_i to one of the generators of F_p in degree r, then the result is an $\mathcal{H}_{n-1}$-linear combination of the generators of F_p in degree $r-1$.

Take $-1 \leqslant r \leqslant (n-2)$, so that $B^r \otimes 1$ is a generator of F_p. Then by Lemma 5.8 below there is an element $x \in \mathcal{H}_{n-1}$ such that

$$\partial^r_j(B^r \otimes 1) = B^r D^r_j \otimes 1 = xB^{r-1}T_{n-r-1} \otimes 1 = xB^{r-1} \otimes T_{n-r-1} \cdot 1 = qx(B^{r-1} \otimes 1).$$

Here, we have used the fact that the tensor products (all but the first) are over $\mathcal{H}_{n-(r-1)-1} = \mathcal{H}_{n-r}$, which contains T_{n-r-1}. Thus $\partial^r_j(B^r \otimes 1)$ is an $\mathcal{H}_{n-1}$-multiple of a generator of F_p (in fact of F_0) as required.

Take $0 \leqslant r \leqslant (n-1)$ and $0 \leqslant t \leqslant \min(r,p)$, so that $C_t \otimes 1$ is a generator of F_p in degree r.

By Lemma 5.9 below we have

$$\partial^r_j(C_t \otimes 1) = C_t D^r_j \otimes 1 = \begin{cases} UC_t \otimes 1 & j \leqslant r-t-1 \\ B^{r-1} \otimes 1 & j = r-t \\ VB^{r-1} + WC_{t-1} & j \geqslant r-t+1 \end{cases}$$

where U, V, W are elements of $\mathcal{H}_{n-1}$. So in each case, the right hand side also lies in F_t. $\square$

Note 5.7. We establish the following notational convention. There are various quantities associated to $\mathcal{D}(n)$, such as ∂^r, ∂^r_j, D^r_j and so on. We will use an underline to denote those quantities when they are associated not to $\mathcal{D}(n)$ but to $\mathcal{D}(n-1)$, giving e.g. $\underline{\partial}^r$, $\underline{\partial}^r_j$, $\underline{D}^r_j$.

Lemma 5.8. *Let $-1 \leqslant r \leqslant (n-2)$. Then*

$$B^r D^r_j = \underline{D}^r_j B^{r-1} T_{n-r-1}.$$

Observe that $\underline{D}^r_j$ lies in $\mathcal{H}_{n-1}$.

Proof.

$$\begin{aligned}
B^r D^r_j &= (T_{n-1}\cdots T_{n-r-1})\cdot(T_{n-r+j-1}\cdots T_{n-r})\\
&= (T_{n-r+j-2}\cdots T_{n-r-1})\cdot(T_{n-1}\cdots T_{n-r-1})\\
&= (T_{n-r+j-2}\cdots T_{n-r-1})B^{r-1}T_{n-r-1}\\
&= \underline{D}^r_j B^{r-1} T_{n-r-1}
\end{aligned}$$

□

Lemma 5.9. *Given $0 \leqslant r \leqslant (n-1)$, $0 \leqslant t \leqslant r$ and $0 \leqslant j \leqslant r$, we have:*

$$C_t D^r_j = \begin{cases} D^r_j C_t & j \leqslant r-t-1 \\ B^{r-1} & j = r-t \\ (q-1)\cdot(T_{n-r+j-2}\cdots T_{n-t})B^{r-1} & \\ \qquad +q\cdot(T_{n-r+j-2}\cdots T_{n-r})C_{t-1} & j \geqslant r-t+1 \end{cases}$$

Observe that one the right hand side the elements D^r_j, $(T_{n-r+j-2}\cdots T_{n-t})$ and $(T_{n-r+j-2}\cdots T_{n-r})$, all lie in $\mathcal{H}_{n-1}$.

Proof. In the first case, all of the letters in C_t commute with all of the letters in D^r_j, so that $C_t D^r_j = D^r_j C_t$ as required. In the second case, we have:

$$\begin{aligned}
C_t D^r_{r-t} &= (T_{n-1}\cdots T_{n-t})(T_{n-t-1}\cdots T_{n-r})\\
&= (T_{n-1}\cdots T_{n-r})\\
&= (T_{n-1}\cdots T_{n-(r-1)-1})\\
&= B^{r-1}
\end{aligned}$$

In the third case, we have

$$\begin{aligned}
C_t D^r_j &= (T_{n-1}\cdots T_{n-t})(T_{n-r+j-1}\cdots T_{n-r})\\
&= (T_{n-1}\cdots T_{n-t})(T_{n-r+j-1}\cdots T_{n-t+1})(T_{n-t}\cdots T_{n-r})\\
&= (T_{n-r+j-2}\cdots T_{n-t})(T_{n-1}\cdots T_{n-t})(T_{n-t}\cdots T_{n-r})\\
&= (T_{n-r+j-2}\cdots T_{n-t})(T_{n-1}\cdots T_{n-t+1})T^2_{n-t}(T_{n-t-1}\cdots T_{n-r})\\
&= (q-1)\cdot(T_{n-r+j-2}\cdots T_{n-t})(T_{n-1}\cdots T_{n-t+1})T_{n-t}(T_{n-t-1}\cdots T_{n-r})\\
&\qquad + q\cdot(T_{n-r+j-2}\cdots T_{n-t})(T_{n-1}\cdots T_{n-t+1})(T_{n-t-1}\cdots T_{n-r})
\end{aligned}$$

and the individual summands can be simplified as

$$\begin{aligned}(T_{n-r+j-2}\cdots T_{n-t})(T_{n-1}\cdots T_{n-t+1})T_{n-t}(T_{n-t-1}\cdots T_{n-r})\\ &= (T_{n-r+j-2}\cdots T_{n-t})(T_{n-1}\cdots T_{n-r})\\ &= (T_{n-r+j-2}\cdots T_{n-t})B^{r-1}\end{aligned}$$

and

$$\begin{aligned}(T_{n-r+j-2}\cdots T_{n-t})(T_{n-1}\cdots T_{n-t+1})(T_{n-t-1}\cdots T_{n-r})\\ &= (T_{n-r+j-2}\cdots T_{n-r})(T_{n-1}\cdots T_{n-t+1})\\ &= (T_{n-r+j-2}\cdots T_{n-r})C_{t-1}\end{aligned}$$

to give the result. □

6. The filtration quotients: Defining Φ

Definition 6.1. We define a map

$$\Phi\colon C(\mathcal{D}(n-1)) \longrightarrow F_0$$

as follows. Observe that $C(\mathcal{D}(n-1))$ in degree r is

$$C(\mathcal{D}(n-1))_r = \mathcal{D}(n-1)_r \oplus \mathcal{D}(n-1)_{r-1} = (\mathcal{H}_{n-1}\otimes_{\mathcal{H}_{n-r-2}}\mathbb{1})\oplus(\mathcal{H}_{n-1}\otimes_{\mathcal{H}_{n-r-1}}\mathbb{1})$$

so that as an $\mathcal{H}_{n-1}$-module it has two generators, namely $(1\otimes 1, 0)$ and $(0, 1\otimes 1)$. And recall that F_0 in degree r is the submodule of $\mathcal{D}(n)_r = \mathcal{H}_n\otimes_{\mathcal{H}_{n-r-1}}\mathbb{1}$ generated by $B^r\otimes 1$ and $C_0\otimes 1 = 1\otimes 1$. We now define Φ in degree r to be the $\mathcal{H}_{n-1}$-linear map defined on generators by

$$\Phi(1\otimes 1, 0) = q^{-r}B^r\otimes 1$$

and

$$\Phi(0, 1\otimes 1) = q\cdot 1\otimes 1.$$

The map Φ defined above is the Iwahori-Hecke analogue of the isomorphism obtained in Lemma 3.7, and indeed reduces to give the map constructed there in the case $q=1$.

Lemma 6.2. *Φ is well defined.*

Proof. We must check that for $1\leqslant k\leqslant n-r-3$, we have

$$\Phi(T_k\otimes 1, 0) = \Phi(1\otimes(T_k\cdot 1), 0)$$

which amounts to showing that $T_kB^r\otimes 1 = qB^r\otimes 1$. This is immediate since T_k commutes with B^r and the tensor product is over $\mathcal{H}_{n-r-1}$, which contains T_k.

We must also check that for $0\leqslant k\leqslant n-r-2$ we have

$$\Phi(0, T_k\otimes 1) = \Phi(0, 1\otimes(T_k\cdot 1)),$$

which amounts to showing that $T_k\otimes 1 = 1\otimes(T_k\cdot 1)$, but since the tensor product is over $\mathcal{H}_{n-r-1}$, which contains T_k, this is again immediate. □

Lemma 6.3. Φ *is a chain map.*

Proof. We must show that $\Phi \circ d^r_{C(\mathcal{D}(n-1))} = \partial^r \circ \Phi$ on the two generators of $C(\mathcal{D}(n-1))_r$. The proof will make use of Lemma 5.8, which states that $B^r D^r_j = \underline{D}^r_j B^{r-1} T_{n-r-1}$. Recall from Note 5.7 that underlines, such as on $\underline{D}^r_j$, indicate the usual quantity, but now associated to $\mathcal{D}(n-1)$ rather than $\mathcal{D}(n)$.

We have

$$
\begin{aligned}
\partial^r(\Phi(1\otimes 1,0)) &= \sum_{j=0}^{r}(-1)^j q^{-j}\partial^r_j \Phi(1\otimes 1,0)\\
&= \sum_{j=0}^{r}(-1)^j q^{-j}q^{-r}\partial^r_j(B^r\otimes 1)\\
&= \sum_{j=0}^{r}(-1)^j q^{-j}q^{-r}(B^r D^r_j\otimes 1)\\
&= \sum_{j=0}^{r}(-1)^j q^{-j}q^{-r}\underline{D}^r_j B^{r-1}T_{n-r-1}\otimes 1\\
&= \sum_{j=0}^{r}(-1)^j q^{-j}q^{-r}\underline{D}^r_j B^{r-1}\otimes T_{n-r-1}\cdot 1\\
&= \sum_{j=0}^{r}(-1)^j q^{-j}q^{-(r-1)}\underline{D}^r_j B^{r-1}\otimes 1\\
&= \sum_{j=0}^{r}(-1)^j q^{-j}\Phi(\underline{D}^r_j\otimes 1,0)\\
&= \sum_{j=0}^{r}(-1)^j q^{-j}\Phi(\underline{\partial}^r_j(1\otimes 1),0)\\
&= \Phi(\underline{\partial}^r(1\otimes 1),0)\\
&= \Phi(d^r_{C(\mathcal{D}(X))}(1\otimes 1,0))
\end{aligned}
$$

For the fourth and fifth lines, note that the elements lie in $\mathcal{D}(n)_{r-1} = \mathcal{H}_n \otimes \mathcal{H}_{n-r}\mathbb{1}$, and T_{n-r-1} lies in $\mathcal{H}_{n-r}$.

And we have

$$\begin{aligned}
\partial^r(\Phi(0,1\otimes 1)) &= \sum_{j=0}^{r}(-1)^j q^{-j}\partial_j^r\Phi(0,1\otimes 1)\\
&= \sum_{j=0}^{r}(-1)^j q^{-j}q\partial_j^r(1\otimes 1)\\
&= \sum_{j=0}^{r}(-1)^j q^{-j}q(D_j^r\otimes 1)\\
&= \sum_{j=0}^{r-1}(-1)^j q^{-j}q(D_j^r\otimes 1)+(-1)^r q^{-r+1}(D_r^r\otimes 1)
\end{aligned}$$

The first term can be simplified as

$$\begin{aligned}
\sum_{j=0}^{r-1}(-1)^j q^{-j}q(D_j^r\otimes 1) &= \sum_{j=0}^{r-1}(-1)^j q^{-j}q(\underline{D}_j^{r-1}\otimes 1)\\
&= \sum_{j=0}^{r-1}(-1)^j q^{-j}\Phi(0,\underline{D}_j^{r-1}\otimes 1)\\
&= \sum_{j=0}^{r-1}(-1)^j q^{-j}\Phi(0,\underline{\partial}_j^{r-1}(1\otimes 1))\\
&= \Phi(0,\underline{\partial}^{r-1}(1\otimes 1))
\end{aligned}$$

where we have used the fact that $D_j^r = \underline{D}_j^r$ for $j \leqslant r-1$. And the second term can be simplified as

$$\begin{aligned}
(-1)^r q^{-r+1}(D_r^r\otimes 1) &= (-1)^r q^{-(r-1)}(B^{r-1}\otimes 1)\\
&= (-1)^r\Phi(1\otimes 1,0)
\end{aligned}$$

where we have used the fact that $D_r^r = B^{r-1}$. So altogether we have

$$\begin{aligned}
\partial^r(\Phi(0,1\otimes 1)) &= \Phi(0,\underline{\partial}^{r-1}(1\otimes 1)) + (-1)^r\Phi(1\otimes 1,0)\\
&= \Phi((-1)^r(1\otimes 1),\underline{\partial}^{r-1}(1\otimes 1))\\
&= \Phi(d_{C(\mathcal{D}(n-1))}(0,1\otimes 1)).
\end{aligned}$$

□

7. The filtration quotients: Defining Ψ

We will now begin to identify the filtration quotients F_p/F_{p-1} for $1 \leqslant p \leqslant (n-1)$. We will usually write elements of F_p/F_{p-1} as elements of F_p, and will not include in the notation the fact that we are working in a quotient.

Definition 7.1. Let $p \geqslant 1$. We define a map

$$\Psi\colon \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \Sigma^{p+1}\mathcal{D}(n-p-1) \longrightarrow F_p/F_{p-1}.$$

Before defining the map, let us elaborate on its domain and codomain. We work in a fixed degree $(p+1)+r$ for $-1 \leqslant r \leqslant n-p-1$.

- In degree $(p+1)+r$, the domain is

$$\begin{aligned}\mathcal{H}_{n-1} &\otimes_{\mathcal{H}_{n-p-1}} \Sigma^{p+1}\mathcal{D}(n-p-1)_{(p+1)+r}\\ &= \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \mathcal{D}(n-p-1)_r\\ &= \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \left(\mathcal{H}_{n-p-1} \otimes_{\mathcal{H}_{n-p-r-2}} \mathbb{1}\right)\end{aligned}$$

  and so it is generated as an $\mathcal{H}_{n-1}$-module by the element $1 \otimes (1 \otimes 1)$.
- In degree $(p+1)+r$, the codomain F_p/F_{p-1} is a subquotient of

$$\mathcal{D}(n)_{(p+1)+r} = \mathcal{H}_n \otimes_{\mathcal{H}_{n-p-r-2}} \mathbb{1},$$

  and it is generated as an $\mathcal{H}_{n-1}$-module by the element $C_p \otimes 1$. (Recall that F_p in this degree is the $\mathcal{H}_{n-1}$ submodule generated by $B^{(p+1)+r}, C_0, \ldots, C_p$, and that F_{p-1} is the $\mathcal{H}_{n-1}$-submodule generated by all these except the last.)

Now we define Ψ to be the $\mathcal{H}_{n-1}$-linear map that is defined on the generator by

$$\Psi(1 \otimes (1 \otimes 1)) = C_p \otimes 1.$$

(We remind the reader that although we are working in the quotient F_p/F_{p-1} we will not indicate that when we are describing the elements of quotient.)

The map Ψ defined above is the Iwahori-Hecke analogue of the isomorphism obtained in Lemma 3.8, and indeed reduces to give the map constructed there in the case $q = 1$.

Lemma 7.2. Ψ *is well defined.*

Proof. Working in degree $(p+1)+r$, we must show that for $1 \leqslant k \leqslant n-r-p-3$, Ψ sends the elements $T_k \otimes (1 \otimes 1)$ and $1 \otimes (1 \otimes (T_k \cdot 1))$ to the same element. Since

$$\Psi(T_k \otimes (1 \otimes 1)) = T_k \cdot \Psi(1 \otimes (1 \otimes 1)) = T_k \cdot (C_p \otimes 1) = (T_k C_p) \otimes 1$$

and

$$\Psi(1 \otimes (1 \otimes (T_k \cdot 1))) = q \cdot \Psi(1 \otimes (1 \otimes 1)) = q(C_p \otimes 1) = C_p \otimes (T_k \cdot 1)$$

it is enough to show that T_k commutes with C_p. And indeed, since $C_p = T_{n-1} \cdots T_{n-r-p-1}$ and $k \leqslant n-r-p-2$, this follows immediately. $\square$

Lemma 7.3. Ψ *is a chain map.*

Proof. In this proof we will use $\underline{\underline{\partial}}^r$ to denote the quantities usually denoted ∂^r, but now associated to $\mathcal{D}(n-p-1)$ rather than $\mathcal{D}(n)$. Similarly for $\underline{\underline{\partial}}^r_j$ and $\underline{\underline{D}}^r_j$.

Consider the differentials going from degree $r+p+1$ to $r+p$. In the domain of Ψ this is simply the map

$$\underline{\underline{\partial}}^r = \sum_{j=0}^{r} (-1)^j q^{-j} \underline{\underline{\partial}}^r_j.$$

In the codomain of Ψ the differential in degree $(p+1)+r$ is

$$\partial^{(p+1)+r} = \sum_{j=0}^{(p+1)+r} (-1)^j q^{-j} \partial^r_j.$$

Now, the codomain is $(F_p/F_{p-1})_{(p+1)+r}$, which is generated as an $\mathcal{H}_{n-1}$-module by the single element $C_p \otimes 1$. For this element we have, by Lemma 5.9:

$$\begin{aligned}
\partial^{r+p+1}_j (C_p \otimes 1) &= C_p D^{r+p+1}_j \otimes 1 \\
&= \begin{cases} D^{r+p+1}_j C_p \otimes 1 & j \leqslant r \\ B^{r+p} \otimes 1 & j = r+1 \\ V B^{r+p} \otimes 1 + W C_{p-1} \otimes 1 & j \geqslant r+2 \end{cases}
\end{aligned}$$

where $V, W \in \mathcal{H}_{n-1}$. The terms in the second and third line all lie in F_{p-1}, so that in the quotient F_p/F_{p-1} we in fact have:

$$\partial^{(p+1)+r}_j (C_p \otimes 1) = \begin{cases} D^{r+p+1}_j C_p \otimes 1 & j \leqslant r \\ 0 & j \geqslant r+1 \end{cases}$$

In particular, since $C_p \otimes 1$ generates F_p/F_{p-1} in this degree, we have $\partial^{r+p+1}_j = 0$ for $j \geqslant r+1$ and

$$\partial^{r+p+1} = \sum_{j=0}^{r} (-1)^j q^{-j} \partial^{r+p+1}_j.$$

This means that in order to verify that $\Psi \circ \underline{\underline{\partial}}^r = \partial^{(p+1)+r} \circ \Psi$, it will suffice to check that $\Psi \circ \underline{\underline{\partial}}^r_j = \partial^{(p+1)+r}_j \circ \Psi$ for $0 \leqslant j \leqslant r$. And since the domain is generated as an $\mathcal{H}_{n-1}$ module by the element $1 \otimes (1 \otimes 1)$, it is enough to verify that

$$\partial^{r+p+1}_j \Psi(1 \otimes (1 \otimes 1)) = \Psi \underline{\underline{\partial}}^r_j (1 \otimes (1 \otimes 1)).$$

And indeed:

$$\begin{aligned}
\partial^{r+p+1}_j \Psi(1 \otimes (1 \otimes 1)) &= \partial^{r+p+1}_j (C_p \otimes 1) \\
&= D^{r+p+1}_j C_p \otimes 1 \\
&= \underline{\underline{D}}^r_j C_p \otimes 1 \\
&= \Psi(1 \otimes (\underline{\underline{D}}^r_j \otimes 1)) \\
&= \Psi \underline{\underline{\partial}}^r_j (1 \otimes (1 \otimes 1))
\end{aligned}$$

Here we have used the fact that $\underline{\underline{D}}^r_j = D^{r+p+1}_j$. $\square$

8. The filtration quotients: Φ and Ψ are isomorphisms

In this section we will prove that the maps

$$\Phi\colon C(\mathcal{D}(n-1)) \longrightarrow F_0$$

and

$$\Psi\colon \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \Sigma^{p+1}\mathcal{D}(n-p-1) \longrightarrow F_p/F_{p-1}$$

are isomorphisms. In order to do so, we will obtain bases for the domain and codomain and prove that Φ and Ψ induce bijections between these. Now, Φ and Ψ are maps of $\mathcal{H}_{n-1}$-modules, whose domains are built out of tensor products of the form $\mathcal{H}_{n-1} \otimes_{\mathcal{H}_k} \mathbb{1}$, and we understand from Proposition 2.7 how to give a basis for $\mathcal{H}_{n-1} \otimes_{\mathcal{H}_k} \mathbb{1}$ as an $\mathcal{H}_{n-1}$-module using the distinguished coset representatives $(X^{S_{n-1}}_{S_k})^{-1}$ for $\mathfrak{S}_{n-1}/\mathfrak{S}_k$. However, the codomains of Φ and Ψ are built from tensor products of the form $\mathcal{H}_n \otimes_{\mathcal{H}_k} \mathbb{1}$, and in order to obtain a basis of this *as an* $\mathcal{H}_{n-1}$-*module*, we will need to study the distinguished double coset representatives of $\mathfrak{S}_{n-1}\backslash\mathfrak{S}_n/\mathfrak{S}_k$.

In what follows we will consider the Coxeter system $(\mathfrak{S}_n, S_n)$ where $S_n = \{s_1, \ldots, s_{n-1}\}$. We will similarly write $S_k = \{s_1, \ldots, s_{k-1}\}$, so that the parabolic subgroup of $\mathfrak{S}_n$ generated by S_k is precisely $\mathfrak{S}_k$. We are interested in understanding generators of $\mathcal{D}(n)_r = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$, which is to say, the distinguished representatives $X^{-1}_{S_{n-r-1}}$ for the left cosets $\mathfrak{S}_n/\mathfrak{S}_{n-r-1}$. In particular, in order to study the filtration $\{F_p\}$ of $\mathcal{D}(n)$, we consider $\mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$ as an $\mathcal{H}_{n-1}$-module, so that we will need to compute the distinguished representatives $X_{S_{n-1},S_{n-r-1}}$ of the double cosets $\mathfrak{S}_{n-1}\backslash\mathfrak{S}_n/\mathfrak{S}_{n-r-1}$.

We begin with the distinguished representatives for $\mathfrak{S}_{n-1}\backslash\mathfrak{S}_n$.

Lemma 8.1. *An element of $\mathfrak{S}_n$ is $(S_{n-1}, \emptyset)$-reduced if and only if it has the form $w = s_{n-1}\cdots s_j$ for some j in the range $1 \leqslant j \leqslant n$. In the case $j = n$ we take the product to be empty so that $w = e$. In other words,*

$$X_{S_{n-1}} = \{e, s_{n-1}, s_{n-1}s_{n-2}, \ldots, s_{n-1}\cdots s_1\}.$$

Proof. First we show that the given elements are all $(S_{n-1}, \emptyset)$-reduced. To do so, we need only show that they have no reduced expression beginning with an element of S_{n-1}. But the given expressions for the elements clearly admit no M-moves, and are therefore reduced, and since they do not begin with elements of S_{n-1}, this makes clear that the elements are $(S_{n-1}, \emptyset)$-reduced.

Now let w be $(\mathfrak{S}_{n-1}, \emptyset)$-reduced and let $n \geqslant j \geqslant 1$ be the smallest element such that w has a reduced expression beginning $s_{n-1}\cdots s_j$. We will show that $w = s_{n-1}\cdots s_j$. Suppose not: then w has a reduced expression beginning $s_{n-1}\cdots s_j s_i$ for some $i = 1, \ldots, (n-1)$. We cannot have $i = j$ for then the expression is not reduced. We cannot have $i = j-1$ by minimality of j. We cannot have

$i < j-1$ for then $s_{n-1}\cdots s_j s_i = s_i s_{n-1}\cdots s_j$ and w is not $(S_{n-1},\emptyset)$-reduced. And finally we cannot have $i > j$ because then $s_{n-1}\cdots s_j s_i = s_{i-1}s_{n-1}\cdots s_j$ is again not $\mathfrak{S}_{n-1}$-reduced on the left. So there is no such i. $\square$

Now we wish to study the distinguished representatives of the double cosets $\mathfrak{S}_{n-1}\backslash\mathfrak{S}_n/\mathfrak{S}_{n-r-1}$. In order to do so we introduce the following analogues of the elements C_t and B^r which were used to define the filtration of $\mathcal{D}(n)$.

Definition 8.2 (The analogues of the generators)**.** For $-1 \leqslant r \leqslant n-2$ we define an element of $\mathfrak{S}_n$ as follows.

$$b^r = s_{n-1}s_{n-2}\cdots s_{n-r}s_{n-r-1}$$

Here the indices decrease from left to right, so that if $r=-1$ then the product is empty and we have $b^{-1} = 1$.

For $0 \leqslant p \leqslant (n-1)$ we define an element of $\mathfrak{S}_n$ as follows.

$$c_p = s_{n-1}s_{n-2}\cdots s_{n-p}$$

Here the indices decrease from left to right, so that if $p=0$ then the product is empty and $c_0 = e$. Thus:

$$\begin{aligned} c_0 &= e \\ c_1 &= s_{n-1} \\ c_2 &= s_{n-1}s_{n-2} \\ &\vdots \\ c_{n-1} &= s_{n-1}s_{n-2}\cdots s_1 \end{aligned}$$

Lemma 8.3. *For r in the range $-1 \leqslant r \leqslant (n-1)$, the elements $c_0,\ldots,c_r$ (if $0 \leqslant r \leqslant (n-1)$), together with b^r (if $-1 \leqslant r \leqslant (n-2)$), form a complete set of distinguished $\mathfrak{S}_{n-1}\backslash\mathfrak{S}_n/\mathfrak{S}_{n-r-1}$ double coset representatives. In other words,*

$$X_{S_{n-1},S_{n-r-1}} = \begin{cases} \{b_0\} & r=-1 \\ \{c_0,\ldots,c_r,b_r\} & 0 \leqslant r \leqslant (n-2) \\ \{c_0,\ldots,c_{n-1}\} & r=(n-1) \end{cases}$$

Proof. The given expressions for the elements clearly admit no M-moves, and therefore are reduced, and are the unique reduced expressions for these elements. Since none of the expressions begin with a generator of $\mathfrak{S}_{n-1}$ or end with a generator of $\mathfrak{S}_{n-r-1}$, they are (S_{n-1},S_{n-r-1})-reduced. They are therefore minimal double coset representatives. (Proposition 2.1.7 of [GP00].)

It remains to show that they are a complete set of minimal double coset representatives. But a minimal double coset representative is $(S_{n-1},\emptyset)$-reduced, so by Lemma 8.1 it has the form $s_{n-1}s_{n-2}\cdots s_j$ for some j. And for w to be $(\emptyset,S_{n-r-1})$-reduced, we must have $j \geqslant n-r-1$, so that w is one of the given elements. $\square$

Now we will apply the Mackey formula for left-cosets in order to understand distinguished representatives of $\mathfrak{S}_n/\mathfrak{S}_{n-r-1}$ in terms of $\mathfrak{S}_{n-1}\backslash\mathfrak{S}_n/\mathfrak{S}_{n-r-1}$:

Lemma 8.4. *Let $-1 \leqslant r \leqslant (n-1)$. Then*
$(X_{S_{n-r-1}})^{-1} = (X^{S_{n-1}}_{S_{n-r-2}})^{-1} b^r \sqcup (X^{S_{n-1}}_{S_{n-r-1}})^{-1} c_0 \sqcup (X^{S_{n-1}}_{S_{n-r-1}})^{-1} c_1 \sqcup \cdots \sqcup (X^{S_{n-1}}_{S_{n-r-1}})^{-1} c_r$
where the b-term is omitted in the case $r = (n-1)$, and the c-terms are omitted in the case $r = 0$.

Proof. We will use the Mackey decomposition of section 2.2, taking $S = S_n$, $J = S_{n-r-1}$ and $K = S_{n-1}$, so that we obtain

$$X^{-1}_{S_{n-r-1}} = \bigsqcup_{d \in X_{S_{n-1}, S_{n-r-1}}} (X^{S_{n-1}}_{S_{n-1} \cap {}^d S_{n-r-1}})^{-1} \cdot d \tag{1}$$

So we must work out $S_{n-1} \cap {}^d S_{n-r-1}$ for $d \in X_{S_{n-1}, S_{n-r-1}}$. For $0 \leqslant k \leqslant r \leqslant (n-1)$ we have

$$\begin{aligned} S_{n-1} \cap {}^{c_k} S_{n-r-1} &= S_{n-1} \cap {}^{s_{n-1} \cdots s_{n-k}} \{s_1, \ldots, s_{n-r-2}\} \\ &= S_{n-1} \cap \{s_1, \ldots, s_{n-r-2}\} \\ &= S_{n-1} \cap S_{n-r-1} \\ &= S_{n-r-1} \end{aligned}$$

since $s_{n-1} \cdots s_{n-k}$ commutes with $s_{n-1}, \ldots, s_{n-k}$. And for $-1 \leqslant r \leqslant (n-2)$ we have

$$\begin{aligned} S_{n-1} \cap {}^{b^r} S_{n-r-1} &= S_{n-1} \cap {}^{s_{n-1} \cdots s_{n-r-1}} \{s_1, \ldots, s_{n-r-2}\} \\ &= S_{n-1} \cap \{s_1, \ldots, s_{n-r-3}, s_{n-1} \cdots s_{n-r-2} \cdots s_{n-1}\} \\ &= \{s_1, \ldots, s_{n-r-3}\} \\ &= S_{n-r-2} \end{aligned}$$

The Mackey decomposition (1) now gives us the required result. □

Lemma 8.5. *Let $-1 \leqslant r \leqslant (n-2)$. Then $(F_0)_r$ has basis*

$$\{T_x B^r \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-2}})^{-1}\} \cup \{T_x \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1}\}.$$

Let $0 \leqslant r \leqslant (n-1)$ and let $1 \leqslant p \leqslant (n-1)$. Then $(F_p/F_{p-1})_r$ has basis

$$\{T_x C_p \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-p-r-2}})^{-1}\}.$$

Proof. $\mathcal{D}(n)_r = \mathcal{H}_n \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}$ has basis $\{T_x \otimes 1 \mid x \in (X_{S_{n-r-1}})^{-1}\}$. By Lemma 8.4, this is equal to

$$\begin{aligned} &\{T_y \otimes 1 \mid y \in (X^{S_{n-1}}_{S_{n-r-2}})^{-1} b^r\} \\ &\qquad \cup \{T_y \otimes 1 \mid y \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1} c_0\} \\ &\qquad \vdots \\ &\qquad \cup \{T_y \otimes 1 \mid y \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1} c_r\}. \end{aligned}$$

Here we omit the b^r-term if $r = (n-1)$, and the c_t-terms if $r = -1$. Observe that if $T_y \otimes 1$ is an element of the first set in the union above, then $y = xb^r$ for some $x \in (X^{S_{n-1}}_{S_{n-r-2}})^{-1}$. Since $\ell(xb^r) = \ell(x) + \ell(b^r)$ as in Theorem 2.4, we then have $T_y = T_x T_{b^r} = T_x B^r$. Similarly for the other sets in the union, so that the basis is given by

$$\begin{aligned}\{T_x B^r \otimes 1 \,| x \in (X^{S_{n-1}}_{S_{n-r-2}})^{-1}\}& \\ &\cup \{T_x C_0 \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1}\} \\ &\vdots \\ &\cup \{T_x C_r \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1}\}.\end{aligned}$$

Again, we omit the B^r-term if $r = (n-1)$, and the C_t-terms if $r = -1$. Comparing with the definition of the filtration, we see that $(F_0)_r$ has basis given by the union of the first two of these sets, and in general that $(F_p)_r$ has basis given by the union of the first $(p+1)$ of the sets, so that $(F_p/F_{p-1})_r$ has basis given by the $(p+1)$st set alone, again with the necessary omissions when $r = -1, (n-1)$. This completes the proof. □

Proof of Theorem 5.4. The domain of Φ in degree r is

$$\begin{aligned} C(\mathcal{D}(n-1))_r &= \mathcal{D}(n-1)_r \oplus \mathcal{D}(n-1)_{r-1} \\ &= (\mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-r-2}} \mathbb{1}) \oplus (\mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-r-1}} \mathbb{1}) \end{aligned}$$

and therefore has basis

$$\{T_x \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-2}})^{-1}\} \cup \{T_x \otimes 1 \mid x \in (X^{S_{n-1}}_{S_{n-r-1}})^{-1}\},$$

with the first term omitted when $r = (n-1)$ and the second omitted when $r = -1$. Moreover, observing the definition of Φ, we see that, up to scaling by powers of q, Φ restricts to a bijection between this basis and the basis of $(F_0)_r$ given in Lemma 8.5, so that Φ is an isomorphism.

Similarly, the domain of Ψ in degree $(p+1)+r$ is

$$\begin{aligned} \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \Sigma^{p+1}\mathcal{D}(n-p-1)_{(p+1)+r} &= \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} \mathcal{D}(n-p-1)_r \\ &= \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-1}} (\mathcal{H}_{n-p-1} \otimes_{\mathcal{H}_{n-p-r-2}} \mathbb{1}) \\ &\cong \mathcal{H}_{n-1} \otimes_{\mathcal{H}_{n-p-r-2}} \mathbb{1} \end{aligned}$$

and therefore has basis

$$\{T_x \otimes (1 \otimes 1) \mid x \in (X^{S_{n-1}}_{S_{n-p-r-2}})^{-1}\}$$

Observing the definition of Ψ in this degree, we see that Ψ induces a bijection between this basis, and the basis of $(F_p/F_{p-1})_r$, and therefore Ψ is an isomorphism. □

9. Obtaining the spectral sequence

Tor groups. Induced maps. Change of ring maps. Induced maps in the case of including a flat thing.

Pick a projective resolution P_* of $\mathbb{1}$ as an $\mathcal{H}_n$-module, and treat it as a resolution of $\mathbb{1}$ as an $\mathcal{H}_{n-s-1}$-module for all $s = -1, \dots, n-1$.

Proposition 9.1. *There is a homological spectral sequence $\{E^r\}_{r\geqslant 1}$ with the following properties:*

- $E^1_{s,t}$ *is concentrated in horizontal degrees $s \geqslant -1$.*
- $E^1_{s,t} = \operatorname{Tor}_t^{\mathcal{H}_{n-s-1}}(\mathbb{1},\mathbb{1})$
- $d^1\colon E^1_{s,t} \to E^1_{s-1,t}$ *is the stabilisation map when s is even, and vanishes when s is odd.*
- $E^\infty_{s,t} = 0$ *in total degrees $s+t \leqslant (n-2)$.*

Similarly, there is a cohomological spectral sequence $\{E^r\}_{r\geqslant 1}$ with the following properties:

- $E_1^{s,t}$ *is concentrated in horizontal degrees $s \geqslant -1$.*
- $E_1^{s,t} = \operatorname{Ext}^t_{\mathcal{H}_{n-s-1}}(\mathbb{1},\mathbb{1})$
- $d_1\colon E_1^{s,t} \to E_1^{s-1,t}$ *is the stabilisation map when s is even, and vanishes when s is odd.*
- $E_\infty^{s,t} = 0$ *in total degrees $s+t \leqslant (n-2)$.*

Lemma 9.2. *There is a homological spectral sequence $\{{}^{II}E^r\}$ with the following properties:*

- ${}^{II}E^1_{s,t}$ *is concentrated in horizontal degrees $s \geqslant -1$.*
- ${}^{II}E^1_{s,t} = \operatorname{Tor}_t^{\mathcal{H}_n}(\mathbb{1},\mathcal{D}(n)_s)$
- $d^1\colon {}^{II}E^1_{s,t} \to {}^{II}E^1_{s-1,t}$ *is induced by $\partial^s\colon \mathcal{D}(n)_s \to \mathcal{D}(n)_{s-1}$.*
- ${}^{II}E^\infty_{s,t} = 0$ *in total degrees $s+t \leqslant (n-2)$.*

Similarly, there is a cohomological spectral sequence $\{{}^{II}E_r\}$ with the following properties:

- ${}^{II}E_1^{s,t}$ *is concentrated in horizontal degrees $s \geqslant -1$.*
- ${}^{II}E_1^{s,t} = \operatorname{Ext}^t_{\mathcal{H}_n}(D(n)_s,\mathbb{1})$
- $d^1\colon {}^{II}E_1^{s-1,t} \to {}^{II}E_1^{s,t}$ *is induced by $\partial^s\colon \mathcal{D}(n)_s \to \mathcal{D}(n)_{s-1}$.*
- ${}^{II}E_\infty^{s,t} = 0$ *in total degrees $s+t \leqslant (n-2)$.*

Proof. We prove the homological version first. Consider the (homological) double complex $P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_*$. This double complex gives two spectral sequences, $\{{}^{I}E^r\}$ and $\{{}^{II}E^r\}$, obtained by filtering the totalization by rows or columns. In our case, the first spectral sequence has E^1 term

$$ {}^{I}E^1_{s,t} = H_t(P_s \otimes_{\mathcal{H}_n} \mathcal{D}(n)_*) $$

with $d^1\colon {}^IE^1_{s,t} \to {}^IE^1_{s-1,t}$ induced by the differential $P_s \to P_{s-1}$. The second spectral sequence has E^1 term

$$^{II}E^1_{s,t} = H_t(P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_s) = \mathrm{Tor}_t^{\mathcal{H}_n}(\mathbb{1}, \mathcal{D}(n)_s)$$

and differential $d^1\colon {}^{II}E^1_{s,t} \to {}^{II}E^1_{s-1,t}$ induced by $\partial^s\colon \mathcal{D}(n)_s \to \mathcal{D}(n)_{s-1}$. Both spectral sequences converge to the homology of the total complex $\mathrm{Tot}(P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_*)$. See section 5.6 of [Wei94] for details.

The E^1-term of $\{{}^IE^r\}$ can be identified using the fact that P_s is projective, so that the functor $(P_s \otimes_{\mathcal{H}_n} -)$ commutes with homology, giving us

$$^IE^1_{s,t} = H_t(P_s \otimes_{\mathcal{H}_n} \mathcal{D}(n)_*) \cong P_s \otimes_{\mathcal{H}_n} H_t(\mathcal{D}(n)_*).$$

But by Theorem 5.5, the right-hand-side vanishes for $t \leqslant (n-2)$. In particular, ${}^IE^1_{*,*}$ vanishes in total degrees $\leqslant (n-2)$. The same therefore holds for all subsequent pages of the spectral sequence, so that $H_*(\mathrm{Tot}(P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_*))$ vanishes in degrees $* \leqslant (n-2)$. Since $\{{}^{II}E^r_{s,t}\}$ also converges to $H_*(\mathrm{Tot}(P_* \otimes_{\mathcal{H}_n} \mathcal{D}(n)_*))$, we obtain the conclusion.

For the second case, we consider instead the (cohomological) double complex $\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{D}(n)_*, I^*)$. One obtains analogous spectral sequences $\{{}^IE_r\}$ and $\{{}^{II}E_r\}$, which are analysed in the same way as before. In the analysis of $\{{}^IE_r\}$ one uses the fact that I^* is injective and therefore $\mathrm{Hom}_{\mathcal{H}_n}(-, I^s)$ commutes with homology to show that

$$^IE_1^{s,t} = H^t(\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{D}(n)_*, I^s)) \cong \mathrm{Hom}_{\mathcal{H}_n}(H_t(\mathcal{D}(n)), I^s).$$

$\square$

Having obtained the spectral sequences $\{{}^{II}E^r\}$ and $\{{}^{II}E_r\}$, we now proceed to turn them into the ones required by Proposition 9.1. Recall that

$$\begin{aligned} {}^{II}E^1_{s,t} &= \mathrm{Tor}_t^{\mathcal{H}_n}(\mathbb{1}, \mathcal{D}(n)_s) = \mathrm{Tor}_t^{\mathcal{H}_n}(\mathbb{1}, \mathcal{H}_n \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}), \\ {}^{II}E_1^{s,t} &= \mathrm{Ext}^t_{\mathcal{H}_n}(\mathcal{D}(n)_s, \mathbb{1}) = \mathrm{Ext}^t_{\mathcal{H}_n}(\mathcal{H}_n \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}, \mathbb{1}). \end{aligned}$$

Recall from Proposition 2.7 that $\mathcal{H}_n$ is free as a right $\mathcal{H}_{n-s-1}$-module, so that in particular $\mathcal{H}_n$ is flat as a right $\mathcal{H}_{n-s-1}$-module, and there is therefore a change-of-rings isomorphisms

$$\Xi_*\colon\ \mathrm{Tor}_t^{\mathcal{H}_{n-s-1}}(\mathbb{1}, \mathbb{1}) \xrightarrow{\cong} \mathrm{Tor}_t^{\mathcal{H}_n}(\mathbb{1}, \mathcal{H}_n \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}) = \mathrm{Tor}_t^{\mathcal{H}_n}(\mathbb{1}, \mathcal{D}(n)_s) = E^1_{s,t}.$$

given on the level of chain complexes by the isomorphism

$$\Xi\colon P_* \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1} \xrightarrow{\cong} P_* \otimes_{\mathcal{H}_n} (\mathcal{H}_n \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}), \qquad \Xi(p \otimes 1) = p \otimes (1 \otimes 1),$$

with inverse $\Xi^{-1}(p \otimes (h \otimes 1)) = ph \otimes 1$. And a change-of-rings isomorphism

$$\Xi^*\colon\ \mathrm{Ext}^t_{\mathcal{H}_{n-s-1}}(\mathbb{1}, \mathbb{1}) \xrightarrow{\cong} \mathrm{Ext}^t_{\mathcal{H}_n}(\mathcal{H}_n \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}, \mathbb{1}) = \mathrm{Ext}^t_{\mathcal{H}_n}(\mathcal{D}(n)_s, \mathbb{1}) = E_1^{s,t}.$$

given on the level of chain complexes by the isomorphism

$$\Xi\colon\ \mathrm{Hom}_{\mathcal{H}_{n-s-1}}(\mathbb{1}, I^*) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{H}_n}(\mathcal{H}_n \otimes_{\mathcal{H}_{n-s-1}} \mathbb{1}, I^*) \qquad \Xi(f)(h \otimes 1) = h \cdot f(1)$$

with inverse $\Xi^{-1}(g)(1) = g(1\otimes 1)$.

We define $\{E^r\}$ to be simply the spectral sequence $\{{}^{II}E^r\}$, but with the E^1-term modified by replacing ${}^{II}E^r_{s,t} = \mathrm{Tor}^{\mathcal{H}_n}_t(\mathbb{1}, \mathcal{D}(n)_s)$ with $\mathrm{Tor}^{\mathcal{H}_{n-s-1}}_t(\mathbb{1},\mathbb{1})$ using the map Ξ_*, and then taking the induced differentials. And we define $\{E_r\}$ to be $\{{}^{II}E_r\}$ but with E_1-term modified by replacing ${}^{II}E^{s,t}_r = \mathrm{Ext}^t_{\mathcal{H}_n}(\mathcal{D}(n)_s,\mathbb{1})$ with $\mathrm{Ext}^t_{\mathcal{H}_{n-s-1}}(\mathbb{1},\mathbb{1})$ using the map Ξ^*, and again taking the induced differentials. Then $\{E^r\}$ and $\{E_r\}$ have all the properties required by Proposition 9.1, except for the description of the differentials.

Lemma 9.3. *The composites*

$$\Xi_*^{-1}\circ d^1\circ \Xi_*\colon\ \mathrm{Tor}^{\mathcal{H}_{n-s-1}}_*(\mathbb{1},\mathbb{1}) \longrightarrow \mathrm{Tor}^{\mathcal{H}_{n-s}}_*(\mathbb{1},\mathbb{1}),$$
$$\Xi^{*-1}\circ d_1\circ \Xi^*\colon\ \mathrm{Ext}^*_{\mathcal{H}_{n-s}}(\mathbb{1},\mathbb{1}) \longrightarrow \mathrm{Ext}^*_{\mathcal{H}_{n-s-1}}(\mathbb{1},\mathbb{1})$$

vanish when s is odd, and are given by the relevant stabilisation map when s is even.

Proof. Recall that d^1 is induced by the differential of $\mathcal{D}(n)$, so that it is given on the level of chains by the map

$$\mathrm{id}\otimes\partial^r\colon P_*\otimes_{\mathcal{H}_n}\mathcal{D}(n)_s \longrightarrow P_*\otimes_{\mathcal{H}_n}\mathcal{D}(n)_{s-1},$$
$$p\otimes(h\otimes 1)\longmapsto \sum_{j=0}^{s}(-1)^j q^{-j}(p\otimes(hD^s_j\otimes 1)).$$

Thus $\Xi_*^{-1}\circ d^1\circ\Xi_*$ is given on the level of chains by the composite

$$P_*\otimes_{\mathcal{H}_{n-s-1}}\mathbb{1} \xrightarrow{\Xi} P_*\otimes_{\mathcal{H}_n}\mathcal{D}(n)_s \xrightarrow{\mathrm{id}\otimes\partial_s} P_*\otimes_{\mathcal{H}_n}\mathcal{D}(n)_{s-1} \xrightarrow{\Xi^{-1}} P_*\otimes_{\mathcal{H}_{n-s}}\mathbb{1},$$
$$p\otimes 1\mapsto p\otimes(1\otimes 1)\mapsto \sum_{j=0}^{s}(-1)^jq^{-j}(p\otimes(D^s_j\otimes 1))\mapsto \sum_{j=0}^{s}(-1)^jq^{-j}(pD^s_j\otimes 1).$$

By Lemma 9.4 below, this composite is chain homotopic to the map

$$P_*\otimes_{\mathcal{H}_{n-s-1}}\mathbb{1}\longrightarrow P_*\otimes_{\mathcal{H}_{n-s}}\mathbb{1}$$
$$p\otimes 1\mapsto \sum_{j=0}^{s}(-1)^jq^{-j}q^j(p\otimes 1) = \sum_{j=0}^{s}(-1)^j(p\otimes 1) = \begin{cases} p\otimes 1 & s \text{ even}\\ 0 & s\text{ odd}\end{cases}$$

and the result follows in the homological case. In the cohomological case the proof is similar, and we leave the details to the reader. □

Lemma 9.4. *The map $P_*\otimes_{\mathcal{H}_{n-s-1}}\mathbb{1}\to P_*\otimes_{\mathcal{H}_{n-s-1}}\mathbb{1}$, $p\otimes 1\mapsto pD^s_j\otimes 1$ is chain homotopic to the map given by multiplication by q^j. Consequently, the map $P_*\otimes_{\mathcal{H}_{n-s-1}}\mathbb{1}\to P_*\otimes_{\mathcal{H}_{n-s}}\mathbb{1}$, $p\otimes 1\mapsto pD^s_j\otimes 1$ is chain homotopic to the reduction map $P_*\otimes_{\mathcal{H}_{n-s-1}}\mathbb{1}\to P_*\otimes_{\mathcal{H}_{n-s}}\mathbb{1}$ multiplied by q^j.*

Analogously, the map $\mathrm{Hom}_{\mathcal{H}_{n-s-1}}(\mathbb{1},I^*)\to \mathrm{Hom}_{\mathcal{H}_{n-s-1}}(\mathbb{1},I^*)$, $f\mapsto(1\mapsto D^s_j\cdot f(1))$ is chain homotopic to the map given by multiplication by q^j. Consequently, *the map* $\mathrm{Hom}_{\mathcal{H}_{n-s}}(\mathbb{1}, I^*) \to \mathrm{Hom}_{\mathcal{H}_{n-s-1}}(\mathbb{1}, I^*)$*,* $f \mapsto (1 \mapsto D_j^s \cdot f(1))$ *is chain homotopic to the restriction map* $\mathrm{Hom}_{\mathcal{H}_{n-s}}(\mathbb{1}, I^*) \to \mathrm{Hom}_{\mathcal{H}_{n-s-1}}(\mathbb{1}, I^*)$ *multiplied by* q^j*.*

Proof. Let us begin with the homological case. The next paragraph will show that right-multiplication by D_j^s on P_* is a map of $\mathcal{H}_{n-s-1}$-modules, and that its effect on homology is multiplication by q^j. Another chain map with the same properties is multiplication by q^j. But since P_* is a projective resolution by $\mathcal{H}_{n-s-1}$-modules, these two maps are chain homotopic.

To see that right-multiplication on P_* by D_j^s is a map of $\mathcal{H}_{n-s-1}$-modules, recall from the proof of Lemma 4.4 that D_j^s commutes with $\mathcal{H}_{n-s-1}$. The effect of the map on homology is the map $\mathbb{1} \to \mathbb{1}$ that is again given by right multiplication by D_j^s, and since D_j^s is a product of j factors T_k, this is multiplication by q^j.

The proof in the cohomological case is similar, left-multiplication by D_j^s on I^* is a map of $\mathcal{H}_{n-s-1}$-modules given on cohomology by multiplication by q^j, and since I^* an injective resolution by $\mathcal{H}_{n-s-1}$-modules, this map is chain homotopic to multiplication by q^j. □

10. The spectral sequence argument

We are now able to prove Theorem 1.1, in the homological case. The following argument is essentially what appears in section 5.2 of [RW13], or in the proof of Theorem 2 of [Ker05], except for changes in indexing and notation.

We prove that $\mathrm{Tor}_d^{\mathcal{H}_{n-1}}(\mathbb{1}, \mathbb{1}) \to \mathrm{Tor}_d^{\mathcal{H}_n}(\mathbb{1}, \mathbb{1})$ is an isomorphism in degrees d satisfying $2d \leqslant n-1$. We do this by induction on n. The cases $n = 1$ and $n = 2$ only make a statement about degree $d = 0$ and therefore hold trivially.

Suppose now that $n \geqslant 3$ and that the induction hypothesis holds for all smaller values of n. In the spectral sequence $\{E^r\}_{r \geqslant 1}$ of Proposition 9.1, we recall that, the differential

$$d^1 \colon E^1_{s,t} \to E^1_{s-1,t}$$

is the stabilisation map

$$\mathrm{Tor}_t^{\mathcal{H}_{n-s-1}}(\mathbb{1}, \mathbb{1}) \to \mathrm{Tor}_t^{\mathcal{H}_{n-s}}(\mathbb{1}, \mathbb{1})$$

when s is even, and vanishes when s is odd. In particular, our aim is to show that the maps $d^1 \colon E^1_{0,t} \to E^1_{-1,t}$ are isomorphisms for $2t \leqslant n-1$, or in other words that $E^2_{0,t} = 0$ and $E^2_{-1,t} = 0$ for $2t \leqslant n-1$.

Now let $u \geqslant 1$ and consider the differential

$$d^1 \colon E^1_{2u,t} \to E^1_{2u-1,t}.$$

Since this is the stabilisation map, our induction hypothesis states that it is an isomorphism for $2t \leqslant n - 2u - 1$. This gives the first property below. The second property follows easily from it.

(1) For $r \geqslant 2$, $E^r_{*,*}$ vanishes in bidegrees $(2u, t)$ and $(2u-1, t)$ for $u \geqslant 1$, $2t \leqslant n - 2u - 1$.
(2) For $r \geqslant 2$, $E^r_{*,*}$ vanishes in bidegrees (s, t) satisfying $2t \leqslant n - s - 2$ and $s \geqslant 1$.

We now claim that for $r \geqslant 2$ there are no differentials d^r affecting terms in bidegrees $(-1, t)$ and $(0, t)$ for $2t \leqslant n - 1$. In the case of bidegrees $(-1, t)$, observe that a d^r landing there must originate in bidegree $(-1 + r, t - r + 1)$, but that $E^r_{-1+r,t-r+1} = 0$ by property (2) above. In the case of bidegrees $(0, t)$ and $r \geqslant 3$, the same reasoning applies. In the case of bidegrees $(0, t)$ and $r = 2$, the differential d^2 landing there must originate in $(2, t - 1)$, which is $(2u, t - 1)$ for $u = 1$, and $E^2_{2u,t-1} = 0$ by property (1) above.

It follows that if $2t \leqslant n - 1$ then $E^\infty_{-1,t} = E^2_{-1,t}$ and $E^\infty_{0,t} = E^2_{0,t}$. These terms lie in total degrees d satisfying $d \leqslant (n-2)$ (this requires our assumption that $n \geqslant 3$). But by Proposition 9.1 we know that E^∞ vanishes in these total degrees, so that these terms vanish, and this completes the proof in the homological case.

The proof in the homological case is entirely similar.

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