

### Ideal gas meteorology

In this supplementary note we'll see how to find three very useful numbers used in meteorology, namely the **rate of change of pressure with height**, the **dry adiabatic lapse rate** and the **depression of the dew point with height**. These numbers are found using first-year maths and the ideal gas law that many science students meet in other first-year courses besides this one. A succeeding section shows that the statement in the lectures and textbook that the relative humidity is determined by the difference between the actual temperature and the dew point is true. From these ideas it is deduced that the relative humidity inside in winter is generally less than it is in summer.

The meteorology that we have been doing in this course sticks close to first-principles. There's nothing wrong with this approach, in fact there's everything right about it. In any science the first-principles of the subject are the bedrock upon which it's based. If you get stuck answering a question, then go back to first-principles and work up from there. It's a bit of a sweeping statement but meteorology books tend to be either fairly descriptive, with key numbers 'pulled out of the air', or a mass of mathematics with lots of vector analysis and calculus. The maths is certainly needed in modern meteorology, for computational forecasting and climate analysis is an implementation of advanced mathematical analysis.

The next step up in mathematical analysis from our course is to introduce more advanced concepts such as 'divergence' and 'vorticity', 'geopotential' and the 'hypsometric equation' and so on. This isn't for us but I thought that some in the class might be interested to see that you can get an idea where a few numbers we have 'pulled out of the air' come from by treating the (dry) atmosphere as an ideal gas.

The ideal gas law says that for a volume  $V$  of air all under the same conditions of pressure  $P$  and temperature  $T$ :

$$PV = nRT,$$

where  $n$  is the number of moles of air in the volume and  $R$  is the so-called 'gas constant' whose value is  $8.3145 \text{ J mole}^{-1} \text{ K}^{-1}$ . The same relationship is more useful in meteorology when written in terms of pressure, density and temperature, remembering the definition of density  $\rho$ :

$$P = \rho RT \quad (1)$$

where in this form  $R$  has the value  $287 \text{ J kg}^{-1} \text{ K}^{-1}$  ( $\equiv \text{m}^2 \text{ s}^{-2} \text{ K}^{-1}$ ).

#### *Pressure reduction with height*

In the lecture we sketched the argument that the atmospheric pressure varied with height  $h$  exponentially according to the relationship:

$$P = P_0 e^{-h/H} \quad (2)$$

Where  $H$  is called the 'scale height'. The following few lines give the derivation a bit more substance. The change in pressure  $dP$  over a small height  $dh$  is equal to the weight of air in a slab of cross-section  $1 \text{ m}^2$  and thickness  $dh$ ; i.e.

$$\begin{aligned} dP &= -g\rho dh \\ &= -gPdh/RT \text{ (using the gas law eq. (1))} \end{aligned}$$

Hence  $dP/P = -gdh/RT$ .  $R$  has the value in eq. (1). The exponential change of eq. (2) is equivalent to  $dP/P = -dh/H$  and therefore by comparing the two expressions for  $dP/P$  the scale height  $H$  is given by  $RT/g$ .

For  $T = 273$  K and  $g = 9.81$  m s<sup>-2</sup> and the value of  $R$  in eq. (1) above this gives  $H$  as  $7.99 \times 10^3$  m, i.e. 8 km, as quoted in the textbooks. The scale height  $H$  gets smaller higher up in the atmosphere, reducing to nearer 6 km at the tropopause where  $T$  is closer to 210 K.

We know now how pressure decreases with height but the ideal gas equation doesn't tell us by itself how the temperature decreases with height. All it says from (1) is that the product of  $\rho T$  decreases as  $P$  decreases, namely approximately exponentially. (The decrease would be exactly exponential if  $H$  were a true constant). We know, for example, that the temperature can actually increase with height in some parts of the atmosphere due to absorption of energy from the Sun by components of the atmosphere, and hence all we can deduce is that the density must decrease very quickly since the product of density and temperature has to follow the decrease of pressure with height.

#### *Adiabatic packets*

In the troposphere, the sphere of mixing, packets of air can move up and down. Although the pressure of the packet adjusts to the outside pressure very quickly, there is little time for heat to flow into or out of a packet and under these circumstances the pressure, temperature and volume change 'adiabatically'. Thermodynamics texts tell us that adiabatic changes are subject to the relationship:

$$PV^\gamma = \text{const} \quad , \quad (3)$$

where  $\gamma$  is a number whose value is close to 1.4 for the atmosphere. For compatibility with the gas law written in terms of density, let's apply this to a volume whose mass is 1 kg. For this mass,  $\rho = 1/V$ . We can now use the adiabatic condition and the gas law along with the known pressure change relationship with height to find out how the temperature changes with height. Eq. (3) becomes  $P/\rho^\gamma = \text{const}$  and when combined with the gas law from eq. (1):

$$\begin{aligned} P &= \text{const} \times (P/RT)^\gamma \\ \text{i.e. } T^\gamma &= \text{const} \times P^{\gamma-1} \\ \text{or } T &= \text{const} \times P^{(\gamma-1)/\gamma} = T_0 \times e^{-(\gamma-1)h/\gamma H} \end{aligned}$$

The constant changes (written as 'const2', 'const3' and finally ' $T_0$ ') as various powers of  $R$  are included but we shan't need its value. The final line gives the expression for the temperature change with height. The rate of change of temperature with height is the negative of the *lapse rate*, namely  $-dT/dh$  and differentiating the final line gives:

$$\text{lapse rate} = -dT/dh = ((\gamma-1)/\gamma H)T = (\gamma-1)g/\gamma R .$$

Hence the adiabatic lapse rate must be  $(0.4 \times 9.81) / (1.4 \times 287) = 9.77 \times 10^{-3}$  degrees  $\text{m}^{-1}$  or close to  $10^\circ \text{C km}^{-1}$ . This is just the textbook value.

The logic above confirms what was said in the lecture, namely that the adiabatic condition and the gas law, along with the relation between pressure change and height, does indeed give the dry adiabatic lapse rate, and its value is as described.

If you have studied the basic chemistry or physics of an ideal gas, you'll recognize  $(\gamma-1)/\gamma R$  as  $1/C_p$ , the reciprocal of the molar specific heat at constant pressure. Hence *lapse rate* =  $g/C_p$ . You can rightly conclude that the lapse rates on Venus and Mars are not quite the same as on Earth, since  $C_p$  for  $\text{CO}_2$  is less than  $C_p$  for air and  $g$  is different too. The Venus lapse rate is  $10.5^\circ \text{C km}^{-1}$  and Mars  $4.4^\circ \text{C km}^{-1}$ .

### *Depression of the dew point with height*

When discussing convective cloud base heights, a figure was 'pulled out of the air' that the dew point decreased with height because of the reduction of pressure with height. The underlying idea is that as a packet expands upon rising, the water vapour pressure within it decreases and hence the temperature at which it will reach the dew point must decrease. A number for this decrease in dew point can be found from the relationship between water vapour pressure and temperature.

You can find some quite complicated formulae that give the water vapour pressure  $P_v$  as a function of temperature  $T$  (in degrees Celsius) but a simple one that fits the textbook data pretty well is:

$$P_v = 6 \times 2^{T/10} . \quad (4)$$

$T$  is the dew point for water vapour at pressure  $P_v$ . Hence the rate of change of water vapour pressure with temperature is:

$$dP_v/dT = \ln 2 \times P_v / 10 .$$

i.e. the change in dew point  $dT$  depends on the fractional change in pressure with height:

$$dT = 10(dP_v/P_v)/\ln 2 .$$

We know that the fractional change of pressure  $dP/P$  in a rising packet is  $-dh/H$  ( $= -dh/8 \text{ km}^{-1}$ ) and hence the *change in dew point with height* is  $dT/dh = -10/(8 \times \ln 2) = -1.8^\circ \text{C km}^{-1}$ . The textbook rounds this to  $-2^\circ \text{C km}^{-1}$  for convenience.

The sections above show how three key numbers in the meteorology course come from constants that occur in physics in general. With this insight you could, for example, find out how the quantities might be different on other worlds.

*The difference between the temperature and the dew point determines the relative humidity*

The bigger this difference, the smaller the relative humidity; the smaller this difference, the bigger the relative humidity. This follows from the expression (4) for saturated vapour pressure given above.

Suppose the measured temperature is  $T$  and the dew point  $T_d$ . The relative humidity  $R$  is defined as:

$$R = \frac{P_{\text{water vap}}}{P_{\text{sat}}} \times 100\% . \quad (5)$$

Now  $P_{\text{water vap}} = 6 \times 2^{T_d/10}$  and  $P_{\text{sat}} = 6 \times 2^{T/10}$  and hence the relative humidity  $R$  is given by

$$R = \frac{6 \times 2^{T_d/10}}{6 \times 2^{T/10}} \times 100\% = 2^{(T_d - T)/10} \times 100\%$$

Hence  $\ln(R) = \ln(100) + \ln(2) \times (T_d - T)/10 = 4.605 - 0.0693 \times (T - T_d)$ .

This verifies the statement. The larger the value of  $(T - T_d)$  the smaller is the logarithm of the relative humidity and hence the smaller is the relative humidity. What determines relative humidity is just the difference between  $T$  and  $T_d$ , not the absolute values of either of the temperatures.

*The effect of heating a packet of air is approximately to halve the relative humidity for every increase in temperature of 10°C*

This follows from the relationships given above. The biggest effect of heating a packet of air is to increase the pressure at which water vapour is saturated, i.e. to change  $P_{\text{sat}}$ . For modest changes in temperature, the pressure of the water vapour in the air doesn't change much and hence the dew point  $T_d$  stays about the same. Hence in the definition of relative humidity  $R$  given in (5) above,  $P_{\text{sat}}$  is the quantity that changes significantly with temperature and from (4),  $P_{\text{sat}}$  doubles for every 10°C rise in temperature. Hence the relative humidity  $R$  halves for every 10°C rise in temperature.

If you prefer to see the formula, then if  $R_1$  is the relative humidity at temperature  $T_1$  and  $R_2$  the relative humidity at temperature  $T_2$  then from the assumptions and relationships above:

$$R_2 = R_1 \times 2^{(T_1 - T_2)/10} .$$

For example, this lets you estimate the effect of heating air in the house as it comes in from outside. If the outside temperature is 6°C with humidity 85% and the air is heated to 20°C then its humidity will naturally fall to about  $85 \times 2^{-14/10} = 32\%$ . Of course the presence in the house of people breathing out water vapour, washing, drinking and any cooking will increase the humidity but in our climate the air in a reasonably heated house is often less humid in winter than in summer, because of this very effect. In Aberdeen, we need to heat our houses for most of the year and hence for most of the year the air inside is less humid than the air outside.

JSR