Physics of Cycling

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This supplementary note is certainly a digression from meteorology but my excuse for including it is not so much that wind and weather affect cycling, which they do, but that more and more people are using a bike these days as a 'buy-in' to the global warming issue and the physics of pedalling a bike along the road can lead to some insights. Personally, I use a bike more than I have done for many years, partly because bikes have got better, partly because I can afford a better bike than I used to own (my previous one having been stolen years ago from within a University building when left unlocked, so take care of yours) and also because a good bike is now seen as something it's worth spending a bit of money on.

The following pages are in the form of a series of numerical questions with answers derived from simple physics. I didn't sit down and write them one evening. They are the accumulation of thoughts that have occurred to me while cycling the bye-ways of Kincardine and Mearns that are conveniently accessible from my home. They are mainly about going forward in a straight line, not about balancing, cornering, efficient pedalling or other topics of interest to the cyclist, though some of these come up in appendices. The notes are long enough as it is and you probably won't read them all. At least you can look at the numbers that come out of the sums. If you're very keen you can substitute numbers that you think better represent your own case and rework the calculations. I claim no formal knowledge of sports science applied to cycling so the sums are just the kind of estimate a physicist might make when faced with a real-world situation. Some aspects of the physics of cycling aren't at all obvious. The bike moves forward yet the part in contact with the road is stationary. Even more odd is that the force moving the bike forward is the force provided by the stationary road on this stationary part of the bike. A few of the conclusions from the physics that come out from the numbers are:

- In most circumstances the hill is mightier than the wind in determining your speed. For example, from the figures given later I can easily free-wheel down a 2% hill as fast as I travel on the flat.
- That said though, a strong wind is as good as a steep hill in slowing you down.
- Wind has the biggest effect on speed on the flat; it has less effect on hills.
- On the flat, a wind behind you will increase your speed a bit more than the same wind ahead decreases your speed. In spite of this, the time lost over a given distance with the wind against exceeds the time gained over the same distance by having the wind behind.
- Being able to generate more power has a much bigger effect on the speed you can travel up a hill than it does on the flat.
- If you want to work at a constant rate, which is good policy on a long trip, use your gears going uphill so that you pedal with the same force at the same cadence (turns per minute). Your speed will drop in proportion to the change in gear ratio. Don't be tempted to pedal a lot harder; that's a recipe for overheating.

Since writing the above, this piece has expanded both by adding additional scenarios and by adding three appendices. The first is an *Appendix on forces and motion*, the second a short *Appendix on the difference between cycling and running* and the third a longer *Appendix on why does a bicycle stay upright when moving?* Obviously skip to near the end to read any of these.

1) *I can travel at 14 mph along the flat on my bicycle, half this speed going up a hill and twice this speed coming down. My route takes me up a hill 1 mile long and down the other side, which is of equal distance. Calculate my average speed over this hill and explain why it is not the same as the speed I travel on the flat. It turns out that working at a constant rate on a round trip over a course that isn't flat (with no wind), my average speed will always be less than the speed I can make on the flat.*

This introductory question is in 'everyday' units of miles and mph. The rest of this piece is in metric units. The time taken going 1 mile uphill at a speed of 7 mph is 1/7 h; time going downhill at 28 mph (pretty fast) is 1//28 h. Hence total time for 2 miles is 5/28 h. Hence average speed over the 2 miles is $2\times28/5 = 11.2$ mph. This is not the same as the speed on the flat because most of the time in the journey is spent going up-hill and averages are calculated over the time spent. In fact it takes as long to reach the top of the hill as it would to travel 2 miles on the flat and hence no matter how fast I go downhill, the combined uphill and down-hill journey will always take longer than an equal distance on the flat. A variation of this theme is to consider the effect of wind. If the wind is against me for half an hour reducing my speed to 10 mph (over a 5 mile stretch) how far will I have to travel at 18 mph with the wind behind me so my average speed is the 14 mph I achieve on the flat? Clearly I'll need to travel for half-an-hour at the faster speed (since average speed is computed over time), which means I'll need to travel for 9 miles with the wind behind me to restore my average.

Another conclusion from such thoughts about average speeds is that if you look at your cyclometer and see that you are travelling at less than your average speed, don't think your average speed will necessarily drop. One spends longer on a journey travelling at less than the average speed than one spends travelling faster than the average speed. For the rest of the questions I've put speed in $m\ s^{-1}$.

My proof of the statement about a round trip uses ideas from a few pages on and takes a page, so I shan't include it here. It's true, though. The same applies to any journey that starts and finishes at the same height, even if not at the same place. If I always work at a constant rate, I'll make the fastest journey if the route is flat. This is true for long or short journeys. The same is true on a velodrome; even more so, in fact, since going up the banking and down introduces extra distance as well.

2) *Cycling into a steady wind feels like cycling uphill. If I'm cycling along the flat on a calm day at 7 $m\ s^{-1}$ (15.66 mph), what is my speed relative to the air? The drag I experience at this speed is 20 N (Newtons). If a headwind of 5 $m\ s^{-1}$ now picks up but I'm able to continue at the same speed, what is my speed relative to the air? The drag has now increased to 60 N. How much extra power must I expend to maintain the speed of 7 $m\ s^{-1}$ into this headwind? If the mass of myself and my bicycle together is 100 kg, what gradient of hill is equivalent to the extra drag force? Take $g = 9.81\ m\ s^{-2}$.*

Initial speed relative to the air is 7 m s^{-1}. Speed relative to air when the headwind picks up is 12 m s^{-1}. Increase in drag is 40 N. A force of 40 N moving at 7 m s^{-1} corresponds to a power of $40\times7 = 280$ W (from the definition of power, namely force multiplied by speed). This is the **extra power** I need to exert to maintain my speed when the wind picks up. It's a lot of power and I, as an average cyclist, couldn't keep this up. The average cyclist isn't going to be able to keep up a steady speed on the flat of 12 m s^{-1} (43 km h^{-1}). Professional cyclists can maintain this speed by a combination of low-drag design of bike and clothes and high personal fitness.

Going uphill, the component of my weight W against me is $W\sin\theta$, where θ is the angle of the hill. Setting $W\sin\theta = 40$ N, with $W = 100\times9.81$ gives $\sin\theta = 0.0408$ and hence a gradient of about 1 in 25, since the gradient is $\tan\theta$, which is almost the same as $\sin\theta$ for the small gradient concerned. In fact air resistance in this case is equivalent to quite a steep hill. In practice I'm not going to maintain my speed into a 5 m s^{-1} wind, just as I can't go up a 1 in 25 hill at 7 m s^{-1}, for I have to work too hard to do so.

3) *On a calm day I cycle along a flat road at 5 m s^{-1} against an air resistance of 12 N. At what rate am I working? If one dietary calorie (usually written as Cal or kcal) equals 4.18×10^3 J and my body's efficiency at producing work from burning food is 20%, how many Cal am I using up per minute?*

Work done per second is $5\times12 = 60$ J (i.e. I'm working at 60 W). This represents 20% of the food energy I am burning up and hence I am burning $60/0.2 = 300$ J per second. In one minute this is equivalent to $60\times300 = 18000$ J $\equiv 18000/4180 = 4.3$ Cal.

The conditions above represent burning food at about 260 Cal per hour cycling at a rate of 18 km h^{-1}. Tour-de-France cyclists, who average over 40 km h^{-1} for at least 4 hours per day, typically burn up 5000 Cal a day, more in the mountains. Although you stop working when you stop pedalling, you don't stop burning calories, which is why one stays hot for quite a while after finishing a cycle run. Put another way, the Calories lost from exercising exceed the Calories used while exercising.

4) *The mass of myself and my bike equals 100 kg. If I maintain my speed while climbing a 1 in 40 hill, how many Cal am I now burning per minute? Take $g = 9.81$ m s^{-2}. At what rate am I generating heat?*

If θ is the angle of the hill, then $\tan\theta = 1/40$. The component of my weight that is now directed against me in addition to the air resistance of 12 N is $Mg\sin\theta = 100\times9.81/40 = 24.53$ N to a good approximation ($\sin\theta \approx \tan\theta$). Hence the total force that I am working against is $(24.53 + 12) = 36.53$ N. I am now burning food at a rate of $4.3\times36.53/12 = 13.1$ Cal min^{-1}. I am working at the high rate of $36.53\times5 = 183$ W which represents 20% of the energy I am burning. Hence I'm generating four times as much heat = 731 W. A young fit person could maintain this energy consumption. I couldn't. [Top grand tour sprint cyclists can generate power of over 1 kW for short periods. The fastest Olympic sprint cyclists can manage over 2 kW, allowing them to cover 200 m in less than 10 seconds].

Two facts come out of these numbers. First the work done against gravity is much more than the work done against air resistance, i.e. most of my effort goes against gravity. [A short stretch of my regular cycle route is a 1 in 7 hill. If I want to maintain a miserable 2.2 m s^{-2} (5 mph) up the hill then I have to work at a rate of $Mg\sin\theta v = 308$ W, ignoring air resistance. I can keep this up on the short stretch but tire noticeably and couldn't maintain this speed for a long hill.] Secondly, if you try to maintain your speed going up a hill, then it's very hot work, for you generate a lot more internal heat that has to be got rid of or you'll likely overheat. If you want to keep your speed up on a bike there's a good physical reason for wearing shorts and short sleeves, namely to make it easier to dissipate the heat you generate.

5) The following multi-part question is all you ever wanted to know about the speed I can maintain on my bike if I work at a steady rate!

When cycling my bike I can maintain a steady rate of working of 100 W. The air resistance I experience creates a force against me of $F_r = 0.5v^2$ where F_r is in Newtons and my speed through the air is v m s^{-1}. Friction within the bike can be ignored (as I did in the working above) and so can 'rolling resistance', the contribution to resistance caused by slippage of tyres, deformation of tyres, etc. usually quoted as a few Newtons. Take g as 9.81 m s^{-2}.

[The figures above are round figures but 100 W is not a lot of power to provide. Expending mechanical energy is just what we do climbing a hill and for comparison if I work at a steady 100 W I could raise myself and a rucksack of total mass 100 kg a height of 400 m in an hour and 5 minutes. This is a slightly simplistic comparison since it ignores the effort of traversing the horizontal distance involved.

Notice also that air resistance depends on the **square** of the speed relative to the air. It's this fact that makes a strong wind so much more noticeable on a bike than a moderate wind. It is this fact that makes it much harder to add a few miles per hour to my speed when I'm going fast compared with the effort of adding a few mph when I'm only going slowly.

It's hard to believe when you're on your bike pedalling along the flat that what limits your speed is just air: that flimsy, tenuous stuff that you hardly notice for most of the day as it glides past you imperceptibly as you walk. Waive your hand around as you read this and you'll just feel it. Can air possible be what stops you going any faster? The answer is '*yes*'.]

a) *What speed will I maintain on a flat road?*

At constant speed, the force (F) I provide propelling the bike forward must equal the air resistance F_r because there is no net force on the bike (Newton's first law). The rate of working done by F is Fv which is given as 100 W. Hence $100 = F_r v = 0.5v^2 \times v$. Therefore $v^3 = 200$ and hence $v = 5.85$ m s^{-1}. This is equivalent to 13 mph, a realistic speed.

If I want to go faster then I have only two choices: reduce my air resistance (the factor 0.5 in the expression for F_r) or increase the rate I expend energy (the power I generate, or rate of working). Race cyclists employ both methods. For instance, with the air resistance I offer, as given above, I'd need to expend 700 W of mechanical energy to travel at 25 mph (11.2 m s^{-1}). This is quite beyond me, and indeed almost anyone except in short bursts, and I'd need to reduce my air resistance a lot, one way or another, to have a hope of travelling at this speed on a bike on the flat with no wind assistance. Professional road-racing cyclists employ the strategy of cycling very close to the person in front to significantly reduce their air resistance factor. They can also generate more power. A top time-trial cyclist would be looking to generate a good 400 W of power for up to an hour.

Riding a bike with straight handlebars, I increase my air resistance by putting my hands at the ends of the handlebars, for this puts my lower arms out beyond the rest of the body, increasing my profile facing the wind. There's a bit less air resistance if I hold my hands near the centre of the handlebars, but of course I've less control and this isn't recommended for speeds in excess of 25 mph (40 km h^{-1}). Sitting bolt upright is another 'no, no' as far as air resistance is concerned unless you have the wind behind. You'll see that professional road-race cyclist have their seat so high that their backs are almost horizontal. While digressing on the topic of air resistance, I should add that air resistance is proportional to the density of the air and this in turn depends on the air pressure. So on days of low pressure, air resistance is less than during fine, sunny anticyclonic weather. The effect is even greater if the air is humid, since water molecules are less dense than either oxygen or nitrogen molecules. So if I'm timing myself over a fixed distance, I should get the best times when the pressure is low, other things being equal (which of course they may well not be).

This piece includes lots of digressions. Another one is to comment that there is commercial pressure to sell us the lightest bike. Light bikes are pretty expensive. On the flat the weight of the bike makes pretty well no difference to the speed you can maintain; what counts is its air resistance and that's determined by its shape and size. Going downhill, you'll go faster with a heavier bike, so no advantage there. The only advantage of a light bike is going uphill but if you and the bike have a mass of 100 kg, then a few kg difference in weight between a very expensive bike and a less expensive one is hardly going to be noticed. By much the biggest weight you're taking up a hill is yourself. Unless you are into racing, it's better to make sure the bike is strong enough to resist hitting a few potholes or falling over without being damaged than aiming for the lightest bike.

(For the racer who thinks in terms of seconds gained or lost, then weight does have some effect. Going up a hill steep enough that most of your work is against gravity, for a given power available the time taken increases in proportion to the weight of cyclist and bike. Going down a hill that is steep enough that there is little need to pedal, the time taken decreases as the square root of the weight. For example, for a 2% increase in weight, it takes 2% longer to go up a hill but only 1% shorter to come down it. Since the going up time is already longer than the coming down, there is a net increase in time with weight. Unless you've got the stopwatch running, it really isn't a big issue for everyday cycling.)

Returning to the question, using the simple relationship above, you can show that if I want to increase my speed by 1 m s^{-1} when I'm going at only 3 m s^{-1}, I need work harder by only 18.5 W. If I want to increase my speed by 1 m s^{-1} when I'm already travelling at 6 m s^{-1} then I need to work harder by 63.5 W. In other words, the faster I travel, the harder it is to go even faster. This is particularly relevant if I want to catch up with someone in front of me who is for the moment going at my speed. Suppose the person ahead is 100 m away (not very far). If I want to catch up in about a minute and a half (100s) then I need to increase my speed relative to the person ahead by 1 m s^{-1}. The figures just given show that if we are both travelling at just 3 m s^{-1} (6.7 mph), then I need only work at a modest extra 18.5 W but if we are both going at 6 m s^{-1} (13.4 mph) then I need to work at an extra 63.5 Watts, a stiff ask. This all applies when my speed is limited by the wind. The figures will be different if we are going up hill. As we'll see, when going up hill most resistance is provided by gravity.

Another factor to notice is that the speed I can attain on the flat depends on the cube root of the power I can provide. Suppose you are pretty fit and can sustain 200 W, twice what I can. Can you travel at twice my speed on the flat? No way. You could only travel at $2^{1/3}$ as fast, namely 1.26 times my speed or 7.37 m s^{-1}. Spending months in the gym increasing your power output is not going to pay nearly such big dividends as reducing your air resistance. Top cyclists need to spend time in a wind tunnel, reducing their drag.

If you have no idea how much power you use regularly on a bike, then measure the speed you can maintain comfortably on a flat road (zero gradient) on a calm day. Convert the speed to m s^{-1}. If the answer is 'v', the power you're expending is about $0.5v^3$. I say 'about' because 0.5 is only a guess at a reasonable air resistance factor. In fact, even for a leanly dressed cyclist about three-quarters of the air resistance comes from the cyclist and a quarter from the bike. 0.5 seems to work well for me in normal clothes but I am 1.88 m tall without cycling helmet. If you can maintain over 10 m s^{-1}, then your air resistance factor is likely to be no more than 0.3 and you must be pretty fit.

Using my data, cycling on the flat on a calm day I'll experience a wind against me of 5.85 m s^{-1} which may seem on paper to be the same as cycling into a wind of the same strength but in real life winds have gusts, lulls and turbulence so the experience of cycling into a real wind is different, noticeably so since the dependence of air resistance on the square of the wind-speed relative to me enhances the perceived effect of natural variations in the wind.

If you would like a fuller discussion of the forces acting on a bicycle, see the appendix to this piece.

b) *What force F must I supply when travelling on the flat at the speed of 5.85 m s^{-1} calculated above?*

Since 100 = F×5.85, then F = 17.1 N. This is the average force that must act on me and my bike to maintain my speed against the air resistance. It's not the force I push downward on the pedals. That force is much larger. The chain transmits my pedal force to a point on the rear wheel not far from the axle. This generates a much smaller force at the rim of the wheel acting on the road and by Newton's 3rd law the road provides the opposite force on the bike. It is this force that the road provides on the bike that is 17.1 N. Yes, it's the road that's pushing you along, courtesy of the friction between road and tyres. Reduce that friction by coating the road with ice, spilt oil, loose sand or plenty of water and the road will no longer provide the required force to move you along or hold you when you turn a corner. The part of the wheel in contact with the road is instantaneously at rest and you can ponder that the reason the bike moves forward at speed is that the road (at rest) applies a force to a point on the bike that's at rest, which sounds crazy. That's physics for you!

Perhaps it's more obvious that the road is pushing you along if you think about braking. Imagine cycling along at say 30 km per hour and then going over a patch of ice or oil. Jam on the brakes and the wheels will stop quickly because the brakes exert their force on the wheels but the wheels can't exert a horizontal force on the road any more because of lack of friction so the road can't exert its reaction back on the tyres and the bike won't stop. Normally, the road stops the bike when you brake and accelerates the bike when you pedal.

The reason that the chain applies its force close to the axle and not close to the rim of the wheel is that a single turn of the pedal moves you a few times the entire circumference of the wheel, a much larger distance than your feet travel. [How much further depends on the ratio of the gear wheels engaged. There is more on this subject later on.] If the chain went to the rim of the rear wheel, then one turn of the pedal would take you forward much less than the wheel circumference and you would go forward less quickly than you could walk. The design of a bike is such that you're providing a comparative large force travelling in a small circle and the bike converts this into a modest force exerted over a large circle.

c) *I now start to climb a hill that is 1 in 20. The combined mass of myself and bike is 100 kg. If I keep working at the same rate, what is my speed up the hill?*

Going up hill, both air resistance is against me and the component of the weight of me and my bike. Because I will be going up slower, the air resistance will be less than the number above but it will still be given by $0.5v^2$, whatever v turns out to be.

If θ is the angle of the hill then "1 in 20" means that the hill rises by 1 m for every 20 m travelled horizontally. This makes tanθ = 1/20 = 0.05. The weight of me and the bike is "Mg"

(where M is our mass and $g = 9.81\ \mathrm{m\,s^{-2}}$) = 981 N. The component of weight down the slope is $981\times\sin\theta$. For comparatively small angles like 1 in 20 (mathematically small, though a hill this steep doesn't seem small on a bike) there is negligible difference between $\tan\theta$ and $\sin\theta$ and hence the component of weight downhill is $981\times0.05 = 49$ N.

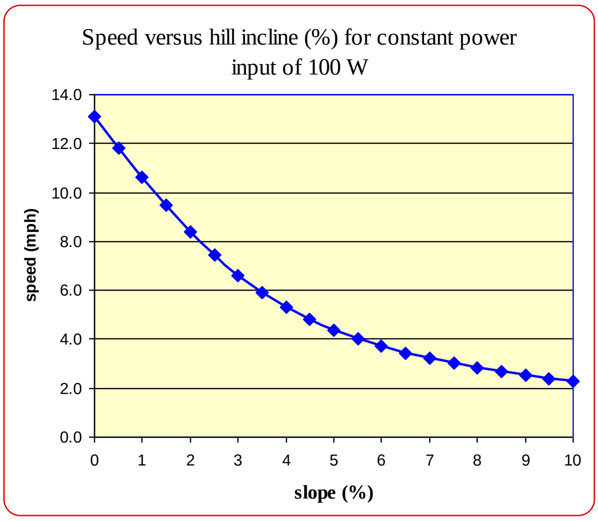
Putting these forces together, the sum of the rate of working done against gravity and working against air resistance must equal 100 W. Hence $100 = 49v + 0.5v^3$. This is a cubic equation which can be solved by a little fiddling around with the calculator. [See later on for how to solve such an equation exactly]. The term $49v$ on the right is going to be much the largest, which means that most of my work is done against gravity. If the other term wasn't there at all the answer would be $v = 100/49 = 2.04\ \mathrm{m\ s^{-1}}$ but adding in the other term makes the right-hand side too big. If you try $v = 2\ \mathrm{m\,s^{-1}}$ (~4.5 mph) then this solves the equation to within 1 decimal place. The hill "1 in 20" is quite a significant hill to cycle up so the reduction to a speed of 2 $\mathrm{m\ s^{-1}}$ is realistic. At this speed, the equations tell us that air resistance makes a negligible difference to the rate I can climb. Professional road-racing cyclists are used to slip-streaming each other to reduce air resistance for the ones behind but going up a steep hill this slip-streaming is almost irrelevant, since most of a cyclist's effort is working against gravity, not air resistance. In short, there is no shelter from a hill. It's for this reason that a breakaway group in road-racing is nearly always caught, for those sheltering in the peloton have fresher legs near the end of the race, since they have not been exposed to so much windage. On a hilly course, though, the breakaway has a better chance, since no-one can shelter from the hills which are in fact using up most of the cyclists' energy. Moreover, the average speed will be less over a hilly course so the sheltering effect of the bunch will be less of an advantage.

Another consequence of the physics above is that the speed I can travel up a hill that's steep enough to make air resistance of little relevance is now directly proportional to the power I can generate. Power now becomes really important. If you can generate 200 W to my 100 W then you'll go up at almost twice my speed. Compare this with the case of travelling on the flat, where doubling the power available only makes a difference of a factor of 1.26.

Yet another consequence is that the speed attainable going up hill is determined mainly by the ratio of power available divided by the combined mass of cyclist and bike. In terms of our symbols, $v \propto$ power/M. My power to mass ratio, including bike, is 1 W/kg, which isn't going to break any records in the animal kingdom or even among fit people. For comparison, a typical car with a couple of passengers might have a figure 50 times as great, which is why it can travel much faster, and a passenger plane will have a power to mass ratio 1000 times as great, which is why it can fly. Since weight is proportional to mass, the figure is usually called 'power to weight ratio' but the 'weight' is quoted in kg. The bottom line for cycling seems to be that lighter fit people have a greater power to weight ratio than heavier fit people so I'm not going to be able to sustain as high a climbing speed as a fit person whose mass with bike is only 80 kg. On the flat, mass is irrelevant to final speed, what is important is the ratio of power to air resistance and this, on the whole, favours the larger fit person. This is very clearly seen amongst 'grand tour' professional cyclists where the mountain finish stage winners are usually light people but the sprint finish stage winners are sturdily built.

To get back to the question, "1 in 20" is often expressed as a 5% slope. I'm not sure that it helps when you're climbing a 5% hill but maybe it's worth knowing that the additional force dragging you back is near enough 5% of the weight of you and the bike. The same is true for any other % of incline you are likely to cycle up: for going up an 8% hill you have 8% of your combined weight against you or, on the downhill side, 8% pulling you along.

The accompanying graph shows the speed (in mph) for a range of slopes up to 10% if I keep my work rate steady at 100 W. The calculation for any slope is similar to that above. Notice that there is a 10% reduction in speed even for a modest slope of 0.5%, or "1 in 200". In real life there is a temptation not to work at a constant rate but to increase your rate of working up a modest slope to keep the speed up. My experience is that it's not worth it. I let the hill decide the speed. Drop down the gears and keep working at the same rate even if the speedo is reading less than I would like. As a ball-park figure, I drop down at least 1 gear for each percent the inclination rises. Yes, one can go up a 6 mph hill at 9 mph but one has to work just over 50% harder and it saps one's energy for the next part of the journey, so the time gained is likely more than lost later on. For hills steeper than 10% though, I really have to work harder than 100 W otherwise the bike goes so slowly that I'm in danger of wobbling off. All that said, professional road cyclists generally increase the work rate going up a hill because the loss of speed by working at their flat-ground cruising rate is too great. You'll see jackets unzipped to the waist as riders try to dissipate the extra heat produced by the extra work rate.



The ability of gravity to eat up the power available is very obvious on a cycle run. I can be zipping along the flat with some wind assistance at 20 mph but as a hill rises up in front, my speed falls rapidly, even with the wind assistance, and I have to drop down the gears quickly. If the hill has a slope of more than 5%, it's not long before I'm going at little more than 5 mph.

Touring on country roads by bike, the changes in slope are conspicuous but on the whole the corners are not. Travel the same roads by car and the corners will be conspicuous but the slopes not.

d) *What force is pushing the bike up the "1 in 20" hill at the speed 2 m s^{-1} just calculated?*

The force F is working at the rate of 100 W and the speed v is 2 m s^{-1}. Hence 100 = F×2, giving a force of 50 N.

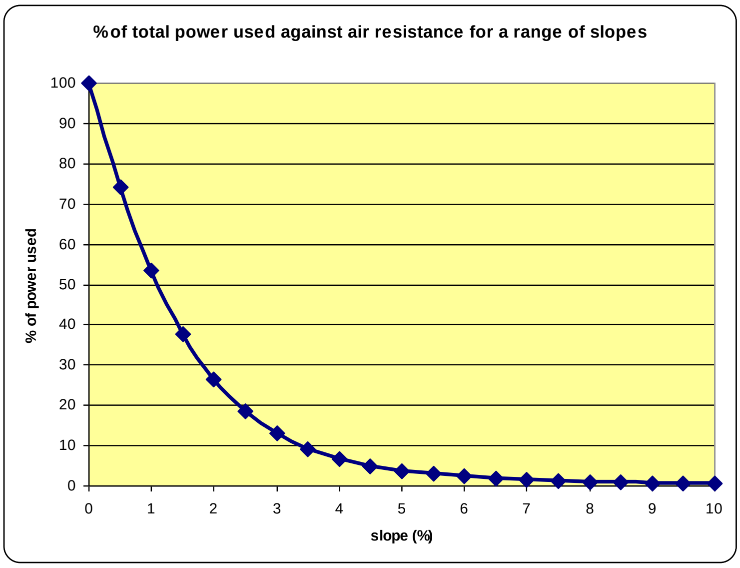
e) *The force F pushing the bike forward is the force of the road on the tyres, as mentioned above. The force originates from me pushing the pedals through the chain and bike gears. What change in gear ratio do I need so that the force I am supplying to the pedals is the same going uphill as it is on the flat?*

On the flat I needed to pedal so that the force of 17.1 N was acting on the bike and now the driving rear wheel must supply a force of 50 N hence the gear ratio must change by 50:17.1 ≈ 3:1. This is the same as the ratio of speeds on the flat and uphill. Going up hill I am turning the pedals at the same speed with the same force as on the flat to achieve only one third of the speed but a larger force exerted on the rear wheel.

If I drop down to too low a gear going uphill then I'm turning the pedals quickly but going up too slowly; if I stay in too high a gear I find it hard to exert the force required to move forward at any speed and end up going too slowly. Selecting the right gear is the key to going uphill at a reasonable speed with a sustainable effort. The reason many people find it's hard going uphill is that bikes don't have a big enough gear ratio to allow you to work at the same rate as on the flat, so you end up working much harder. Even if they did have a big gear ratio, for steep hills you would be travelling so slowly that balance would be a problem and impatience is another problem!

I'm tempted to recap here. Going at constant speed on the flat or uphill there is a force acting against the bike, in general a combination of wind resistance and gravity. There has to be an equal and opposite force on the bike to sustain constant speed. This is what Newton's first law of motion says. This force is the component of force the road exerts on the tyre in the direction of motion. It is created by friction between tyre and road and, from Newton's third law, equals the backward force the tyre exerts on the road. (No friction, no forward motion, however hard I pedal). That force is derived from the force I exert on the pedals. The conversion between the road force, I'll call it F, and the force I exert on the pedals, I'll call it P, is governed by two ratios. The first is the ratio of rear wheel radius (assuming rear wheel drive) to the radius of the rear gearwheel. For an intermediate gear this works out at 10:1 on my bike. So the force exerted on the road is a tenth of the tension in the chain for this gearwheel. Changing to lower gears (larger diameter), decreases this ratio and increases the force on the road for the same chain tension. The other determining factor is the ratio of crank radius to chainwheel radius. For my middle chainwheel this is 178 mm/70 mm = 2.54, implying a force P exerted by my foot perpendicular to the pedal is converted into a chain tension of 2.54×P. For my selected gear and chainwheel F = 2.54×P/10 = 0.254×P. Or, to put it the other way around, if the resistance to the bike moving forward (F) is 17.1 N then with the selected gears I need to pedal with a force P of 17.1/0.254 = 67.3 N. In reality the useful force exerted on the pedal varies from much higher down to zero (when the pedals are straight up and down) so the *average* force must be 67.3 N.

f) *Coming back to the issue that as a hill becomes steeper less of available power is used against air resistance and more against gravity, what % of my 100 W is used against air resistance for hills of increasing slope?*



The calculation in *c)* can be worked for any slope. The total power of 100 W is used partly against gravity (981×sinθ×v) and partly against air resistance (0.5v^3). Hence the % used against air resistance is 100×(0.5v^3/100) = 0.5v^3. The speed v is found by solving the equation 100 = 981×sinθ×v + 0.5v^3. This equation was solved in *c)* by trial and error but a general method is discussed later on. Implementing this in an Excel spreadsheet gives the accompanying graph. The % of my total power used against air resistance falls very quickly. Going up a 1 in 200 hill (slope 0.5%) only 75% of the power is used against air resistance and 25% against gravity. For a 1 in 20 hill (slope 5%) less than 4% of the power is used against air resistance.

g) *On the flat I normally pedal using the crank middle gearwheel (i.e. middle chainwheel) that has 36 teeth leading to the rear-wheel gear of 11 teeth. With my lowest gear, the chain runs from the small chainwheel with 26 teeth to the largest rear-wheel gear with 32 teeth. What is the steepest hill I can climb (in my lowest gear) keeping the same cadence (turns per minute) and exerting the same force as on the flat?*

On the flat one turn of the pedal takes me forward by 36/11 turns of the rear wheel (i.e. 36/11 times the rear wheel circumference). 36/11 is the "gear ratio" of the drive system. In the lowest gear this is reduced to 26/32 times the rear wheel circumference. If my pedal is turning at the same cadence then my speed has dropped by the fraction (26/32)/(36/11) = 0.248. Hence my speed has dropped from 5.85 m s^{-1} (part a) to 1.45 m s^{-1}. If θ is the angle of the hill and I'm still generating 100 watts then $100 = 981\sin\theta\times1.45 + 0.5\times1.45^3$. This relation can be re-arranged to give me $\sin\theta = 0.0692$. The hill I can go up therefore has an angle, θ = 4º. Alternative one could express it as about 1 in 14 or a gradient of 6.9%. For any steeper hill I'll need to work harder to keep up the same speed (i.e. exert more force on the pedals) or travel slower (reduce my cadence). 1.45 m s^{-1} is the same as 5.2 km h^{-1} or 3.2 mph, so I can't go much slower. If the cadence is the same as on the flat and the power expended the same, then the force on the pedals must be the same. See the next paragraph.

Picking up the gear story two questions back, in general I can either directly choose the speed I want to cycle or choose the power I wish to exert, which determines the speed for me. Either way, my speed determines the resistance to motion (F) resulting from a combination of gravity and speed dependent air resistance. The bike travelling at speed v against this force must do work W where $F\times v = W$. The overall gear ratio (G) of the bike is the distance travelled by the bike divided by the distance moved by the pedals. It equals the product of (radius of rear wheel/radius of rear gearwheel) × (radius of chainwheel/radius of pedal crank). Assuming as usual no losses within the bike, the pedal force perpendicular to the crank P = FG. The pedal speed = $2\times\pi\times r_p\times$cadence/60, where r_p is the radius of the crank and the cadence is always measured in turns per minute. Because of the gearing, the pedal speed must also equal v/G. The power I expend on the pedals is $P\times v/G = Fv = W$, just the power need to push the bike forward, as it should be with no losses. Once a gear is selected then G is determined. This fixes both the force I exert on the pedals and the cadence. Or, within limits, I can choose a gear to give me either a desired cadence or a desired force on the pedals and the one I didn't choose is then fixed. On my bike by changing chainwheel and rear gear, G can be varied from 9.35 in the highest gear to 1.69 in the lowest gear.

I'll add another aside. Cycle makers should take a leaf out of the old millwright's practice of making sure intersecting wheels don't have a common factor in the number of their teeth. This ensures uniform wear of the teeth. My chain has 56 links and my large chainwheel 48 teeth. Clearly 7 turns of the chainwheel corresponds to exactly 6 turns of the chain and if one link is slightly long, for example, with every 7 turns of the pedals it re-engages awkwardly with exactly the same tooth on the chainwheel, thereby increasing the wear on this one tooth. As I cycle, only 7 teeth on the chainwheel have to take all the extra wear whereas if the chain had 57 links, for example, the wear would be distributed over the entire chainwheel. The reverse is also true if a defective tooth is exerting undue wear on the chain. Having no common factor would also ensure that lubrication spreads evenly over chain and gearwheel. Issues of wear perhaps don't get so much attention when replacement is cheaper than it was in the past but in general wear certainly affects performance on a bike.

h) *Having reached the top of the hill in part c, if I free-wheel down the other side, which is also 1 in 20, what speed will I achieve?*

The component of the weight of 49 N is now equal to the air resistance $0.5v^2$. Hence $v^2 = 98$, giving $v = 9.9$ m s^{-1} (22 mph), a decent speed on a bike. Of course I'm not working at all when free-wheeling. Gravity is providing a motive force of 49 N moving at 9.9 m s^{-1} and hence is doing work on me and the bike at a rate of 49×9.9 = 485 W. Nice work if you can get it.

It's easy to work out the slope of the hill that I can free-wheel down at the same speed (5.85 m s^{-1}) as I pedal on the flat. At that speed I'm working at 100 W and hence the hill I can free-wheel down at the same speed is one for which the component of my weight down the hill works at 100 W. i.e. $mg\sin\theta \times v = 100$. This gives $\sin\theta = 100/(5.85\times981) = 0.0174$. This is equivalent to a slope of only 1 degree or just less than 2%. It's not much but remember that the air resistance against me when I'm cycling on the flat is only 17.1 N and hence all that's needed is the component of my weight of 981 N to equal this.

My free-wheel speed down a known hill (in calm conditions) can be used to let me work out the air-resistance factor that I set at 0.5 at the beginning of this section of questions. You can try this yourself. For example, suppose you and your bike of total mass 110 kg free-wheel at 27 mph (12.07 m s^{-1}) down a hill that you know from an accurate map or from your cyclometer is a 1 in 20 hill. If your air resistance is given by αv^2, then what is the factor α? At free-wheel speed, the component of your weight down the hill is equal to the air resistance. Hence $Mg\sin\theta = \alpha v^2$, where θ is the angle of the hill. But $\sin\theta \approx \tan\theta = 1/20$ and you therefore know everything except α in the previous expression. $110\times9.81/20 = \alpha\times12.07^2$, from which $\alpha = 0.37$. So your air resistance is given by $0.37v^2$, which is less than mine.

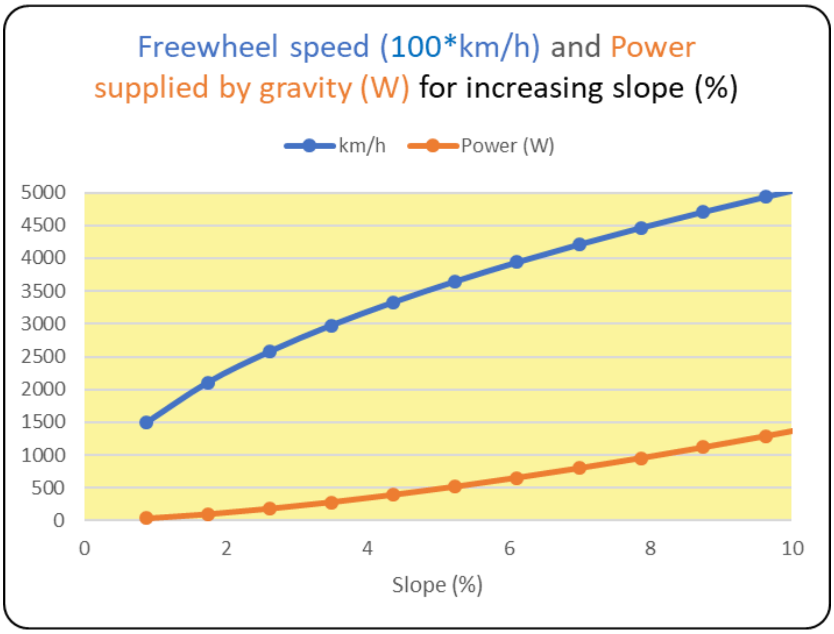
i) If I decide to pedal downhill exerting 100 W as before, what speed will I reach?

If I supply a force F (in addition to the component of weight) that makes the bike go at a speed of v, then $100 = F\times v$. i.e. the force I supply = $100/v$. The sum of the two forces together must balance the air resistance. Hence $49 + 100/v = 0.5v^2$. i.e. $100 = 0.5v^3 - 49v$. This is a cubic equation. The quickest way to solve it is by trial and error with a calculator, though mathematicians wouldn't approve of this technique. (You can also use the graphics facility in your calculator, if you have one, or evaluate the function using Excel.) $v = 10$ makes the right-hand side too small and $v = 11$ makes it too big. You very quickly find $v = 10.8$ m s^{-1} is about right. This is just over 24 mph. Therefore, pedalling down a decent hill doesn't make much difference to my speed so I may as well rest and free-wheel. This makes sense because you can see from part *g*) that gravity is working at some 500 W so the extra 100 W that I can supply isn't going to make much difference, particularly as air resistance increases as the square of my speed. Even on modest hills of one or two percent, rolling downhill without pedalling and watching the ever-changing scenery of the countryside glide past is one of the joys of cycling.

j) How much power does gravity provide when free-wheeling down a hill?

It takes power to move a cyclist through the air. When free-wheeling downhill this power is provided by gravity. If θ is the slope of the hill, as before $mg\sin\theta$ is the component of my weight that is doing the work and this equals the force of air resistance $0.5v^2$ when free-wheeling. Equating the two expressions gives the free-wheeling speed v for a hill of slope θ. The power exerted by gravity is therefore $mg\sin\theta\times v = mg\sin\theta\times(2mg\sin\theta)^{1/2} = 2^{1/2}(mg\sin\theta)^{3/2}$. The free-wheeling speed depends on the square root of the mass of the cyclist plus bike for the same air resistance. There will not be a great deal of difference between members of a group. The work done by gravity increases quickly as the slope increases. Free-wheeling down a 5%

slope, gravity takes me down at 35 km h^{-1} with a power of about half a kilowatt. By the time the slope has increased to 8%, my speed has increased to 45 km per hour and gravity is providing a kilowatt of power, far beyond what I can produce by leg power. Taking the figures beyond the graph shown here, free-wheeling down a 20% slope, had I the nerve to do it, would have me travelling at 70 km h^{-1} with gravity exerting a power of 3.8 kW, more than is in the legs of any cyclist.



k) *I have a regular cycle route of 25 km on the country roads between the Dee valley and the coast. It's not very hilly but it's not often flat either. Over the circular route I go uphill a total of 250 m according to my cyclometer and obviously downhill by the same amount. Making the simplification that half the route is uphill with a gradient of 2% and half downhill and I work at a steady rate of 100 W, how should my average speed compare with my speed on the flat of 5.85 m s^{-1}?*

The physics of going uphill and downhill has been explored in the previous questions so I'll just steal the results. The component of the weight of me and the bike down a 2% hill is 2% of 981 N, namely 19.62 N. Hence going uphill, $100 = 19.62v + 0.5v^3$, giving a speed v of 3.75 m s^{-1} (8.4 mph). If I travelled 12.5 km at this speed it would take about 55.5 minutes. Downhill, my speed v is given by $100 = -19.62v + 0.5v^3$ and hence $v = 8.01$ m s^{-1} (17.9 mph). Travelling the remaining 12.5 km would take only 26 minutes. Hence my total journey time will be 1 hour 21.5 minutes and average speed 5.1 m s^{-1}. If the whole route were flat, my time would be 25000/5.85 = 4273 secs (1 hour 11.2 minutes). The hills therefore increase my time by just over 10 minutes. This just gives an idea of how much difference hills make, for of course the route has a wide variation of slopes. On most days there's a wind blowing, too, which adds additional complications.

l) *On the following day, I cycle when there is a wind of 7 m s^{-1} (about 16 mph, Beaufort force 4, sufficient to kick up dust and move small branches of trees). If the wind is behind me on the flat, what speed do I make?*

Now my speed through the air is not the same as my speed over the ground. Let v be my speed over the ground and hence $(v - 7)$ the speed that air comes at me. The force (F) I exert still works at 100 W and air resistance limits my speed. Hence $F = 0.5(v - 7)^2$ and $100 = Fv$. Therefore $100 = 0.5(v - 7)^2 \times v$, giving $v(v - 7)^2 = 200$ which is another cubic equation. Clearly $v = 10$ is too small and 12 is too big. By trial and error 11.2 m s^{-1} is close, flying along at 25 mph. This is about 5.4 m s^{-1} faster than without the wind. This is in agreement with experience. It's easy to keep up a speed of over 20 mph with a reasonable wind behind. In reality, though, with a strongish wind behind I'll take the opportunity of relaxing and catching my breath and hence won't continue to work so hard.

If I stopped pedalling altogether then the wind would eventually get me up to a speed of 7 m s^{-1} if there really was no significant friction in the bearings of my wheels. At a slower speed there would be a force pushing me forward and Newton's second law says that that produces a forward acceleration which would increase my speed a bit until the wind speed is reached, at which point there is no net forward force. I guess in practice the friction within the bike would reduce the achieved speed a little on a dead level road.

m) *What is the equivalent hill on which I would achieve the same speed by pedalling down in calm weather?*

The calculation is the same as in part *h* only this time the unknown is the slope of the hill θ and the speed v = 11.2 ms^{-1} is known. As in part *h*, Wsinθ + 100/v = 0.5v^2 and putting in the known weight of 981 N and the speed gives 981sinθ = 62.7 – 8.93 and hence sinθ = 0.0548. The angle of the hill is 5.5% or 3.14º or about 1 in 18.

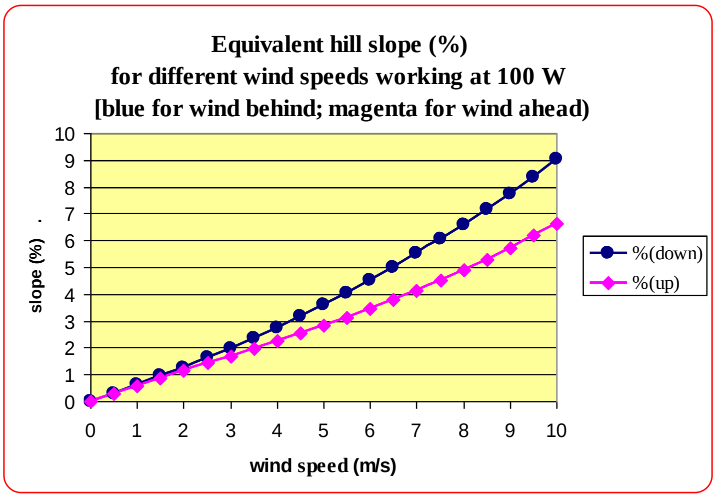
[Another 'hill' concept associated with the wind is to ask what hill can I go up with the wind behind me at the same speed as I travel on the flat with no wind? The answer is a smaller slope than the 'equivalent hill' defined above. With the example figures at hand, gravity is against me and my pedalling force and a slight residue of the wind behind me is pushing me up this hill. Hence Wsinθ = 100/5.85 + 0.5×(7-5.85)^2, giving sinθ = 0.01152. The angle of the hill is therefore only 1.1% or about 0.66º.

Sometimes one can go uphill with the wind behind at the same speed V_w as the wind, so that no wind is felt. i.e. $v = V_w$. This is very satisfying. If I work with the usual 100 W power, what inclination of hill fulfils this condition? In this circumstance, my power is used by the vertical component of my speed, $V_w sin\theta$, to raise my weight, not fight the wind. 100 = $V_w sin\theta \times mg$ for a hill of slope θ. Taking m =100 kg, as before, sinθ = 1/(9.81× V_w). The hill angle decreases proportionally to the increase in wind speed. For a light wind of 2 m s^{-1} behind me, I can travel up a hill of angle 2.9°, 1 in 20, at 2 m s^{-1}. For a strongish wind of 7 m s^{-1}, I can travel up a hill of only about 1 in 70 at 7 m s^{-1}. The earlier results show that with no wind I'd travel up such a hill at just over 4 m s^{-1}, so the extra speed with the wind behind is very noticeable.]

n) *If the wind is against me on the flat, what speed do I make?*

The force the wind exerts on me when I'm stationary and facing it is 0.5×7^2 = 24.5 N. Cycling at speed v over the ground the resistance against me is 0.5(v + 7)^2 and hence 0.5(v + 7)^2×v = 100. Again, trial and error shows quickly that 2 m s^{-1} is too small but 2.3 m s^{-1} is about right. This is 5 mph.

Notice that although I make only 5.85 m s^{-1} without a wind, I can still cycle into a 7 m s^{-1} wind. The 'equivalent hill' can be found from a calculation similar to part *c* except the unknown is now the slope of the hill. The answer is 1 in 24 in this case. The last two questions show that a strongish wind makes a huge difference on the flat, bowling me along when it's behind me and really slowing me when it's ahead. To take the figures a little further, if 2 m s^{-1} is the slowest speed I can travel into a wind without my balance being compromised, what is the strongest wind I can cycle into on the flat given I can sustain 100 W? For this calculation, with v_w as the wind speed, 0.5(2 + v_w)^2 × 2 = 100. Solving for v_w quickly gives the wind speed as 8 m s^{-1}. This is not a lot of wind, about 18 mph, at the top of the Force 4 range for a 'moderate breeze' where small trees in leaf begin to sway. It is barely half the speed of a gale. The conclusion is that if I come to a stretch where a stronger wind is against me, then I need to up my pedalling power from 100 W. Into a gale of 20 m s^{-1}, I could just maintain 1.5 m s^{-1} if I provided a power = $0.5(1.5 + 20)^2 \times 1.5 = 350$ W. I've no idea how long I could sustain this, but not for long. Cycling into a wind isn't all about speed reduction. The wind noise in one's ears and the turbulence of the wind make it noticeably harder to hear traffic, particularly traffic coming up behind.



The good news from these figures, and the same is true for other wind speeds, is that the gain in speed on the flat with the wind behind is slightly more than the loss in speed when you have the wind against you. Put another way, the slope of the equivalent downhill with the wind behind is slightly greater than the slope of the equivalent uphill with the wind ahead. Using the same method of calculation as above for different wind-speeds, I've used Excel to calculate the effect for a range of wind-speeds between 0 and 10 m s^{-1} and plotted the result in the nearby graph. The vertical axis is the slope (expressed as %) of a hill that is equivalent to the wind-speed shown along the bottom (in m s^{-1}). The top curve is for when the wind is behind; the bottom for when the wind is ahead. If I work harder, say at 200 W, then my speed on the flat is faster and both curves are slightly higher. This isn't immediately obvious but comes out of the new figures for the higher workload.

To pick one example from the graph: if there is a 5 m s^{-1} wind against me then the equivalent hill is about 3% which means if I cycle for 10 km into such a wind the extra time taken is the same as climbing 300 m, quite a bit. I'm not working any harder into the wind, since the assumption is that I work at a constant power of 100 W, but I'm going slower and it takes longer, quite a lot longer, so the fatigue is more conspicuous. To put it graphically, the presence of a wind is like tilting the landscape up in the direction the wind is coming from. The amount of tilt depends on the strength of the wind.

While on the topic of wind and the 'equivalent hill', there is one notable difference between wind and hills. If I increase the mass of myself and bike without changing my air resistance, for example by loading the bike panniers with camping gear or perhaps shopping, or even personally just putting on a few kg of additional weight, then this doesn't change the speed I'll achieve on the flat, where speed is determined simply by air resistance and the rate I work. It will, however, reduce the speed I can make up a hill because the component of my weight acting against me is now larger, so working at a given rate, like 100 W, means travelling at a slower speed against a larger force. Correspondingly, I'll free-wheel faster down the other side of the hill, since the force pulling me downhill is greater. From all that's been said before, a journey with as much 'up' as 'down' will take longer if me and my bike are heavier. It's no surprise then that road racing cyclists don't carry any spare weight on themselves or their bikes. Carrying more weight on the velodrome is a different issue. There, more weight but exerting the same force means it will take longer to reach your maximum speed, and time means places in a race.

o) *If I travel 1 km on the flat with the wind behind me, at 11.2 m s^{-1} as in part 'j', and 1 km with the same wind against me, at a speed of 2.3 m s^{-1} as in part 'l', calculate my average speed over the 2 km and compare this with my average speed when no wind is blowing.*

This is a similar question to the very first question that compared the average speed going up and down a hill with the speed on the flat, only the numbers are a bit different because the slopes of the 'equivalent hill' with the wind against and behind are a bit different. Time taken for 1 km with the wind behind = 1000/11.2 = 89.3 s. Time taken with the wind against = 1000/2.3 = 434.8 s. Hence total time for 2 km is 524 s. Hence average speed = 2000/524 = 3.8 m s^{-1}, appreciably less than my average speed of 5.85 m s^{-1} without a wind. Although it's true that I gain more speed with the wind behind than I lose with the wind ahead, it's not enough to make up for the time I lose when the wind is ahead of me.

p) *If the same wind of 7 m s^{-1} is behind me going up the 1 in 20 hill, what speed will I make?*

The downhill force is 49 N, as before. Uphill I now have the force I supply (F) and the wind force of $0.5(7-v)^2$ on my back. You might think the wind would help a lot but the faster I go uphill the less this wind-force is. Remembering that at constant speed there is no net force on me and my bike: $49 = F + 0.5(7-v)^2$ and as before, $F = 100/v$. Hence we get a cubic equation again and multiply both sides by $2v$ gives $98v - (7-v)^2v = 200$. The same trial and error method gives a solution of $v = 2.4$ m s^{-1} (to one decimal place). This is only 0.4 m s^{-1} faster than with no wind. What's going on is that most of the effort going uphill is against gravity. At 2.4 m s^{-1}, the wind is helping me with a force of 6.5 N but I need to overcome the 49 N gravity component and hence 6.5 N from the wind doesn't make much difference.

You'll find that with the wind behind you, you can go up a modest hill at the same speed as you would travel on the flat in calm conditions while working at the same rate. I'll leave this issue as an exercise for you to try yourself. The problem is quite 'do-able'. You'll find the answer is a lot less than the 'equivalent hill' corresponding to a particular wind-speed. For example even in a stiff 10 m s^{-1} wind, sufficiently strong that I probably wouldn't choose to cycle in it unless I had to, I calculate that I will get up a hill of slope 2.6% at the speed of 5.85 m s^{-1} that I can achieve on the flat in the calm, while continuing to work at 100 W. Going uphill with a good wind behind you is like having an invisible hand in your back, helping you along. The best scenario on a windy day is to have the wind behind you going uphill and against you going downhill.

q) *If the same wind of 7 m s^{-1} is against me going uphill, how much does it slow me down?*

A variant of the argument above gives the relationship for v as $49 + 0.5(v+7)^2 = 100/v$ and multiplying by $2v$ as before gives $98v + (v+7)^2v = 200$. The solution for this using any of the techniques above is 1.2 m s^{-1} (2.7 mph), again to 1 decimal place.

In brief, the wind reduces me to almost a speed of wobbling and getting off. If the hill is 1 km long it takes me 8 min 20 secs to climb at 2 m s^{-1} but 13 minutes 53 seconds at only 1.2 m s^{-1} so the reduction in speed caused by the wind means that my uphill effort has to be applied for noticeably longer.

The numbers in the previous sections illustrate what difference a strongish wind makes on the flat and up a steep hill by ordinary cycling standards. The effect on speed is greatest on the flat. The wind will have a larger effect on speed for hills less steep than the one cited here.

r) *Suppose I reach the bottom of a hill travelling at 10 m s^{-1} (see part 'i)') and free-wheel up the other side which has a slope of 1 in 40. How far will I get before gravity and air resistance bring me to a halt? How far would I have got in the absence of air resistance?*

On the upslope two forces cause me to decelerate, namely Mg/40 and $0.5v^2$. As before, I have taken m = 100 kg. Hence Newton's law of motion tells me that $m\frac{dv}{dt} = -mg/40 - 0.5v^2$ and hence $\frac{dv}{dt} = -g/40 - 0.5v^2/m$. This equation with a variable acceleration (*dv/dt*) is more complicated than motion with constant acceleration.

We'll not worry that we don't know formally how to solve this equation because we do know what it is saying. It's saying that in a short interval of time, say 1 second, the change in velocity (write it δv) is $-(g/40 + 0.5v^2/m)$. Hence we can follow the decreasing velocity in an Excel spreadsheet by recalculating the velocity every second. $v(+1 \text{ sec}) = v - g/40 - v^2/200$. Having calculated *v* one second later, the formula is repeated to calculate v another second later using the previous result as input. Do this in Excel, taking g = 9.81 m s^{-2}, and look at the result.

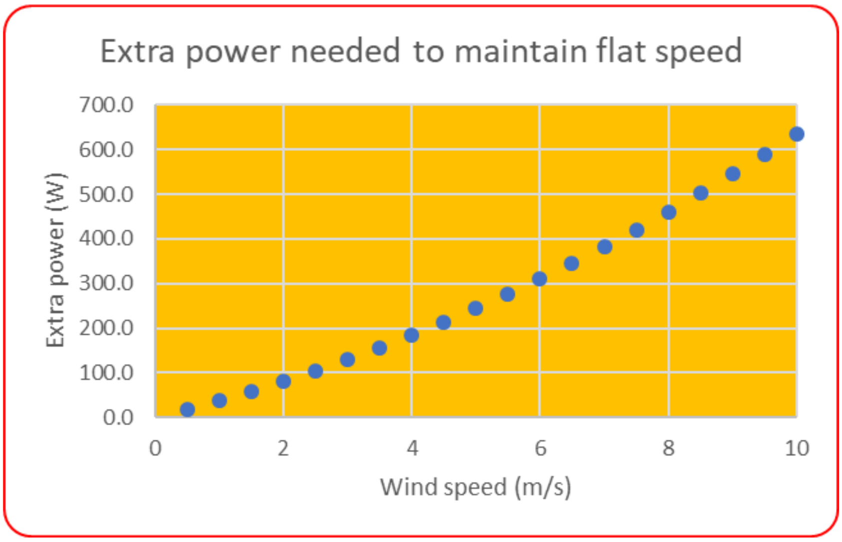
I will come to rest in 29 s, having travelled about 108 m up the hill. Air resistance takes my velocity down quite quickly, in agreement with experience.

You can try the same problem in the absence of air resistance. This is now an easy problem, using the equations for a constant deceleration of g/40 and initial velocity of 10 m s^{-1}. Without air resistance I would have travelled just over 200 m in just over 40 s before coming to rest. However, only in textbooks can you get rid of air resistance (on Earth) and you can see in this case that it in reality air resistance reduces the free-wheel stopping time by 25% and the stopping distance to about half of what you would otherwise have got from the kinetic energy you have gained coming downhill.

You can use the 'little interval at a time' approach (here 1 second) with Excel in more complicated situations such as with a wind blowing (ahead or behind) or even with some given pedalling. Don't be put off finding out what is going on just because an equation looks too difficult to solve by the maths you know.

s) *Suppose I try to maintain my speed on the flat into an increasing headwind. How much extra power must I supply for increasing wind speeds?*

If the wind is against me, then I must exert more power to maintain the same speed. The wind is draining some of my resources. Let the wind speed be v_w. Travelling at speed *v*, the air resistance against me is $0.5(v + v_w)^2$.

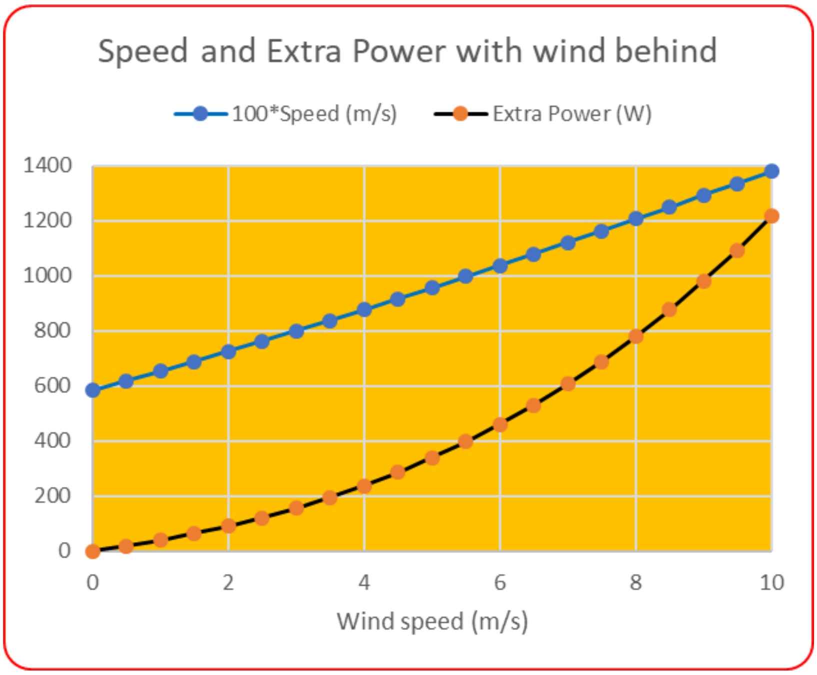


Hence the extra power needed is $0.5v(v + v_W)^2 - 0.5v^3 = v^2v_W + 0.5vv_W^2$. With the earlier example figure of the speed on the flat determined by my power output of 100 W, $v = 200^{1/3} = 5.85$ ms^{-1}. The accompanying graph shows the extra power needed to maintain my speed on the flat when travelling into a range of headwinds. As expected, if the headwind is 5.85 ms^{-1}, I need to provide 4 times my normal power and hence an extra power of 300 W. The obvious conclusion is that even to maintain my speed on the flat into a headwind stronger than 2.5 m s^{-1} (Force 3 and above) I need to more than double my normal power output and this can't be sustained.

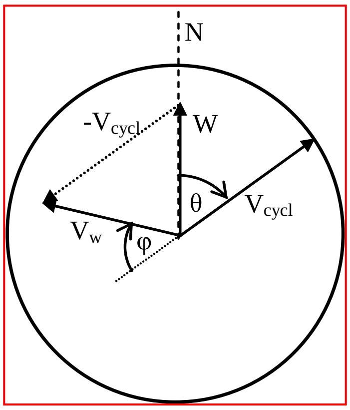
t) If the wind is behind me, what power does it contribute?

If the wind is behind me, then I would be able to free-wheel at the speed of the wind if there were no 'internal' friction in the bike. In practice, there always is some friction, including 'rolling resistance' generated by the flexure of the tyres so I will travel a bit slower than the wind. However, if I were travelling with the wind, there would be no force on me and hence the wind would be exerting no power in spite of my motion forwards. [This is the same as the curious physics of clouds. If a cloud of mass many hundreds of tonnes is being blown along by the wind, no energy is needed to do this].

In reality, with a wind behind I may use it to travel faster than my normal flat speed. How much faster, if I maintain my power output of 100 W? $100 = 0.5\, v\,(v_W - v)^2$ giving $0.5v^3 - v^2v_W + 0.5v_W^2 - 100 = 0$. The solution of this cubic equation gives my speed v with the wind behind, shown in blue in the accompanying graph (divide the y axis figures by 100 to get speed in m s^{-1}). The extra power the wind is supplying is the power I would have needed to reach this speed, less the 100 Watts I'm expending, i.e. $0.5v^3 - 100$. This is shown in the graph by the black line. With a 10 m s^{-1} 'fresh breeze' behind me (Beaufort force 5), I will be travelling at 13.8 m s^{-1}, a speed that without the wind I would need to exert almost 1.5 kW to achieve. The wind is 'saving' me 1.38 kW. See the graph for savings at other wind speeds. Even a 2 m s^{-1} breeze gets me to a speed that I would need almost double my power to achieve.



u) It's a common experience that when there is a wind blowing it seems to be against me more than behind me. Is this just an illusion or is it true? Suppose I am cycling with a wind blowing from the South at 4 m s^{-1}. If I can maintain a speed of 5 m s^{-1} in whatever direction I cycle, for which directions of the compass do I feel a stronger wind than on a calm day?

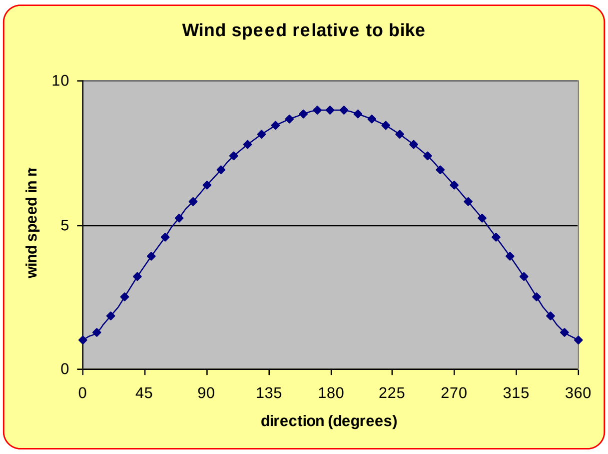


This is a question about relative velocities. The velocity of me relative to the road is 5 m s^{-1} in any compass direction. The velocity of the wind is 4 m s^{-1} due North. Hence the velocity of the wind relative to me ($\mathbf{V}_w$) is the velocity of the wind relative to the road ($\mathbf{W}$) minus the velocity of me relative to the road (V_{cycl}). Let θ be the compass direction of my travel (θ = 0 is North). Using the symbols just introduced:

$$\mathbf{V}_w = \mathbf{W} - V_{cycl}$$

$\mathbf{V}_w$ gives the wind speed as it seems to me on my bike.

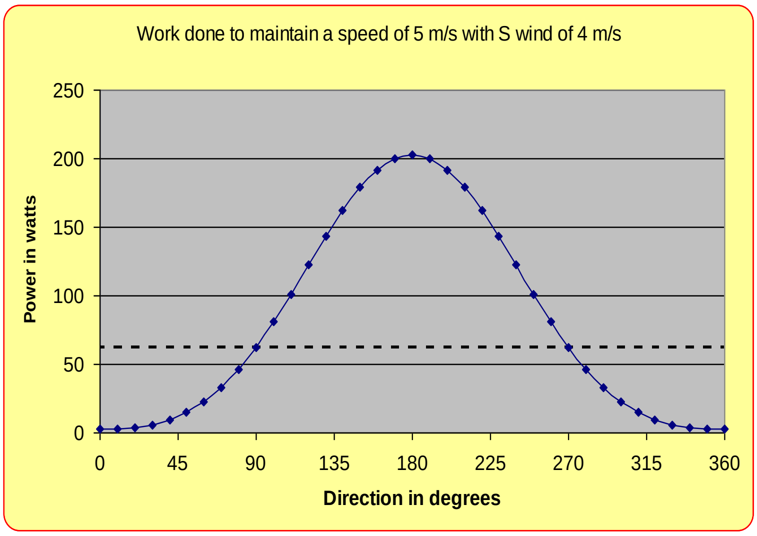
With no wind, I always experience a 5 m s^{-1} headwind due to my own speed over the ground. With the wind as given, the net wind-speed I experience is given in the accompanying graph. For angles from 66.5° around to 293.5° (about two-thirds of all directions), I experience a stronger wind than 5 m s^{-1} but it usually has a sideways component as well as a component against me. A sideways component won't help or hinder my speed and can be ignored unless it is very strong. As you would expect, the component against me is less than 5 m s^{-1} for all directions of travel North of an East-West line and greater for all directions South of this line.



v) *The given wind will help or hinder me by varying amounts depending on my direction of travel. How does the power I need to exert to maintain the constant speed of 5 m s^{-1} vary with direction of travel?*

The sideways component of the wind-speed relative to me does no work since my motion is at right angles to the sideways force. The air resistance force against which I'm working is $F_r = 0.5v^2$, where v is the component of wind speed against me in the direction of my travel. The rate of working is $F_r \times 5$, since I always travel at 5 m s^{-1}. The component of wind-speed in my direction of travel is Wcosθ, with W = 4 and hence the power I need to exert when travelling at a compass bearing of θ degrees is $0.5\times(5 - 4\cos\theta)^2\times 5$. This is plotted nearby. The power varies from very little indeed with the wind behind (heading North at θ = 0°) to a stiff 200 W when heading South.

The power I need to exert in calm conditions is $0.5v^2\times 5 = 62.5$ W (in whatever direction I travel). This is shown by the dashed line on the nearby graph. If the wind speed were something other than 4 m s^{-1} then the curve on the graph would be a similar shape but a different size. For example in a 2 m s^{-1} wind, the power varies from 22.5 W when travelling due North to 122.5 W when heading South.



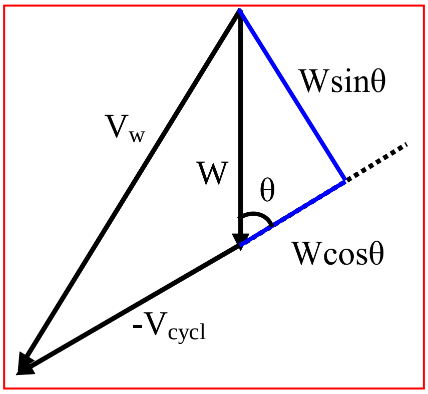
w) *Free-wheeling downhill is one of the pleasures of cycling. Compare the speeds I expect when I'm free-wheeling down the 1 in 20 hill with a) 7 m s^{-1} prevailing wind against me b) no prevailing wind and c) the 7 m s^{-1} prevailing wind behind me.*

As earlier, the downward component of my weight is "Mgsinθ" which equals 49 N whatever the wind. At equilibrium free-wheel speed, this force must equal the air resistance. Hence *a)* $49 = 0.5\times(v + 7)^2$ where v is my speed. This quickly gives $v = 2.9$ m s^{-1} (6.5 mph). The wind makes a big difference when free-wheeling. If the hill was less than half as steep, I couldn't free-wheel down it into a wind as strong as 7 m s^{-1} but would have to pedal. *b)* This problem was solved earlier. $49 = 0.5\times v^2$, giving $v = 9.9$ m s^{-1} (22 mph). *c)* With the wind behind, $49 = 0.5\times(v - 7)^2$, giving $v = 16.9$ m s^{-1} (38 mph). This is sufficiently fast that I'd probably put the brakes on, for one doesn't wasn't to hit a pot-hole at that speed or be travelling at 38 mph when you reach the inevitable corner at the foot of the hill. Falling off at nearly 40 mph would be scary and probably shoulder and arm-breaking, not to mention bike-bending. In short, the wind makes a big difference to my free-wheeling speed.

You may be wondering how long it would take to get up to the 'equilibrium free-wheel speed' as I called it above without any pedalling. In section *l)* we saw that on the flat with a 7 m s^{-1} wind behind me I could pedal at 11.2 m s^{-1}. Suppose I reach the top of the downslope and stop pedalling. How long does it take before the wind speeds me up to the final figure of 16.9 m s^{-1}? This is another variable acceleration problem. The acceleration I have down the hill is provided by a net force that is the downhill component of my weight (Mgsinθ) and the residual resistance of the wind ($0.5\times(v - 7)^2$ up the hill). Hence $49 - 0.5\times(v - 7)^2 = M\, dv/dt$. Using the same idea as in section *r)*, I can follow my increasing speed in an Excel spreadsheet. In fact it takes quite a long time to reach even 16.8 m s^{-1} (within 0.1 m s^{-1} of the final speed), about 50 s during which I travel a distance of 0.75 km. So it needs quite a long hill at 1 in 20, dropping about 750/20 = 37.5 m. The reason, of course, is that the accelerating force gets less and less the closer I approach the 'equilibrium speed'. Mathematically I never quite get there but within 0.1 m s^{-1} is near enough in practice.

x) *Finally, this is a tougher but interesting question. Supposing the wind is blowing at a constant speed and I travel around a circular course working at a constant rate, how is the time I take to complete the circular course influenced by the wind speed?*

To solve this problem I need to find my cycling speed for any compass direction θ, then add up the time it will take me to go around the circle covering all directions. I said this was a tougher problem because although the physics is exactly the same as we've met before, the maths is harder. However, if we can do it for a circle, then it can be done for any shape of track, for example a real course whose shape and direction is extracted from a map. There isn't much extra physics involved in including hills, as in other examples here, so with more sophisticated programming than I'm going to give, a program could be written that extracted information about a real route (e.g. from Google maps?) and calculated how long it is expected to take for a given wind direction and rate of working.



To see how the physics goes, I'll take the wind direction as coming from the North (it doesn't matter for a circular track), find out what the wind speed against me is in general relative to my bike, then work out the speed I'll reach at my usual rate of working. Finally, I'll put in specific numbers to see what comes out.

From the earlier diagram and symbols, the velocity of the wind I experience, $\boldsymbol{V}_w$, at any direction of travel is given by

$$\boldsymbol{V}_w = \boldsymbol{W} - V_{cycl} \, .$$

If my direction of travel is in compass direction θ, then the strength of the wind against me is $V_{cycl} + W\cos\theta$.

The wind resistance I experience is, as before, $F_w = 0.5\times(V_{cycl} + W\cos\theta)^2$ and the rate I work at $F_w\times V_{cycl}$ and hence we finally get the relationship that determines my speed in any direction as determined by the rate I can work against the wind resistance:

$$100 = F_w\times V_{cycl} = 0.5\times(V_{cycl} + W\cos\theta)^2\times V_{cycl}$$

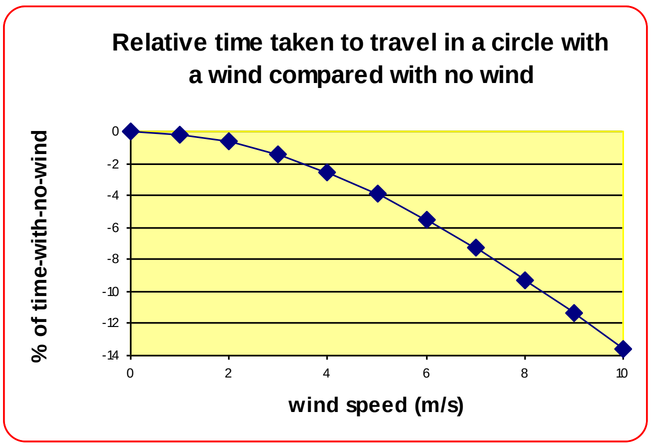
$$\text{i.e. } 0.5\times V_{cycl}^3 + WV_{cycl}^2\cos\theta + 0.5\times W^2\cos^2\theta\times V_{cycl} - 100 = 0$$

If you look at the relationship above, then remember that the wind speed W will be given and hence for any angle θ a cubic equation will determine my cycling speed V_{cycl}. For example if the wind speed is 4 m s^{-1} and I am cycling NE (θ = 45°), then the equation becomes:

$$0.5\times V_{cycl}^3 + 2.833V_{cycl}^2 + 4V_{cycl} - 100 = 0$$

Many people reading this won't be able to solve a cubic equation on their own. On other occasions when cubic equations have come up here we've resorted to an approximate answer. There is a 'cubic equation solver' on the web (at http://www.1728.com/cubic2.htm at the time of typing) and using the numbers above the solution given is 4.13 m s^{-1}, a very reasonable answer. [For the mathematicians, a cubic equation has 3 roots but in this case two of them are imaginary and only one real]. Looking back, you can see that the physics needed to solve the problem is intelligible but the mathematics needed to get a number out is not trivial.

You can check that the equation above gives the right answer if there is no wind (W = 0), in which case $100 = 0.5\times V_{cycl}^3$, as expected, giving V_{cycl} = 5.85 m s^{-1}, as was found earlier.



What's the grand answer to the question? I programmed an Excel spreadsheet to solve the cubic equation above at 1° intervals around a circle, added up the total time to travel around and compared that with the time taken when there was no prevailing wind. Perhaps surprisingly, the wind doesn't make much difference but what difference it does make allows me to travel quicker than with no wind. The reason for this is that the percentage reduction in speed against the wind is less than the % gain in speed when I'm travelling in the opposite direction. For example, with a 4 m s^{-1} Northerly wind, my speed into the wind (travelling North) is reduced by about 40%, from 5.85 m s^{-1} with no wind to 3.85 m s^{-}

[1], but travelling South I travel at 8.77 m s^{-1}, an increase of speed of 50%. So there's a surprise. On a circular route on the flat, working at a constant rate I'll get around faster with a constant wind than with no wind. To achieve this I must pedal full-on (100 W) with the wind behind and not be tempted to sit back and rest then. Of course this doesn't take any account of the kind of buffeting that a real, unsteady, wind introduces.

On reflection, perhaps the result is not so surprising if we go back to the question of the 'equivalent hill' introduced by a wind. For half a circular path, there is some component of the wind behind and for half it is ahead. A wind behind is equivalent to a downhill that is steeper than the uphill of the wind ahead. Hence the introduction of a wind is equivalent to the finishing point being lower than the starting point and the path being effectively downhill a bit. From this viewpoint, it's not surprising that one can go round a bit faster with a wind.

You might think that the same would apply to athletic track runners but I think races are slower with a wind present because athletes can't move their legs fast enough to get the full benefit of the wind behind them. In effect, there is internal friction within the athlete and all my sums have no internal friction in the powerhouse (the bike) and I'm assuming the bike has gears enough so that there is no need to turn one's legs much faster to achieve higher speeds with the wind behind. That's something to think about. Runners have no gears. You are welcome to develop my concept of telescopic stilts, easily adjustable by the wearer while in motion - effective gears for runners and walkers!

y) This is an after-thought.

Quite recently, as I write, I had the experience of walking up the steepest street in the world (according to the *Guinness Book of Records*). At the time it was Baldwin Street in Dunedin and it's 1 in 2.86. It is impressively steep (35%). On the spot it was obvious that I couldn't have cycled up but it got me wondering if a fit cyclist could. There are just two important factors involved: how slowly can I ride without falling off and how much power can I provide?

Supposing the slowest speed I can cycle is v =3 mph (4.8 km per hour, 1.34 m s^{-1}) and for the time necessary to climb I can exert a power of p = 200 W, twice my average power. Air resistance isn't an issue here, only gravity. If θ is the slope of the hill (physicists like their Greek letters) then in 1 second the power I exert raises me a height $h = v\sin\theta$ (speed v in m s^{-1}). This gives me an extra gravitational energy of mgh. Hence $p = mgv\sin\theta$. This relationship allows me to work out the greatest slope of hill I can climb. $\sin\theta = p/(mgv) = 200/(100\times9.81\times1.34) = 0.152$. The equivalent slope is $\tan\theta$ (0.153, almost the same) or 1 in 6.5, a lot less steep than Baldwin Street. That agrees with what my eyeballs were telling me.

How much power is needed to climb Baldwin Street at 3 mph? The relationship above tells us that $p = mgv\sin\theta$, with $\tan\theta = 1/2.86$. $\theta = 19.3°$ and hence $\sin\theta = 0.33$. Hence for a mass of 100 kg, $p = 100\times9.81\times1.34\times0.33 = 434$ W. This is within the compass of a professional cyclist. A professional will also be more adept at balancing a bike at slow speeds so could go slower than 3 mph if needed.

z1) If I free-wheel down a hill and on the return journey pedal up the same hill, do I use more energy or less energy on the hill than travelling the same distance on the flat?

The energy used is equal to the power exerted, P, times the time, t, it is exerted for. Suppose, as usual, the same power is available for going uphill as for going along the flat. No energy is used free-wheeling downhill so the energy used free-wheeling and going back up will exceed the energy of a flat journey if the time taken going uphill is more than twice the time taken going on the flat. This will be the case for all slopes θ where the speed going uphill is less than half the speed along the flat.

The 'break-even' slope (the one where the hilly journey and the flat journey take the same energy) is the one where the uphill speed is half the flat speed. The flat speed is given by $P = 0.5v^3$ (0.5 is the assumed drag coefficient). Hence flat speed $v = (2P)^{1/3}$. On the break-even slope the speed will be $0.5(2P)^{1/3}$ and as before on going uphill the power relation is $P = mg\sin\theta\times0.5(2P)^{1/3} + 0.5\times2P/8$. This determines the break-even slope θ since $\sin\theta = (2P)^{2/3}\times(7/8)/mg$.

For P = 100 W and m = 100 kg this gives a slope of 3%. For lesser slopes, the speed is greater and hence the time to cover a given distance is less. The power is the same and hence less energy is expended going down and up than for the same distance as on the flat; for slopes greater than the break-even value, more energy is expended.

If I cycle uphill using a power α times the power on the flat, then some further lines of calculation will show that the break-even speed is now $\alpha(2P)^{1/3}/2$ and the break-even slope is given by $\sin\theta = (1 - \alpha^2/8)\times(2P)^{2/3}/mg$. So if I try to use more power going uphill ($\alpha > 1$), the break-even slope is less and although I'll take less time going uphill I end up using more energy.

z2) *Watching the World Championship cycling road races recently got me reflecting on questions of optimisation. In the interests of sustainable performance I have emphasised working at a constant rate over changing terrain, within the limits of the bike's gears. Will that strategy get me from A to B fastest?*

The answer is 'no'. Consider only a circular course, or at least one that starts and finishes at the same height. If the course were flat I could cycle at a constant power, at a constant speed and arrive in a given time. With hills (up and down) if I could maintain the same speed all the time over the same distance I would arrive at the same time which, from a comment made in question 1, is sooner than I would arrive using the constant power strategy. I would also use the same amount of energy. This is because for every hill climbed I gain potential energy but I get it all back when I go downhill, assuming I never use the brakes. Brakes waste energy so what I'm going to say about energy assumes no braking. The constant speed strategy requires working extra hard going uphill but less hard going downhill. The constant speed strategy with hills is the minimum energy strategy (this isn't obvious) but it works only for shallow hills. With steeper hills I will free-wheel downhill faster than my speed on the flat and some of the energy saved is spent pushing air quickly out of the way and is not available to maintain my speed uphill. However, even with steeper hills by varying my effort I can arrive sooner than I would using the constant power strategy.

It's easier to investigate the effect of hills in the approximation that gravity is the dominant force controlling speed. This is pretty well true if the hills are steeper than 3%. In this case the energy needed to climb a given altitude h is independent of slope. If I can exert a given power then the speed I make and the time taken to climb that altitude will depend on the slope but the total energy used will be just Mgh. Even better is the fact that I get all this energy back again, not a Joule less, when I drop down the same height h. The energy is of course transferred to the air as I go down but at least I don't have to supply the work done to stir the air. Gravity does the work. This downhill energy recovery is just the reason why it's worth a passenger plane expending a lot of energy climbing 10 km to where air resistance is significantly less.

The energy moving forward at high altitude is less and the climbing energy is recovered on the 'downhill' glide. On a bicycle, unfortunately some potential energy will be lost in the interests of safety if I have to brake at sharp corners. You can also work out that if I free-wheel down a hill and put the energy I have saved into doubling the power in going up an equivalent slope for the same length of time that I free-wheeled, then I'll use the same amount of energy as if I had pedalled at constant power but I will arrive sooner. The reason is basically that pedalling downhill doesn't increase my speed much but if I use the energy instead to increase my speed uphill then this makes quite a big difference to my uphill speed and hence, overall, I arrive sooner. This is therefore a better strategy for using one's energy if time is important, better that is than cycling at constant power. The downs and ups don't need to follow each other immediately. In a circular route that starts and finishes at the same place there will always be a down for every up.

z3) *How can I estimate the power I have available?*

Since I haven't got a power meter on my bike, you might wonder where the round figure of being able to sustain a power of 100 W comes from. It started off as a guess that produces results consistent with my experience. However, on a typical morning or afternoon ride of 25 km or so I'm aware that I do put in more effort going uphill, effort that I couldn't sustain for the whole trip. It's easy to reverse the sums I have done and use the equilibrium speed I can sustain to estimate my power output. On the flat, the power needed to cycle against the wind is found as simply $0.5v^3$, as much earlier, but on the country roads I cycle on it is hard to find more than 100 m of flat and that is likely to be full of micro-undulations, so not truly flat, and even on calmer days the residual wind is likely to be variable. I think a more reliable figure and a more interesting figure is to ask what power I have available to climb hills.

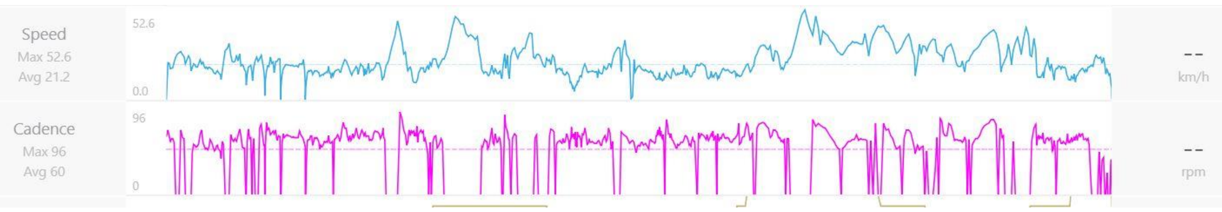
To begin with, take it that on a calm day the power needed to climb a hill of gradient p% is that used against gravity alone. This is reasonably so for gradients of 4% or more (I said 3% earlier. It depends how approximate you want to be). To climb at speed v, the power P in Watts = $mgpv/100$, where v is in m s^{-1}. $v(\text{m s}^{-1}) = v(\text{km h}^{-1})/3.6 = v(\text{mph})*0.447$. Hence, for example with my figures, $P = 100*10*p*v/(100*3.6) = 2.78*p*v$ with v in km h^{-1}. With a bit of rounding you can make a mental calculation while on the bike. I tried this on several hills today. One is a fairly steady incline of 5% lasting about 0.5 km. I could sustain 12 km h^{-1} at climbing effort. I should add that I had two fairly full paniers on my bike, creating enough total mass to round the power needed to a bit more than $3pv$. The power works out at 200 W. I got a similar figure on other hills. It is mid-February, the temperature was 3°C. I was glad of the freewheel down the other side and couldn't have sustained the effort. Maybe I could when younger. If I exert this same power climbing other steepish hills, then notice that pv is a constant. Twice the slope, half the speed, and so on. Adding in extra power to push away the air in front of me at 12 km h^{-1} doesn't change the result much: $0.5v^3 = 18.5$ W. Adding that in makes little difference to the result.

The figures deduced in all the previous questions ring pretty true to life. The analysis is based on the assumption that the internal friction in the bike and rolling resistance are negligible in comparison with the forces of wind and gravity. For this you can thank our 19th century forebearers for the invention of ball-race bearings on wheels, crank and handlebars. It pays to keep your bike clean, oiled, tyres well pumped up and nothing rubbing on the wheels.

Pneumatic tyres were invented and tested by Robert Thomson of Stonehaven (a few miles down the road from where I live) in the 1840s, decades before Dunlop. You might think that a soft tyre would give you a better ride but think of dropping a tennis ball, which is rubber filled with air, onto a hard surface. It loses height with every bounce. At each bounce when the rubber recovers from being squashed some kinetic energy is lost (as heat). The same is true of a bicycle tyre each time it is deformed in contact with the ground. The harder the tyre, the less the deformation and the less the loss of energy in this way. The main job a tyre does on a road bike is gripping the road, not smoothing out the ride. If you want shock absorption, then it should be suspension built into the forks. Under inflated tyres are more likely to deform when you hit an obstacle. The greater the deformation the more stress it puts on the seal and the inner tube, if there is one. Hitting the front edge of a pothole or ridge on a poorly maintained bit of road at speed is a recipe for getting a snake-bite puncture if your tyre is not hard. I try to keep my tyres at 80 psi (5.5 bar). The racing fraternity use greater pressures.

Getting back to the topic of assumptions, I've also taken it that internal friction in the cyclist isn't an issue because the examples have me cycling at a constant cadence and using my gears on hills and with the wind. If I start pedalling much faster, e.g. with a cadence of over 100 turns per minute, then most of the effort I can muster goes into heating me up and a smaller fraction is available to push the bike forward.

Although the numbers in the sections above have generally assumed constant power, with the gears being changed to produce a roughly constant cadence, in practice this doesn't happen accurately. The graphs below show speed and cadence over a 26 km a route I often take around my local roads. The cadence (purple) falls to zero on many occasions due to junctions, traffic and most often the need to limit my speed downhill to a value consistent with the road surface and corners. My speed is in blue in the range 0 to 52.6 km h^{-1}.



Notice in summary the different ways speed v enters the physics, depending on the circumstances. The power needed to travel at a speed v on the flat in the absence of wind increases as v^3. The power needed to travel at a constant speed into a headwind v depends on v^2. The power necessary to go up a reasonable hill depends mainly on v, only in this case the multiplying factor is rather large and depends on the slope of the hill.

It's also worth remembering that the underlying model of what is happening that is being used in the analysis is very simple and reality is more complicated. You may have noticed that when I'm travelling at speed v the whole bike isn't going at the same speed. The tops of the wheels are travelling at $2v$ and the bottoms are stationary. The front, rear and hubs are travelling at v but if air resistance depends on v^2 then it will not quite average out to the same as if the whole wheel was travelling a speed v. The difference is small in the overall scheme of things. Racing folk, though, concern themselves about small details. For example, the more spokes there are in your wheel the more air resistance they create as they flash around. If you can afford stiff light wheel rims that need fewer spokes to keep them round then you'll reduce the air resistance of your bike, though remember that wheels have to do other things as well such as maintain their shape after hitting a pothole at speed.

Another factor not explicitly mentioned is the rotational energy of the wheels. When I'm bowling along at speed v, the kinetic energy of me and the bike is not just $½mv^2$, where m is our combined mass but it is $½mv^2 + ½(I_f + I_r)\omega^2$, where I is the moment of inertia of a wheel (front and rear) and ω its angular velocity. Now $\omega = v/r$, where r is the radius of a wheel and on the assumption that most of the mass m_w of a wheel is in its rim and tyre then $I \approx m_w r^2$. This boils down to the rotational energy of each wheel adding an amount that is approximately $½m_w v^2$. In other words, the kinetic energy of a moving bike is that of the combined mass of rider and bike plus an additional amount determined by the mass of the wheels' rims and tyres. It's not a lot since in my examples *m is 100 kg* and m_w is about 1.5 kg for a good aluminium wheel. None of this matters when considering the conditions that apply at constant speed, as in most of these pages, but it is relevant to the effort required to change speed.

Other simplifications have been that a wind against me produces a force parallel to the ground and that the pedalling force is steady. The wind will also produce a force with a vertical component, probably upwards if I'm leaning forward, that effectively alters my downward force on the road but not my mass. The pedalling force exerted isn't a constant but varies periodically from a minimum to a maximum as the pedals go round a quarter of a turn. This fact has led to the introduction of the elliptical chainwheel and variants that are said to make more efficient use of converting leg motion to power output, smoothing the profile of leg force exerted with time on the pedals. And so on.

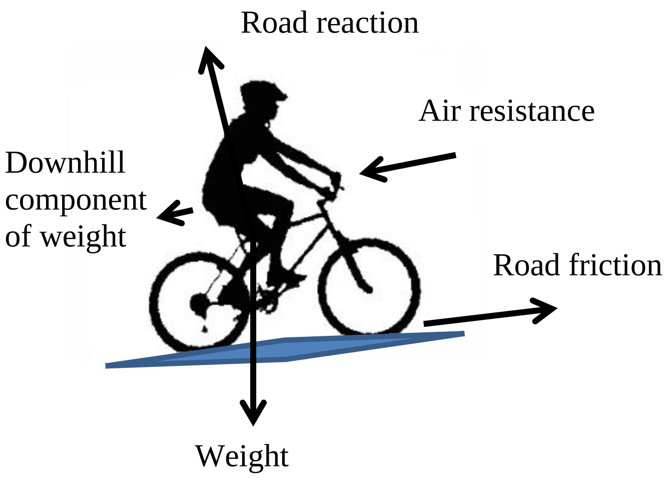
So in real life there are complications but the simple model used above captures the gist of what is happening. It can take some time and effort to know whether a more refined model will make a difference of 1%, 5%, 10% or more. The same approach is used again and again in Physics. Capture the gist of what is happening and then add refinements to the argument if it looks as if they might be relevant. I would expect someone analysing the physics of cycling in order to support top-level racing to be much more sophisticated than I have been but for the purposes of leisure cycling, the arguments above give very plausible results.

Appendix on forces and motion

Imagine drawing on a blank sheet of paper a sketch of yourself on a bicycle. There you are, virtually suspended in space. Now start pedalling. The rear wheel goes round but you wouldn't expect the bicycle to move. It can no more go forward than you can lift yourself up into the air by hauling on the belt round your waste. To move you and the bike from rest needs a force from outside. Pedalling in space creates no outside force on you and the bike. The bicycle needs a road to sit on. The road provides that outside force for the bicycle. That force is the forward reaction of the road as the bottom of your rear wheel tries to move backwards because of your pedalling. Force produces acceleration and your bicycle starts away from rest, thanks to the road or, to be more exact, the friction provided by the road.

If you think about it there are other forces acting on you and the bike. By far the biggest is the weight of you and the bike. If your combined mass is 100 kg then that weight is about 981 Newtons. On the flat this force doesn't cause any motion since it's opposed by an equal and opposite upward reaction from the road. This is just as well, otherwise you would disappear into the ground as if on quicksand. On a slope, though, a fraction of your weight that is not counteracted by the reaction of the road acts down the slope. This is what helps going downhill and hinders going uphill. On a banked road the reaction may have a sideways component too, which can either help in cornering or cause balancing complications. The air provides a further outside influence, a force that depends on the direction and strength of the wind on you. A head-on force provides resistance to motion. A sideways force (due to wind) may affect your balance. So there we have it. The forces affecting the going of the bike come from the road, from gravity and from the air.

If you look at the picture you'll see that the road friction and air resistance don't act in the same line. They therefore produce a twist on you and the bike tending to lift the front wheel up. The countering twist that keeps both wheels on the road is produced by the road reaction at the back wheel and your weight. Obviously the road reaction is exerted on the tyres where they contact the ground but the reaction at the front wheel is less than that at the back. The line of action of your weight needs to be far enough forward of where the rear wheel contacts the ground to produce enough twist to stop yourself falling backwards. The twisting effect of road friction is easily seen on a powerful motor bike where the front wheel lifts off the ground if the throttle is opened quickly.



In addition to the three outside influences on the bike, the effort you have to put in to make the bike go is affected by a number of other factors that can be bundled together as 'internal friction'. This includes friction in the moving bits of the bicycle, the effort required to partly flatten the tyres where they contact the road and even the effort required to change the shape of your clothes as you pedal. Internal friction just generates a little heat and sometimes some squeaky sounds. Its only contribution to the motion of the bicycle is to sap some of the work you put into pedalling.

Finally, here, I'll mention that using an exercise bike is not the same as riding a 'real bike'. I can't speak from experience since the gym teacher I had at school, an ex-army man, succeeding in instilling in me a life-long distaste for the gym. However, it's clear that in all of the above one can treat the bike plus rider as a unit that interacts with the outside world. The force working against the unit that must be equalled in order to travel at constant speed is a force parallel to the road, horizontal on the flat. On an exercise bike, the 'bike' is the 'outside world' and the force of interest is the vertical reaction at the pedals. There are no horizontal forces of relevance. An analysis of the forces is an analysis of the variable force on the pedals as their positions and the angle of your leg change.

Appendix on the difference between cycling and running

It doesn't take advanced science to tell us that cycling is more energy efficient than walking or running. A few supremely fit individuals in the world can run a marathon in a little over two hours. Hundreds of thousands of old-age pensioners can cycle the same distance faster. What's going on?

Millions of years of evolution have made mankind one of the best endurance running animals on the planet, so we must do that pretty well. We do, but at every step we use some energy. In walking and particularly in running our centre of mass rises as we spring forward successively on the balls of our feet. It's not much, 2 or 3 centimetres at a time, but it takes energy. At around 1500 steps per kilometre a rise of 2.5 cm per step adds up to a total rise of about 40 m. Over the marathon's 42.2 km that adds up to a rise of over 1500 m, more than the height of Ben Nevis from sea level. To raise a mass of 70 kg (less than my mass) up 1500 m in 2 hours requires working at a rate of 140 W. When each step is finished almost all the potential energy gained is lost. Basically we don't bounce up again like a rubber ball but have to re-create the upward movement on the next step. That's 140 W of power wasted over and above the power needed to push one through the air while running. There is a further loss if one thinks of the upward kinetic energy of our swinging legs. Most of it is lost as our leg hits the road again to make the next step and the soles of our shoes heat up a little.

Sitting on a bike there is very little up and down movement of our centre of mass as we pedal. This is an immediate gain. There's no heating up of our shoe leather at every pedal stroke either. The energy used is that of pushing ourselves through the air. A runner has this too of course. There is a big bonus as well on a bike when cycling up hills. The energy expended working against gravity is often completely recovered when going down the other side and used to propel you forward. The only energy 'lost' is that dissipated if you have to brake. Climb up a hill on your legs and you still have to propel yourself forward going downhill, albeit it's easier than the climb. Most of your potential energy is 'lost' as you brake on each step with your foot. You lose even more if you carry a rucksack. There's also the physiological difference of sitting on a bike and standing when walking or running. Although merely standing or sitting no work is done as work is defined in physics, we all know that standing is more tiring than sitting. Energy is in fact expended internally to keep one standing that is not expended in sitting.

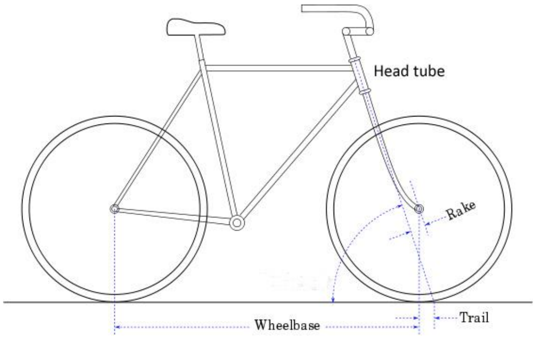
In short, cycling is a much more efficient means of transport than walking or running. A few exceptional individuals can cycle 750 km in 24 hours; the equivalent runners manage 250 km. On this very simple comparison of the limits of human ability, cycling is about three times more efficient than running.

Appendix on why does a bicycle stay upright when moving?

Staying upright on an object that falls over when you are given it might seem odd if it weren't so familiar. If you told a 4-legged animal it is possible to stand with only two legs, they might be equally amazed, if they had the cognitive power. 3 legs, maybe; but only 2 legs? I have to admit to not thinking very much about why a bicycle stays upright in spite of having had a bike since childhood – perhaps not as young as many learn to ride these days but young enough to career around the local village streets and country roads. I guess that the blacksmiths and carpenters who made the original 'dandy horses' (with no pedals) and bikes were quite surprised that it didn't take much speed for you to stay upright on them even though they fell over immediately when they were stationary. Surely it's the rotation of the wheels that makes the difference? Is bicycle stability like that of toy hoops that fall over when still but can be kept upright at some speed? Personally I can keep upright when doing about 5 km h^{-1} up a steep hill with some wiggling of the front wheel but need to be travelling about 20 km h^{-1} before I can take my hands off the handlebars of my touring bike and keep going without tipping over.

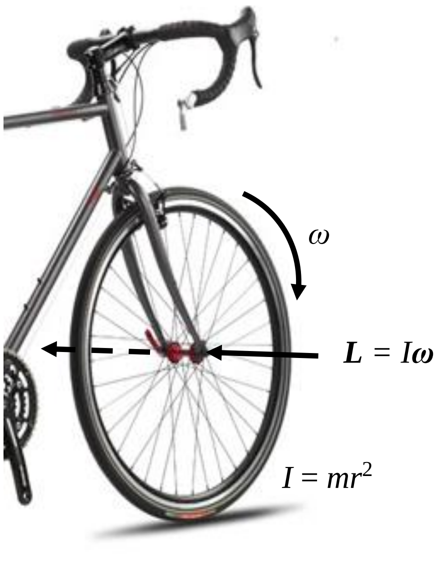
When we can ride a bike we think little about doing it, yet I remember it took months of trying before one of our sons could manage on two wheels without stabilisers. People have been writing technical articles about how bicycles stay upright at least since the 1869 series by the famous Scottish engineer Macquorn Rankine. Curiously enough, it’s still quite a hot topic. Even though a bike is an everyday object, it’s clearly not that simple and there are several things going on

So how does a bicycle stay upright? The following few pages are my take on it, derived from basic principles. If you stick it out I hope it makes sense. There are two factors that contribute to a bicycle staying upright – one is the rider and the other is forward motion. It is possible to balance a bike that is stationary or almost so by turning the handlebars, which is what many cyclists do when pausing at a road junction. The ease of doing this depends on the steering ‘trail’ that is built into the bike. The handlebars, forks and front wheel twist about an axis that is defined by the head tube. When the handlebars are turned, every bit of the bike on that axis stays in the same line. However, that axis doesn’t intercept the place where the front wheel touches the ground but extends ahead of it by an amount called the *trail*. The diagram here, modified from wiki, shows what the trail is. When the handlebars are turned to the right, the point of contact of the front wheel with the road moves left. Now for balance the centre of gravity of rider and bike must lie on the line between the points of contact of front and rear wheels. If you start to tip slightly to the right so your centre of mass moves right, then turning the handlebars left can correct the imbalance. If you’re absolutely stationary then the point of contact of the front wheel will be scraped along the ground, which isn’t good so a very slight forward motion makes this manoeuvre easier. The typical trail on bikes isn’t that big. My touring bike has a trail of about 50 mm so this method of maintaining static balance works only for small wobbles of the rider and you need quite big turns of the handlebars. It is not the main method of balancing when the bike is moving at a reasonable speed. I think it does work because it’s easier and more controllable to alter the line of action of the upward force exerted by the road than to make precise sideways changes to the location of the centre of gravity of rider and bike. You can see now why bikes have curved front forks that aren’t straight like the back forks, and have done since the 1890s. It is to control the trail.



(It’s a different effect that makes it easy to steer a bike when pushing it from the side by the saddle only. To steer the bike towards you all that is needed is to tilt the bike towards you and the front wheel comes over and the bike starts to move round onto your side: the reverse to steer the bike away from you.)

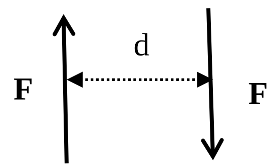
Before leaving the topic of trail I should mention that positive trail as just described provides steering stability. This isn’t an issue of balance directly but does concern ‘ride ability’. As you are travelling forward, if the handlebars move slightly to the right, say, either due to you or an irregularity in the road, then with positive trail the friction between front tyre and road will have a sideways component that twists the handlebars back to the straight ahead position. For just the same reason, both motorbikes and cars are designed with positive trail.



Now we get to the subject of the conspicuous increase in stability of a bicycle moving at normal speed. There are some more concepts that we really need but you may be familiar with them already. The difference between a stationary wheel and a turning wheel is that the turning wheel has *angular momentum*, $\boldsymbol{L}$. Angular momentum is the rotational analogue of linear momentum, which is a product of the body's mass, m, and velocity $\mathbf{v}$. Angular momentum is generally more complicated than linear momentum but for the simple case of a rotating bicycle wheel the angular momentum is the product of its *moment of inertia*, I, about the axis of rotation and its *angular velocity*, almost always represented by the symbol $\boldsymbol{\omega}$ (*omega*, the last letter of the Greek alphabet). In size ω is given by the *number of radians per second* a spoke is rotating. Like linear velocity, $\boldsymbol{\omega}$ is a vector with a direction along the axis of rotation.

To get a feel for size, my bike travels 2.155 m for one rotation of its wheels. Hence 20 km h^{-1} is 9281 rotations per hour or 2.578 rotations per second, equivalent to an angular speed of the wheels of 16.2 radians per second. The moment of inertia, I, plays the role of mass and is given for a spoked wheel by the mr^2, where r is a radius that we'll take as the distance from the axis of the wheel to the outer edge of the rim, 0.32 m in my case. I'll also take the mass of the wheel, m, as 1.75 kg in round numbers, including the tyre, giving $I = 1.5\times0.32^2 = 0.18$ kg m^2. Spare a thought for the youngster's bike. The one in my shed has a wheel rim distance of 0.11 m and mass 0.8 kg. The moment of inertia is therefore 18 times smaller. These concepts are well known to physics students. In short, turning wheels have an angular momentum that is readily calculated. Oh, there's one more thing. Angular momentum is a directed quantity, a vector, like linear momentum. For the bicycle wheel it is directed along the axis of rotation of the wheel, normally parallel to the road and pointing from right to left as seen by the cyclist.

*Torque, $\boldsymbol{\tau}$, given by a pair of forces **F**, is the product Fd*

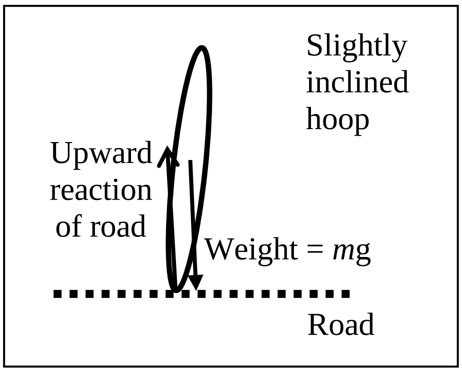


The torque is directed into the page

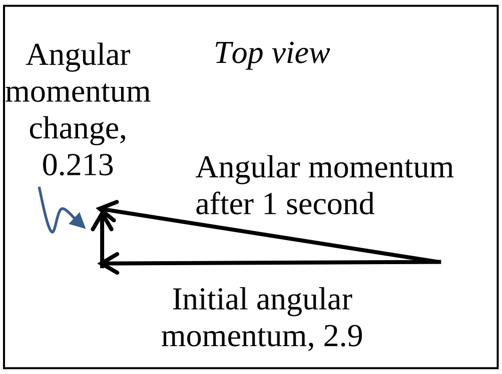
The picture to have in mind is that with your bike upright and going forward the linear momentum is in the direction of travel but the angular momentum of the wheels is at right angles, directed as just stated. Imagine the worst-case scenario that you are about to hit the road on your side while travelling forward at some speed. The angular momentum of the wheels will now be pointing right up in the air. What circumstances could lead to that? We know that linear momentum is changed by applying a force, such as brakes to slow the wheels or air resistance to slow the bike. To change angular momentum an external twist (or *torque*, τ, to use the right technical word) must be applied. If there is no torque then angular momentum is conserved and doesn't change.

The result of applying a torque is to change the angular momentum according to the relationship $d\boldsymbol{L}/dt = \boldsymbol{\tau}$. In words, the rate of change of angular momentum is given by the applied torque. What this implies is that if a torque of size τ is applied quickly for a short time Δt, then the angular momentum L will change abruptly by $\tau\Delta t$. The equation also includes the direction of the change because it is a vector equation. The torque $\boldsymbol{\tau}$ is directed along the axis of rotation of the twist. The change in angular momentum will be in the same direction. It is this relationship that produces the counter-intuitive behaviour of spinning objects. To move the angular momentum from horizontal to up in the air requires an upward pointing torque. Such a torque is provided by horizontal forces. The weight of bike and cyclist is a downward force and the reaction of the road to this weight is upward so even if these two forces get out of alignment then they aren't going to change the angular momentum in a detrimental way.

I'll explain what's going on more fully by pondering why a rolling hoop doesn't fall to the ground quickly. Clearly a bicycle wheel is a bit like a hoop and we know that if a hoop is set rolling it stays upright a lot longer than if it isn't rolled. Why is this?



What's going on with a hoop, or the even more mysterious case of a gyroscope, looks complicated and some people make it sound complicated but it's not really so. I'll make an analogy with linear motion. The Sun pulls directly on the Earth. If the Earth were stopped in its tracks then the Sun would pull it directly inwards and we'd all be burnt to a crisp. However, the Earth has a linear orbital velocity and the very same pull of the Sun just alters the direction of this velocity, pulling the Earth round its orbit a bit but not making it fall into the Sun. The velocity of the Earth (that it was given when it was formed) is what prevents the attractive force of the Sun from causing the Earth to fall into the Sun. The reason the rolling hoop stays upright and doesn't fall to the ground is analogous to the reason the Earth doesn't fall into the Sun. If the hoop is caused to lean a little then if it had no angular velocity, the torque of its weight and the reaction of the road would make it fall over quickly. This is the falling into the Sun equivalent. If it has angular momentum, then the torque simply changes the direction of the angular momentum and the hoop turns its direction of roll but doesn't fall. The Earth is travelling around in the next best thing to a vacuum and has very little to slow it down. The hoop, though, slows in its rolling due to air resistance and so on. Its angular momentum gradually reduces. It wobbles on any road irregularities and finally topples over as it loses its own rotational angular momentum. I'll put some numbers into this argument with my bicycle wheel.



A bicycle wheel on its own when rolled behaves just like a hoop. Take one of my wheels with angular velocity 16.2 radians per second and moment of inertia 0.18 kg m^{-2}. Its angular momentum is therefore the product of these, 2.9 (kg m^2 s^{-1}). Taking the wheel axle to be 0.355 m off the ground (yes, I got out the tape measure), when the wheel is tilted 2 degrees (an example figure) then its weight and the reaction of the road are out of line by 12.4 mm and hence the torque is 1.75*9.81* 0.0124 = 0.213 (kg m^2 s^{-2}). This torque is horizontal and in the forward direction. It moves the angular momentum of the wheel forward, resulting in the wheel turning round to the right **but not falling**. In one second the angular momentum of the wheel is changed by 0.213 in the forward direction. Since angular momentum is a directed quantity, then the vector diagram here shows what is happening.

The hoop effect, if I can call it that, doesn't right the bike but left unchecked it would turn the bike round over a time measured in many seconds. This gives plenty of time for the rider to make good any off-balance by adjusting his or her weight and making handlebar corrections. If you lean to the right, then the natural compensation is to press down with your left arm and tilt the bike over to the left to keep the centre of gravity central. Since the bike isn't going to topple, there is time to do this and the faster one goes, the more time there is and the more stable the bike seems. So it's not the bicycle that stays upright but the bicycle and the person. After many hours of practice, the corrections become second nature and almost invisible both to the cyclist and anyone watching. The wheels aren't the only thing turning on a bike, for the chainset, chain, pedals and cyclist's legs all turn but the wheels will have the biggest moment of inertia so these other things will only make a minor contribution. You don't notice any change in stability when you stop peddling.

The idea of torque changing angular momentum is seen in the act of leaning over to corner a bike with hardly any turning of the handlebars. This works well if you're cornering at over 30 km h^{-1}. Do this on a bicycle, not a tricycle! You lean right. Your weight is now over the side but the balancing reaction of the road holding you up is still a vertical force coming up through the points where the tyres meet the road. These two vertical forces are out of alignment with your weight and the resulting torque is horizontal and facing forward. The angular momentum of the wheels is itself now twisted forwards which means the wheels move round to the right, as needed to round the corner. To move round in a curve a sideways force is needed and this is provided by the sideways friction of tyre on road. If you tilt too far, or try to turn too sharply, or there is loose grit, wet mud or leaves on the road then the friction force won't be enough and you will crash onto the road, as every cyclist has probably done once or twice. Not to be repeated if possible.

Finally, at a good speed you can take your hands off the handlebars and steer by adjusting only your balance on the saddle. Two effects make this possible. First is the steering stability due to the positive trail that corrects wobble of the steering. Second, the hoop effect, sometimes called gyroscopic stability, gives you enough time to correct an involuntary lean without the bicycle falling sideways.

If I'd taken the trouble to work this out before our sons learnt to ride, then I'd have realised that a small bike is harder to ride than a full-sized one. From my earlier figures, the angular speed of the youngster's wheels for a given speed across the ground is 3 times larger but the tiny moment of inertia of the youngster's wheels means that their angular momentum is about a sixth of that of the full-sized bike for the same speed. As a learning cyclist they will be at least half as tall as an adult. Perhaps with some scientific insight I could have cut down the amount of time it took them to find out how to ride a bike instinctively.

[At the very beginning of this piece, many pages ago, I said it would be all about travelling in a straight line and not about cornering. The topic of *balance*, though, inevitably introduces changing direction. I'll add here that changes in speed are produced by acceleration **in the direction of travel**. Changes in direction are produced by accelerations **at right angles to the direction of travel**. These two aspects of acceleration are always quite separate. The acceleration needed to change direction increases with the square of the forward speed and also with the sharpness of the corner (inversely as the radius of the bend). The sideways acceleration is provided by a sideways force, generated by the friction between tyre and road, as said above. If the road is at all slippery, it helps to 'straighten out the corner' as much as other traffic and safety allows but it makes a bigger effect to slow down, because of the dependence of sideways acceleration on the square of the speed. For example, slowing down by 30% roughly halves the sideways force needed to turn. I'm updating this piece in autumn, the season that can see plenty of leaves on wet corners. Anyone taking such a corner at summer speeds is likely to find themselves sliding along the road losing skin, or worse. Slowing down quickly needs good brakes. My bike has rim brakes, better than very early brakes that rubbed on (solid) tyres but nonetheless easily rendered less good than when new. I suspect that in a few decades all bikes will have disk brakes.]

JSR