Hypersequent-based Argumentation: An Instantiation in the Relevance Logic RM

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Abstract. In this paper we introduce *hypersequent-based* frameworks for the modeling of defeasible reasoning by means of logic-based argumentation. These frameworks are an extension of sequent-based argumentation frameworks, in which arguments are represented not only by sequents, but by more general expressions, called *hypersequents*. This generalization allows us to overcome some of the weaknesses of logical argumentation reported in the literature and to prove several desirable properties, stated in terms of rationality postulates. For this, we take the relevance logic RM as the deductive base of our formalism. This logic is regarded as "by far the best understood of the Anderson-Belnap style systems" (Dunn & Restall, Handbook of Philosophical Logic, Vol.6). It has a clear semantics in terms of Sugihara matrices, as well as sound and complete Hilbert- and Gentzen-type proof systems. The latter are defined by hypersequents and admit cut elimination. We show that hypersequent-based argumentation yields a robust defeasible variant of RM with many desirable properties.

1 Introduction

Argumentation theory has been described as "*a core study within artificial intelligence*" [11]. Among others, it is a standard method for modeling defeasible reasoning. Logical argumentation (sometimes called deductive or structural argumentation) is a branch of argumentation theory in which arguments have a specific structure. This includes rule-based argumentation, such as the ASPIC$^+$ framework [26] and methods that are based on Tarskian logics, like Besnard and Hunter's approach [12], in which classical logic is the deductive base (the so-called *core logic*).

The latter method was generalized to *sequent-based argumentation* [4]. Here Gentzen's sequents [19], extensively used in proof theory, are incorporated for representing arguments, and attacks are formulated by special inference rules called *sequent elimination rules*. The result is a generic and modular approach to logical argumentation, in which any logic with a corresponding sound and

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complete sequent calculus can be used as the underlying core logic. A dynamic proof theory as a computational tool for sequent-based argumentation was introduced in [6]. This allows for reasoning with these argumentation frameworks in a fully automatic way.

In this paper we further extend sequent-based argumentation to *hypersequents* [7, 22, 24]. This is a powerful generalization of Gentzen's sequents which was used for providing cut-free Gentzen-type systems for the relevance logic RM, its 3-valued version RM_3 and the modal logic S5. It allows a high degree of parallelism in constructing proofs and has some applications in the proof theory of fuzzy logics (see, e.g., [21]). In the context of argumentation, the incorporation of hypersequents enables to split sequents into different components, and so different rationality postulates [1] can be satisfied, some of which are not available otherwise.

The usefulness of logical argumentation with hypersequents is demonstrated here on frameworks whose core logic is RM. This logic was introduced by Dunn and McCall and later extensively studied by Dunn, Meyer [17] and Avron [7, 9] (see further [3, 18]). In [18, p.81], RM is regarded as "*by far the best understood of the Anderson-Belnap style systems*". The basic idea behind this logic (and relevance logics in general) is that the set of premises should be 'relevant' to its conclusion. This way some problematic phenomena of classical logic, such as the paradoxes of material implication, are avoided. In addition, it was shown that RM is semi-relevant, paraconsistent, decidable and has the Scroggs' property [3, §29.4]. Furthermore, RM has a clear semantics in terms of Sugihara matrices [3, §29.3] and sound and complete Hilbert- and Gentzen-type proof systems are available for RM (see, e.g., [7, 9]). The latter admit cut elimination and are expressed in terms of hypersequents, a fact which makes RM particularly suitable for our purpose.

We will show that hypersequent-based frameworks, with RM as the core logic, satisfy the logic-based rationality postulates from [1] and non-interference and crash-resistance from [14]. In particular, this proves that such formalisms avoid the problem of logical argumentation raised in [15], and further discussed in [2] (to which we shall refer below). A byproduct of our approach is a defeasible variant of RM with many desirable properties.

The rest of the paper is organized as follows. The next two sections contain some preliminary material: in Sect. 2 we recall some basic notions of sequent-based argumentation, and in Sect. 3 we review the notion of hypersequents and the logic RM. Then, in Sect. 4 we extend sequent-based argumentation frameworks to hypersequent-based ones, and in Sect. 5 we consider some properties of these frameworks, instantiated in RM. Finally, in Sect. 6 we conclude.

2 Sequent-based Argumentation

We start by recalling the setting of sequent-based argumentation [4]. Throughout the paper we consider propositional languages, denoted by $\mathcal{L}$, that may contain connectives in $\{\neg, \wedge, \vee, \supset, \leftrightarrow\}$. Sets of formulae are denoted by $\mathcal{S}, \mathcal{T}$, finite sets of formulae are denoted by Γ, Δ, formulae are denoted by ϕ, ψ and atomic formulae are denoted by p, q, all of which can be primed or indexed. We denote by $\bigwedge \Gamma$ (respectively, by $\bigvee \Gamma$), the conjunction (respectively, the disjunction) of all the formulae in Γ. Furthermore, we let $\neg\mathcal{S} = \{\neg\phi \mid \phi \in \mathcal{S}\}$.

Definition 1. *A* logic *for a language $\mathcal{L}$ is a pair $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathcal{L}$, satisfying, for every $\mathcal{T}, \mathcal{T}'$ in $\mathcal{L}$:* reflexivity: *if $\phi \in \mathcal{T}$, then $\mathcal{T} \vdash \phi$;* transitivity: *if $\mathcal{T} \vdash \phi$ and $\mathcal{T}', \phi \vdash \psi$, then $\mathcal{T}, \mathcal{T}' \vdash \psi$; and* monotonicity: *if $\mathcal{T}' \vdash \phi$ and $\mathcal{T}' \subseteq \mathcal{T}$, then $\mathcal{T} \vdash \phi$.*

As usual in logical argumentation (see, e.g., [12, 23, 25, 27]), arguments have a specific structure based on the underlying formal language. In the current setting arguments are represented by the well-known proof theoretical notion of a *sequent*.

Definition 2. *Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{S}$ be a set of formulae in $\mathcal{L}$.*

- *An* $\mathcal{L}$-sequent *(*sequent *for short) is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulae in $\mathcal{L}$ and $\Rightarrow$ is a symbol that does not appear in $\mathcal{L}$.*
- *An* L-argument *(*argument *for short) is an $\mathcal{L}$-sequent $\Gamma \Rightarrow \psi$,*[3] *where $\Gamma \vdash \psi$. Γ is called the* support set *of the argument and ψ its* conclusion.
- *An* L-argument based on $\mathcal{S}$ *is an L-argument $\Gamma \Rightarrow \psi$, where $\Gamma \subseteq \mathcal{S}$. $Arg_{\mathsf{L}}(\mathcal{S})$ denotes the set of all the L-arguments based on $\mathcal{S}$.*

The formal systems used for the constructions of sequents (and so of arguments) for a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, are *sequent calculi* [19]. In what follows we shall assume that a sequent calculus C is sound and complete for its logic (i.e., $\Gamma \Rightarrow \psi$ is provable in C iff $\Gamma \vdash \psi$). One of the advantages of sequent-based argumentation is that any logic with a corresponding sound and complete sequent calculus can be used as the core logic.[4] The construction of arguments from simpler arguments is done by the *inference rules* of the sequent calculus [19].

Argumentation systems contain also attacks between arguments. In our case, attacks are represented by *sequent elimination rules*. Such a rule consists of an attacking argument (the first condition of the rule), an attacked argument (the last condition of the rule), conditions for the attack (the conditions in between) and a conclusion (the eliminated attacked sequent). The outcome of an application of such a rule is that the attacked sequent is 'eliminated'. The elimination of a sequent $s = \Gamma \Rightarrow \Delta$ is denoted by $\overline{s}$ or $\Gamma \not\Rightarrow \Delta$.

Definition 3. *A* sequent elimination (attack) rule *(or* attack rule*) is a rule $\mathcal{R}$ of the form:*

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \ldots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma_n \not\Rightarrow \Delta_n}\ \mathcal{R} \tag{1}$$

Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic with corresponding sequent calculus C, $\Gamma \Rightarrow \psi, \Gamma' \Rightarrow \psi' \in Arg_{\mathsf{L}}(\mathcal{S})$ and let $\mathcal{R}$ be an elimination rule as above. If $\Gamma \Rightarrow \psi$ is an instance

[3] Set signs in arguments are omitted.
[4] See [4] for further advantages of this approach.

of $\Gamma_1 \Rightarrow \Delta_1$, $\Gamma' \Rightarrow \psi'$ is an instance of $\Gamma_n \Rightarrow \Delta_n$ and all the other conditions of $\mathcal{R}$ (i.e., $\Gamma_i \Rightarrow \Delta_i$ for $i = 2, \ldots, n-1$) are provable in C, then we say that $\Gamma \Rightarrow \psi$ $\mathcal{R}$-attacks *$\Gamma' \Rightarrow \psi'$.*

Example 1. We refer to [4, 29] for a definition of many sequent elimination rules. Below are two of them, (assuming that $\Gamma_2 \neq \emptyset$):

$$\text{Defeat:} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \quad \Rightarrow \psi_1 \supset \neg \bigwedge \Gamma_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2} \ \text{Def}$$

$$\text{Undercut:} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \quad \Rightarrow \psi_1 \leftrightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2} \ \text{Ucut}$$

A sequent-based argumentation framework is now defined as follows:

Definition 4. *A* sequent-based argumentation framework *for a set of formulae $\mathcal{S}$ based on a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and a set AR of sequent elimination rules, is a pair $\mathcal{AF}_{\mathsf{L}}(\mathcal{S}) = \langle Arg_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$, where $\mathcal{A} \subseteq Arg_{\mathsf{L}}(\mathcal{S}) \times Arg_{\mathsf{L}}(\mathcal{S})$ and $(a_1, a_2) \in \mathcal{A}$ iff there is an $\mathcal{R} \in \mathsf{AR}$ such that a_1 $\mathcal{R}$-attacks a_2.*

Example 2. Suppose $\{p, \neg p\} \subseteq \mathcal{S}$. When classical logic ($\mathsf{CL}$) is the core logic, the sequents $p \Rightarrow p$ and $\neg p \Rightarrow \neg p$ attack each other according to defeat and undercut (see Ex. 1). The tautological sequent $\Rightarrow \psi \vee \neg \psi$ is not defeated or undercut by any argument in $\mathrm{Arg}_{\mathsf{CL}}(\mathcal{S})$, since it has an empty support set.

Given a (sequent-based) argumentation framework $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, Dung-style semantics [16] can be applied to it, to determine what combinations of arguments (called *extensions*) can collectively be accepted from it.

Definition 5. *Let $\mathcal{AF}_{\mathsf{L}}(\mathcal{S}) = \langle Args_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ be an argumentation framework and let $\mathsf{S} \subseteq Args_{\mathsf{L}}(\mathcal{S})$ be a set of arguments. It is said that:*
- S attacks *an argument a if there is an $a' \in \mathsf{S}$ such that $(a', a) \in \mathcal{A}$;*
- S defends *an argument a if S attacks every attacker of a;*
- S *is* conflict-free *if there are no arguments $a_1, a_2 \in \mathsf{S}$ such that $(a_1, a_2) \in \mathcal{A}$;*
- S *is* admissible *if it is conflict-free and it defends all of its elements.*

An admissible set that contains all the arguments that it defends is a complete extension *of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$. Below are definitions of some other extensions of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$:*
- *a* preferred extension *of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$ is a maximal (with respect to $\subseteq$) admissible subset of $Arg_{\mathsf{L}}(\mathcal{S})$;*
- *a* stable extension *of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$ is an admissible subset of $Arg_{\mathsf{L}}(\mathcal{S})$ that attacks every argument not in it;*
- *the* grounded extension *of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$ is the minimal (with respect to $\subseteq$) complete extension of $Arg_{\mathsf{L}}(\mathcal{S})$.*

In what follows we shall refer to either complete (cmp), grounded (gr), preferred (prf) or stable (stb) semantics as completeness-based semantics*. We denote by $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$ the set of all the extensions of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$ under the semantics $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{gr}, \mathsf{prf}, \mathsf{stb}\}$. The subscript is omitted when this is clear from the context.*

Example 3. Let $\mathcal{AF}_{\mathsf{CL}}(\mathcal{S})$ be a sequent-based argumentation framework for $\mathcal{S} = \{p, \neg p, q\}$, based on CL, with Ucut as the sole attack rule. Then, as noted in Ex. 2, the sequent $\Rightarrow p \vee \neg p$ belongs to every complete extension of $\mathcal{AF}_{\mathsf{CL}}(\mathcal{S})$, since it cannot be undercut-attacked. Similarly, $q \Rightarrow q$ also belongs to every complete extension of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, since $\Rightarrow p \vee \neg p$ counter-attacks any attacker of $q \Rightarrow q$ that belongs to $\mathrm{Arg}_{\mathsf{CL}}(\mathcal{S})$.[5] This implies that both $\Rightarrow p \vee \neg p$ and $q \Rightarrow q$ are in the grounded extension of $\mathcal{AF}_{\mathsf{CL}}(\mathcal{S})$.[6]

Definition 6. *Given a sequent-based argumentation framework $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, the semantics as defined in Def. 5 induces corresponding (nonmonotonic)* entailment relations*: $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{sem}} \phi$ ($\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{sem}} \phi$) iff for every (some) extension $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, there is an argument $\Gamma \Rightarrow \phi \in \mathcal{E}$ for some $\Gamma \subseteq \mathcal{S}$.*

Example 4. Note that, since the grounded extension is unique, $\mid\!\sim^{\cap}_{\mathsf{gr}}$ and $\mid\!\sim^{\cup}_{\mathsf{gr}}$ coincide (so both can be denoted by $\mid\!\sim_{\mathsf{gr}}$). For instance, in Ex. 3, $p, \neg p, q \mid\!\sim_{\mathsf{gr}} q$, while $p, \neg p, q \not\mid\!\sim_{\mathsf{gr}} p$ and $p, \neg p, q \not\mid\!\sim_{\mathsf{gr}} \neg p$.

3 Hypersequents and RM

Unfortunately, ordinary sequent calculi do not capture all the interesting logics. For some logics, which have a clear and simple semantics, no standard cut-free sequent calculus is known. Notable examples are the Gödel–Dummett intermediate logic LC, the relevance logic RM and the modal logic S5. A large range of extensions of Gentzen's original sequent calculi have been introduced for providing decent proof systems for different non-classical logics. Here we consider a natural extension of sequent calculi, called *hypersequent calculi.* Hypersequents were independently introduced by Mints [22], Pottinger [24] and Avron [7], nowadays Avron's notation is mostly used (see, e.g., [8]). Intuitively, a hypersequent is a finite set (or sequence) of sequents, which is valid if and only if at least one of its component sequents is valid. This allows to define new inference (and elimination) rules for "multi-processing" different sequents. These types of rules increase the expressive power of hypersequents compared to ordinary sequent calculi, and as a result the corresponding argumentation systems have some desirable properties that are not available for ordinary sequent-based frameworks.

To illustrate the application of hypersequents in argumentation, we take RM as the core logic and use a hypersequent calculus for it, as well as extended versions of the attack rules for standard sequents. In this section we formally define what a hypersequent is and present a hypersequent calculus for RM.

3.1 Hypersequents and Inference Rules for Them

Definition 7. *An $\mathcal{L}$-*hypersequent *[8] is a finite multiset of sequents: $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$, where $\Gamma_i \Rightarrow \Delta_i$ ($1 \leq i \leq n$) are $\mathcal{L}$-sequents and $\mid$ is a new symbol, not appearing in $\mathcal{L}$.[7] Each $\Gamma_i \Rightarrow \Delta_i$ is called a* component *of the hypersequent.*

[5] This follows since any attacker of $q \Rightarrow q$ has an inconsistent support.

[6] It is well-known [16] that the grounded extension is unique for a given framework.

[7] The common, intuitive interpretation of the sign "|" is disjunction.

Note that every ordinary sequent is a hypersequent as well. In what follows, hypersequents are denoted by $\mathcal{G}, \mathcal{H}$, primed or indexed if needed. Given a hypersequent $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$, the *support* of $\mathcal{H}$ is the set $\mathsf{Supp}(\mathcal{H}) = \{\Gamma_1, \ldots, \Gamma_n\}$ and the *consequent* of $\mathcal{H}$ is the formula $\mathsf{Conc}(\mathcal{H}) = \bigvee \Delta_1 \vee \ldots \vee \bigvee \Delta_n$. Given a set Λ of hypersequents, we let $\mathsf{Concs}(\Lambda) = \{\mathsf{Conc}(\mathcal{H}) \mid \mathcal{H} \in \Lambda\}$.

Example 5. Like in Gentzen's sequent calculi, hypersequent axioms have the form $A \Rightarrow A$. Consider the right implication rule of Gentzen's calculus LK for classical logic (on the left below). The corresponding hypersequent rule is similar, now with added components (on the right below):

$$\frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B} \Rightarrow\supset \qquad\qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta, B \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \supset B \mid \mathcal{H}} \Rightarrow\supset$$

As noted in [8], many sequent rules can be translated like this. However, it can be that there are two versions (an additive form and a multiplicative form), which are equivalent if contraction, exchange and weakening are all part of the system. Take for example the right conjunction rule of LK. The dual hypersequent rule in an additive form:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \mid \mathcal{H} \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, B \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \wedge B \mid \mathcal{H}} \Rightarrow\wedge$$

and the multiplicative form of the same rule:

$$\frac{\mathcal{G}_1 \mid \Gamma_1 \Rightarrow \Delta_1, A \mid \mathcal{H}_1 \quad \mathcal{G}_2 \mid \Gamma_2 \Rightarrow \Delta_2, B \mid \mathcal{H}_2}{\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, A \wedge B \mid \mathcal{H}_1 \mid \mathcal{H}_2} \Rightarrow\wedge$$

3.2 The Logic RM and the hypersequent calculus GRM

As noted previously, we will demonstrate hypersequent-based argumentation by the core logic RM. This is the best understood and researched logic among the relevance logics from the Anderson-Belnap approach [3].[8] Moreover, it is paraconsistent, decidable [9], has a simple semantics [3, §29] and is characterized by a Hilbert-style system [3, §27] (see also [9]). Like other relevance logics (such as R), RM does not satisfy the classical implication paradoxes $\phi \supset (\psi \supset \phi)$, $\neg\phi \supset (\phi \supset \psi)$, $(\phi \wedge \neg\phi) \supset \psi$ and $\phi \supset (\psi \supset \psi)$.[9] This makes RM suitable for the modeling of defeasible reasoning and hence an appropriate core logic for argumentation-based reasoning.

An ordinary cut-free sequent calculus for RM is not known. Fig. 1 presents a hypersequent proof system for RM, called GRM.

In [7] it is shown that GRM admits cut-elimination and that it satisfies the following soundness and completeness result for RM:

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[8] Strictly speaking, RM is a *semi-relevance logic*: it does satisfy the basic relevance criterion (introduced in [3]) and the minimal semantic relevance criterion [9], but it does not have the variable sharing property (introduced in [3]), see, e.g., [9].

[9] Unlike R, RM does satisfy the *mingle axiom* $\phi \supset (\phi \supset \phi)$.

Axioms: $\mathcal{G} \mid \psi \Rightarrow \psi$

Logical rules:

$[\neg\Rightarrow]\ \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi}{\mathcal{G} \mid \neg\varphi, \Gamma \Rightarrow \Delta}$ $\quad [\Rightarrow\neg]\ \dfrac{\mathcal{G} \mid \varphi, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \neg\varphi}$

$[\supset\Rightarrow]\ \dfrac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \mathcal{G} \mid \psi, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2, \varphi \supset \psi \Rightarrow \Delta_1, \Delta_2}$ $\quad [\Rightarrow\supset]\ \dfrac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta, \psi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \supset \psi}$

$[\wedge\Rightarrow]\ \dfrac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ \dfrac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta}$ $\quad [\Rightarrow\wedge]\ \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \qquad \mathcal{G} \mid \Gamma \Rightarrow \Delta, \psi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \wedge \psi}$

$[\vee\Rightarrow]\ \dfrac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta \qquad \mathcal{G} \mid \Gamma, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \vee \psi \Rightarrow \Delta}$ $\quad [\Rightarrow\vee]\ \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \vee \psi}\ \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \psi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \vee \psi}$

Structural rules:

$[\mathrm{EC}]\ \dfrac{\mathcal{G} \mid s \mid s}{\mathcal{G} \mid s}$ $\quad [\mathrm{EW}]\ \dfrac{\mathcal{G}}{\mathcal{G} \mid s}$

$[\mathrm{Sp}]\ \dfrac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2}$ $\quad [\mathrm{Mi}]\ \dfrac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$

$[\mathrm{Cut}]\ \dfrac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \mathcal{G} \mid \varphi, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$

Fig. 1. The proof system GRM [7]

Theorem 1. *Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a hypersequent, where for each $1 \leq i \leq n$, $\Gamma_i = \{\gamma_1^i, \ldots, \gamma_{m_i}^i\}$ and $\Delta_i = \{\delta_1^i, \ldots, \delta_{l_i}^i\}$. Then $\mathcal{H}$ is derivable in GRM if and only if the following formula $\tau(\mathcal{H})$ is a theorem of RM:*

$$\begin{aligned}\tau(\mathcal{H}) = \big(&\neg\gamma_1^1 \vee \ldots \vee \neg\gamma_{m_1}^1 \vee \delta_1^1 \vee \ldots \vee \delta_{l_1}^1\big) \vee \\ &\ldots \vee \Big(\neg\gamma_1^n \vee \ldots \vee \neg\gamma_{m_n}^k \vee \delta_1^n \vee \ldots \vee \delta_{l_n}^k\Big)\end{aligned} \tag{2}$$

To define hypersequent-based argumentation frameworks, it is not enough to simply take the hypersequent inference rules to create arguments. A new definition of arguments is required and sequent elimination rules should be turned into hypersequent elimination rules. This is what we will do in the next section.

4 Hypersequent-based Argumentation

Given a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ with a sound and complete hypersequent calculus H, from now on, an *argument* (or an L-hyperargument) is an $\mathcal{L}$-hypersequent (i.e., whose components are $\mathcal{L}$-sequents) that is provable in H.[10] In the remain-

[10] Since a sequent is a particular case of a hypersequent and hypersequent calculi generalize sequent calculi, arguments in the sense of the previous sections are particular cases of the arguments according to the new definition.

ing of the paper, an argument based on a set $\mathcal{S}$ (of formulae in $\mathcal{L}$), is an L-hyperargument $\mathcal{H}$ such that $\Gamma \subseteq \mathcal{S}$ for every $\Gamma \in \mathsf{Supp}(\mathcal{H})$. We shall continue to denote by $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ the set of arguments that are based on $\mathcal{S}$.

As before, arguments are constructed by the inference rules of the hypersequent calculus under consideration (see Sect. 3). For the elimination rules, we continue to use the same notation: $\overline{\mathcal{H}}$ denotes the elimination of the hypersequent $\mathcal{H}$. The structure of such rules remains the same as before as well: the first hypersequent in the conditions of the rule is the attacking argument, the last hypersequent in the conditions is the attacked argument and the rest of the conditions are to be satisfied for the attack to take place.

Example 6. The elimination rules Def_H and Ucut_H are the hypersequent counterparts of the rules in Ex. 1. Additionally, a third rule, Consistency Undercut, is defined as follow. Let $\mathcal{G}, \mathcal{H}$ be two arguments, where $\mathsf{Supp}(\mathcal{H}) = \{\Delta_1, \ldots, \Delta_m\}$ and $\Delta_j \neq \emptyset$ for Def_H, $\emptyset \neq \Delta'_j \subseteq \Delta_j$ for Ucut_H and $\bigcup_{i=1}^{m} \Delta_i \neq \emptyset$ for $\mathrm{ConUcut}_H$.

$$\frac{\mathcal{G} \quad \Rightarrow \mathsf{Conc}(\mathcal{G}) \supset \neg \bigwedge \Delta_j \quad \mathcal{H}}{\overline{\mathcal{H}}}\ \mathrm{Def}_H \qquad \frac{\mathcal{G} \quad \Rightarrow \mathsf{Conc}(\mathcal{G}) \leftrightarrow \neg \bigwedge \Delta'_j \quad \mathcal{H}}{\overline{\mathcal{H}}}\ \mathrm{Ucut}_H$$

$$\frac{\Rightarrow \neg \bigwedge \bigcup_{i=1}^{m} \Delta_i \quad \mathcal{H}}{\overline{\mathcal{H}}}\ \mathrm{ConUcut}_H$$

The notion of attack between hypersequents is the same as in Def. 3, except that sequents are replaced by hypersequents and the sequent calculus C is replaced by a hypersequent calculus H. Now, a hypersequent-based argumentation framework can be defined in a similar way as that of a sequent-based argumentation framework (cf. Def. 4).

Definition 8. *A* hypersequent-based argumentation framework *for a set of formulae $\mathcal{S}$ based on a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and a set* AR *of hypersequent elimination rules, is a pair $\mathcal{AF}_{\mathsf{L}}(\mathcal{S}) = \langle Arg_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$, where $Arg_{\mathsf{L}}(\mathcal{S})$ is the set of* L*-hyperarguments (arguments), $\mathcal{A} \subseteq Arg_{\mathsf{L}}(\mathcal{S}) \times Arg_{\mathsf{L}}(\mathcal{S})$ and $(a_1, a_2) \in \mathcal{A}$ iff there is an $\mathcal{R} \in \mathsf{AR}$ such that a_1 $\mathcal{R}$-attacks a_2.*

Given a hypersequent-based argumentation framework $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, Dung-style semantics are defined in an equivalent way to those in Def. 5.

Example 7. Let $\mathcal{AF}_{\mathsf{RM}}(\mathcal{S})$ be a hypersequent-based argumentation framework for $\mathcal{S} = \{p, q, \neg p \vee \neg q\}$, based on RM, with Ucut_H as the sole attack rule. From the axioms $p \Rightarrow p$ and $q \Rightarrow q$, by the Mingle Rule [Mi] (see Fig. 1) the sequent $p, q \Rightarrow p, q$ can be derived in GRM and by the Splitting Rule [Sp] the hypersequent $p \Rightarrow q \mid q \Rightarrow p$ is derivable in GRM as well. The hypersequent $p, q \Rightarrow p, q$ is Ucut_H-attacked by the axiom $\neg p \vee \neg q \Rightarrow \neg p \vee \neg q$, but the hypersequent $p \Rightarrow q \mid q \Rightarrow p$ is not Ucut_H-attacked by this axiom. However, both hypersequents are Ucut_H-attacked by the hypersequents $p, \neg p \vee \neg q \Rightarrow \neg q$ and $q, \neg p \vee \neg q \Rightarrow \neg p$.

Definition 9. *Given a hypersequent-based argumentation framework $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, the following* entailment relations *can be defined as in Def. 6: $\mathcal{S} \mid\!\sim^{\cap}_{H,\mathsf{sem}} \phi$ ($\mathcal{S} \mid\!\sim^{\cup}_{H,\mathsf{sem}} \phi$) iff for every (some) extension $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, there is an argument $\mathcal{H} \in \mathcal{E}$ such that $\mathsf{Conc}(\mathcal{H}) = \phi$ and $\bigcup \mathsf{Supp}(\mathcal{H}) \subseteq \mathcal{S}$. The subscript H is omitted when this is clear from the context.*

5 Discussion of Some Properties

In this section we consider some useful properties of hypersequent-based argumentation. We begin by showing that in some cases this kind of argumentation overcomes a shortcoming of some other frameworks (including sequent-based ones) that under some completeness-based semantics (Def. 5) extensions may not always be consistent [2, 15].

Example 8. This example is based on [2, Ex. 2]. Let $\mathcal{AF}_{\mathsf{CL}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{CL}}(\mathcal{S}), \mathcal{A} \rangle$ for $\mathcal{S} = \{p, q, \neg p \vee \neg q, t\}$ and the attack rules Def and/or Ucut. Then the following sequents are in $\mathrm{Arg}_{\mathsf{CL}}(\mathcal{S})$:

$$a_1 = t \Rightarrow t \qquad a_2 = p \Rightarrow p \qquad a_3 = q \Rightarrow q \qquad a_4 = \neg p \vee \neg q \Rightarrow \neg p \vee \neg q$$

$$a_5 = p \Rightarrow \neg((\neg p \vee \neg q) \wedge q) \qquad a_6 = q \Rightarrow \neg((\neg p \vee \neg q) \wedge p)$$

$$a_7 = p, q \Rightarrow p \wedge q \qquad a_8 = \neg p \vee \neg q, q \Rightarrow \neg p \qquad a_9 = \neg p \vee \neg q, p \Rightarrow \neg q$$

It can be proven that $\mathcal{E} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ is admissible in $\mathcal{AF}_{\mathsf{CL}}(\mathcal{S})$, for either of the attack rules Def or Ucut. However, $\mathsf{Concs}(\mathcal{E})$ is inconsistent.

Next, we show that the problem of the last example may be avoided by using a hypersequent-based framework.[11]

Example 9 (Ex. 8, continued). Let $\mathcal{AF}'_{\mathsf{L}}(\mathcal{S}) = \left\langle \mathrm{Arg}'_{\mathsf{L}}(\mathcal{S}), \mathcal{A}' \right\rangle$ be a hypersequent-based argumentation framework (Def. 8) for $\mathsf{L} \in \{\mathsf{CL}, \mathsf{RM}\}$, the attack rules Def_H and Ucut_H, and $\mathcal{S}$ as in Ex. 8. With the possibility of splitting components, we get $\mathrm{Arg}'_{\mathsf{L}}(\mathcal{S}) \supseteq \mathrm{Arg}_{\mathsf{CL}}(\mathcal{S}) \cup \{a_{10}, a_{11}, a_{12}\}$ where:

$$a_{10} = \neg p \vee \neg q \Rightarrow \neg p \mid q \Rightarrow \neg p \qquad a_{11} = \neg p \vee \neg q \Rightarrow \neg q \mid p \Rightarrow \neg q$$

$$a_{12} = p \Rightarrow p \wedge q \mid q \Rightarrow p \wedge q$$

The following three sets of arguments are part of different complete extensions: $\mathcal{E}_1 = \{a_1, a_2, a_3, a_5, a_6, a_7, a_{12}\}$, $\mathcal{E}_2 = \{a_1, a_3, a_4, a_6, a_8, a_{10}\}$ and $\mathcal{E}_3 = \{a_1, a_2, a_4, a_5, a_9, a_{11}\}$. Furthermore, although $\mathcal{E} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ is conflict-free, it defends the arguments a_{10}, a_{11} and a_{12}, but if one of these were added to $\mathcal{E}$, $\mathcal{E}$ would no longer be conflict-free. Hence $\mathcal{E}$ is not part of a complete extension. Additionally, each extension contains a_1, therefore, the system $\mathcal{AF}'_{\mathsf{L}}(\mathcal{S})$ does not only avoid inconsistent extensions, it provides extensions from which the free arguments follow.[12]

In the next subsection it will be shown that the outcome of the last example is not a coincidence.

5.1 Rationality Postulates

In this section we will prove that, for a hypersequent-based argumentation framework for the attack rules Def_H and Ucut_H and core logic RM, the logic-based rationality postulates from [1] hold.

[11] Intuitively, this is so due to the possibility of *splitting* hypersequents into different components. A formal justification will be given in the next subsection.

[12] Where free arguments are those arguments that are based only on premises that are not involved in minimally inconsistent subsets of the premises (see Definition 10).

Definition 10. *Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{T}$ be a set of $\mathcal{L}$-formulae, where $\mathcal{L}$ contains the connectives $\neg$ and $\wedge$.*

- *The* closure *of $\mathcal{T}$ is denoted by $\mathsf{CN_L}(\mathcal{T})$ (thus, $\mathsf{CN_L}(\mathcal{T}) = \{\phi \mid \mathcal{T} \vdash \phi\}$).*
- *$\mathcal{T}$ is* consistent *(for $\vdash$), if there are no formulae $\phi_1, \ldots, \phi_n \in \mathcal{T}$ such that $\vdash \neg \bigwedge_{i=1}^{n} \phi_i$.*[13]
- *A subset $\mathcal{C}$ of $\mathcal{T}$ is a* minimal conflict *of $\mathcal{T}$ (w.r.t. $\vdash$), if $\mathcal{C}$ is inconsistent and for any $c \in \mathcal{C}$, $\mathcal{C} \setminus \{c\}$ is consistent. We denote by $\mathsf{Free}(\mathcal{T})$ the set of formulae in $\mathcal{T}$ that are not part of any minimal conflict of $\mathcal{T}$.*

Let $\mathcal{AF}_\mathsf{L}(\mathcal{S}) = \langle \mathrm{Arg}_\mathsf{L}(\mathcal{S}), \mathcal{A} \rangle$ be a hypersequent-based argumentation framework and let $\mathcal{H}, \mathcal{H}' \in \mathrm{Arg}_\mathsf{L}(\mathcal{S})$ such that $\mathcal{H} = \Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Gamma_n \Rightarrow \phi_n$ and $\mathcal{H}' = \Gamma'_1 \Rightarrow \phi'_1 \mid \ldots \mid \Gamma'_m \Rightarrow \phi'_m$. Then $\mathcal{H}'$ is a *sub-argument* of $\mathcal{H}$ if for each $i \in \{1, \ldots, m\}$ there is a $j \in \{1, \ldots, n\}$ such that $\Gamma'_i \subseteq \Gamma_j$. The set of sub-arguments of a is denoted by $\mathsf{Sub}(\mathcal{H})$.

Definition 11. *Let $\mathcal{AF}_\mathsf{L}(\mathcal{S}) = \langle Arg_\mathsf{L}(\mathcal{S}), \mathcal{A} \rangle$ be an argumentation framework for the logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, the set $\mathcal{S}$ of $\mathcal{L}$-formulae and a fixed (set of) semantics* sem*. $\mathcal{AF}_\mathsf{L}(\mathcal{S})$ satisfies:*

- closure of extensions: $\forall \mathcal{E} \in \mathsf{Ext_{sem}}(\mathcal{AF}_\mathsf{L}(\mathcal{S}))$, $\mathsf{Concs}(\mathcal{E}) = \mathsf{CN_L}(\mathsf{Concs}(\mathcal{E}))$.
- closure under sub-arguments: $\forall \mathcal{E} \in \mathsf{Ext_{sem}}(\mathcal{AF}_\mathsf{L}(\mathcal{S}))$ *if* $\mathcal{H} \in \mathcal{E}$ *so* $\mathsf{Sub}(\mathcal{H}) \subseteq \mathcal{E}$.
- consistency: $\forall \mathcal{E} \in \mathsf{Ext_{sem}}(\mathcal{AF}_\mathsf{L}(\mathcal{S}))$, $\mathsf{Concs}(\mathcal{E})$ *is consistent.*
- exhaustiveness: $\forall \mathcal{E} \in \mathsf{Ext_{sem}}(\mathcal{AF}_\mathsf{L}(\mathcal{S}))$, *for all* $\mathcal{H} \in Arg_\mathsf{L}(\mathcal{S})$ *such that* $\bigcup \mathsf{Supp}(\mathcal{H}) \cup \{\mathsf{Conc}(\mathcal{H})\} \subseteq \mathsf{Concs}(\mathcal{E})$, $\mathcal{H} \in \mathcal{E}$.
- free precedence: $\forall \mathcal{E} \in \mathsf{Ext_{sem}}(\mathcal{AF}_\mathsf{L}(\mathcal{S}))$, $Arg_\mathsf{L}(\mathsf{Free}(\mathcal{S})) \subseteq \mathcal{E}$.

Note 1. For proving the above postulates, we shall use (sometimes implicitly) the following admissible rules of GRM:

- Transitivity: if $\mathcal{G}_1 \mid \Gamma \Rightarrow \phi_1 \mid \mathcal{H}_1$ and $\mathcal{G}_2 \mid \phi_1 \Rightarrow \phi_2 \mid \mathcal{H}_2$ are derivable, then $\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma \Rightarrow \phi_2 \mid \mathcal{H}_1 \mid \mathcal{H}_2$ is derivable.
- From $\mathcal{G} \mid \Gamma \Rightarrow \phi \supset \psi, \Delta \mid \mathcal{H}$ derive $\mathcal{G} \mid \Gamma, \phi \Rightarrow \psi, \Delta \mid \mathcal{H}$.
- From $\mathcal{G} \mid \Delta \Rightarrow \phi \mid \mathcal{H}$ derive $\mathcal{G} \mid \ \Rightarrow \neg\phi \supset \neg \bigwedge \Delta \mid \mathcal{H}$.
- For any $\Gamma' \subseteq \Gamma$, if $\mathcal{G} \mid \Rightarrow \phi \supset \neg \bigwedge \Gamma' \mid \mathcal{H}$ is derivable then $\mathcal{G} \mid \Rightarrow \phi \supset \neg \bigwedge \Gamma \mid \mathcal{H}$ is derivable.
- $\Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Gamma_n \Rightarrow \phi_n$ is derivable iff $\Gamma_1, \ldots, \Gamma_n \Rightarrow \phi_1, \ldots, \phi_n$ is derivable.
- $\phi_1 \vee \ldots \vee \phi_n \Rightarrow \phi_1 \mid \ldots \mid \phi_1 \vee \ldots \vee \phi_n \Rightarrow \phi_n$ is derivable.

Theorem 2. *Any argumentation framework $\mathcal{AF}_\mathsf{RM}(\mathcal{S})$ with attack relation Def_H or $Ucut_H$, under any completeness-based semantics, satisfies closure of extensions, closure under sub-arguments, consistency and exhaustiveness. Moreover, when $ConUcut_H$ is part of the attack rules, it satisfies free precedence as well.*

Proof. Let $\mathcal{AF}_\mathsf{RM}(\mathcal{S}) = \langle \mathrm{Arg}_\mathsf{RM}(\mathcal{S}), \mathcal{A} \rangle$ be an argumentation framework, with the attack rules Def_H and/or Ucut_H and let $\mathcal{E}$ be a complete extension of $\mathcal{AF}_\mathsf{RM}(\mathcal{S})$.

[13] Note that if $\mathcal{T}$ is consistent, then so are $\mathsf{CN_L}(\mathcal{T})$ and $\mathcal{T}'$ for every $\mathcal{T}' \subseteq \mathcal{T}$. If $\mathcal{T}$ is inconsistent, then so is every superset of $\mathcal{T}$.

sub-argument closure: for both Def_H and Ucut_H it can be shown that any attacker of $\mathcal{H}' \in \mathsf{Sub}(\mathcal{H})$ is also an attacker of $\mathcal{H}$. If $\mathcal{H} \in \mathcal{E}$, for any completeness-based $\mathcal{E}$ there is a $\mathcal{G} \in \mathcal{E}$ that defends $\mathcal{H}$ against this attack. Thus $\mathcal{E}$ defends $\mathcal{H}'$ as well. Therefore, $\mathcal{H}' \in \mathcal{E}$.

closure of extensions: showing that $\mathsf{Concs}(\mathcal{E}) \subseteq \mathsf{CN}_{\mathsf{RM}}(\mathsf{Concs}(\mathcal{E}))$ is trivial. For the other direction, assume that $\phi \in \mathsf{CN}_{\mathsf{RM}}(\mathsf{Concs}(\mathcal{E}))$. Then there are arguments $\mathcal{H}_1, \ldots, \mathcal{H}_n \in \mathcal{E}$ such that $\mathcal{H}_i = \Gamma^i_1 \Rightarrow \psi^i_1 \mid \ldots \mid \Gamma^i_{m_i} \Rightarrow \psi^i_{m_i}$, with $\phi_i = \psi^i_1 \vee \ldots \vee \psi^i_{m_i}$ and $\phi_1, \ldots, \phi_n \vdash_{\mathsf{RM}} \phi$.

It can be shown that the argument $\mathcal{H}' = \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{m_k} \bigwedge \Gamma^k_j \Rightarrow \phi_1 \wedge \ldots \wedge \phi_n$ is derivable in GRM. By transitivity and splitting we get that $a = \Gamma^1_1 \Rightarrow \phi \mid \ldots \mid \Gamma^1_{m_1} \Rightarrow \phi \mid \ldots \mid \Gamma^n_1 \Rightarrow \phi \mid \ldots \mid \Gamma^n_{m_n} \Rightarrow \phi$ is provable in GRM. For both attack rules Def_H and Ucut_H, any attacker of $\mathcal{H}$ is an attacker of one of the arguments $\mathcal{H}_1, \ldots, \mathcal{H}_n$. Hence $\mathcal{H} \in \mathcal{E}$ and so $\phi \in \mathsf{Concs}(\mathcal{E})$.

consistency: assume, towards a contradiction, that $\mathsf{Concs}(\mathcal{E})$ is not consistent. Then there are $\phi_1, \ldots, \phi_n \in \mathsf{Concs}(\mathcal{E})$ such that $\Rightarrow \neg \bigwedge_{j=1}^{n} \phi_j$ is derivable. Let $\psi = \phi_1 \wedge \ldots \wedge \phi_n$. Furthermore, like the proof of closure, there are arguments $\mathcal{H}_1, \ldots, \mathcal{H}_n \in \mathcal{E}$, such that $\mathcal{H}_i = \Gamma^i_1 \Rightarrow \psi^i_1 \mid \ldots \mid \Gamma^i_{m_i} \Rightarrow \psi^i_{m_i}$ and $\phi_i = \psi^i_1 \vee \ldots \vee \psi^i_{m_i}$. From these, arguments $\mathcal{H}'_i = \Gamma^i_1, \ldots, \Gamma^i_{m_i} \Rightarrow \phi_i$, for each $i \in \{1, \ldots, n\}$, can be derived. By applying ($\Rightarrow\wedge$) to the $\mathcal{H}'_i$'s, we get $\Gamma^1_1, \ldots, \Gamma^1_{m_1}, \ldots, \Gamma^n_1, \ldots, \Gamma^n_{m_n} \Rightarrow \psi$.

Then, for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m_j\}$, $\neg\psi, \Gamma^1_1, \ldots, \Gamma^1_{m_1}, \ldots, \Gamma^n_1, \ldots, \Gamma^n_{m_n} \Rightarrow \neg \bigwedge \Gamma^j_k$ is derivable, where Γ^j_k is taken out of $\Gamma^1_1, \ldots, \Gamma^1_{m_1}, \ldots, \Gamma^n_1, \ldots, \Gamma^n_{m_n}$. By transitivity from $\Rightarrow \neg\psi$ and splitting $\mathcal{G} = \Gamma^1_1 \Rightarrow \neg \bigwedge \Gamma^j_k \mid \ldots \mid \Gamma^1_{m_1} \Rightarrow \neg \bigwedge \Gamma^j_k \mid \ldots \mid \Gamma^n_1 \Rightarrow \neg \bigwedge \Gamma^j_k \mid \ldots \mid \Gamma^n_{m_n} \Rightarrow \neg \bigwedge \Gamma^j_k$ is derivable. Note that, for both attack rules Def_H and Ucut_H, any attacker of $\mathcal{G}$ is an attacker of one of the arguments $\mathcal{H}_1, \ldots, \mathcal{H}_n$, therefore $\mathcal{G} \in \mathcal{E}$. However, $\mathcal{G}$ attacks (defeats/undercuts) $\mathcal{H}_j$, a contradiction with the assumption that $\mathcal{E}$ is conflict-free. Thus $\mathsf{Concs}(\mathcal{E})$ is consistent.

exhaustiveness: let $\mathcal{H} \in \mathrm{Arg}_{\mathsf{RM}}(\mathcal{S})$ such that $\bigcup \mathsf{Supp}(\mathcal{H}) \cup \{\mathsf{Conc}(\mathcal{H})\} \subseteq \mathsf{Concs}(\mathcal{E})$. It easily follows that $\mathcal{E} \cup \{\mathcal{H}\}$ is conflict-free. Assume that some $\mathcal{G} = \Delta_1 \Rightarrow \psi_1 \mid \ldots \mid \Delta_n \Rightarrow \psi_n \in \mathrm{Arg}_{\mathsf{RM}}(\mathcal{S})$ defeats $\mathcal{H}$ (the case for undercut is similar and left to the reader).

Then $\Rightarrow \mathsf{Conc}(\mathcal{G}) \supset \neg \bigwedge \Gamma$, for some $\Gamma \in \mathsf{Supp}(\mathcal{H})$. Assume $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$, thus there are $\mathcal{H}_1, \ldots, \mathcal{H}_m \in \mathcal{E}$ such that $\mathsf{Conc}(\mathcal{H}_j) = \gamma_j$ and $\bigcup \mathsf{Supp}(\mathcal{H}_j) = \{\delta^j_1, \ldots \delta^j_{k_j}\}$ $(1 \leq j \leq m)$. By Thm. 1, $\delta^j_1, \ldots, \delta^j_{k_j} \vdash_{\mathsf{RM}} \gamma_j$, from this $\delta^1_1, \ldots, \delta^1_{k_1}, \ldots, \delta^m_1, \ldots, \delta^m_{k_m} \vdash_{\mathsf{RM}} \gamma_1 \wedge \ldots \wedge \gamma_m$ and thus $\neg \bigwedge \Gamma, \delta^1_1, \ldots, \delta^1_{k_1}, \ldots, \delta^m_1, \ldots, \delta^m_{k_{m-1}} \vdash_{\mathsf{RM}} \neg\delta^m_{k_m}$ can be derived.

By transitivity from $\mathsf{Conc}(\mathcal{G}) \Rightarrow \neg \bigwedge \Gamma$, Thm. 1, and splitting, $\mathcal{G}' = \Delta_1 \Rightarrow \neg\delta^m_{k_m} \mid \ldots \mid \Delta_m \Rightarrow \neg\delta^m_{k_m} \mid \delta^1_1 \Rightarrow \neg\delta^m_{k_m} \mid \ldots \mid \delta^m_{k_{m-1}} \Rightarrow \neg\delta^m_{k_m} \in \mathrm{Arg}_{\mathsf{RM}}(\mathcal{S})$. But then $\mathcal{G}'$ defeats $\mathcal{H}_m \in \mathcal{E}$, thus there is some $\mathcal{H}^* \in \mathcal{E}$ which defeats $\mathcal{G}'$. This attack has to be on some Δ_i, $i \in \{1, \ldots, n\}$, otherwise $\mathcal{E}$ would not be conflict-free. Hence $\mathcal{H}^*$ defeats $\mathcal{G}$ as well.

Since, by assumption, $\mathcal{E}$ is complete, $\mathcal{E} \cup \{\mathcal{H}\}$ is conflict-free and $\mathcal{E}$ defends $\mathcal{H}$, it follows that $\mathcal{H} \in \mathcal{E}$.

free precedence: suppose that ConUcut_H is among the attack rules in $\mathcal{AF}_{\mathsf{RM}}(\mathcal{S})$ as well. It can be shown that Def_H, Ucut_H and ConUcut_H are *conflict-dependent* in the sense of [1]: if $\mathcal{G}, \mathcal{H} \in \text{Arg}_{\mathsf{RM}}(\mathcal{S})$ such that $\mathcal{G}$ attacks $\mathcal{H}$, then $\bigcup \mathsf{Supp}(\mathcal{G}) \cup \bigcup \mathsf{Supp}(\mathcal{H})$ is inconsistent.

Assume that some $\mathcal{G} \in \text{Arg}_{\mathsf{RM}}(\mathcal{S})$ attacks an argument $\mathcal{H} \in \text{Arg}_{\mathsf{RM}}(\mathsf{Free}(\mathcal{S}))$. Since each of the considered attack rules is conflict-dependent, $\bigcup \mathsf{Supp}(\mathcal{H}) \cup \bigcup \mathsf{Supp}(\mathcal{G})$ is inconsistent. However, $\bigcup \mathsf{Supp}(\mathcal{H}) \subseteq \mathsf{Free}(\mathcal{S})$, thus $\mathcal{G}$ has an inconsistent support set. Then there is an argument $\Rightarrow \neg \bigwedge \mathsf{Supp}(\mathcal{G}) \in \mathcal{E}$ that attacks $\mathcal{G}$ via the ConUcut_H rule. Since any attacker of $\mathcal{H}$ is defended by $\mathcal{E}$, it follows that $\mathcal{H} \in \mathcal{E}$.

We have shown that $\mathcal{AF}_{\mathsf{RM}}(\mathcal{S})$, for the given attack rules, satisfies the five postulates under complete semantics. From this it follows (see, e.g., [1, Prop. 26]) that $\mathcal{AF}_{\mathsf{RM}}(\mathcal{S})$ satisfies the five postulates also under grounded, preferred and stables semantics. □

Note 2. Consider the following weakening of the definition of sub-arguments: $\mathcal{H}'$ is a *weak sub-argument* of $\mathcal{H}$, if $\bigcup \mathsf{Supp}(\mathcal{H}') \subseteq \bigcup \mathsf{Supp}(\mathcal{H})$. Clearly, any sub-argument of $\mathcal{H}$ is in particular a weak sub-argument of $\mathcal{H}$. Interestingly, closure of extensions and exhaustiveness imply closure under weak sub-arguments (and so closure under sub-argument).

5.2 Crash-resistance and Non-Interference

Two additional postulates were introduced in [14] concerning crash-resistance, the problem that a system collapses when it is based on inconsistent information. Before defining these, some other definitions and notations are necessary.

Let $\mathcal{AF}_{\mathsf{L}}(\mathcal{S}) = \langle \text{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ be an argumentation framework for the logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and the set $\mathcal{S}$ of $\mathcal{L}$-formulae. Then, $\text{atoms}(\mathcal{S})$ denotes the set of atoms that occur in the formulae in $\mathcal{S}$ and $\text{atoms}(\mathcal{L})$ denotes the set of all the atoms of the language. Let $\mathcal{AT}$ be a set of atoms, then $\mathcal{S}_{|\mathcal{AT}}$ denotes the set of formulae in $\mathcal{S}$ that contain only atoms from $\mathcal{AT}$. Furthermore, for a set $\mathcal{F}$ of sets of $\mathcal{L}$-formulae, $\mathcal{F}_{|\mathcal{AT}} = \{\mathcal{S}_{|\mathcal{AT}} \mid \mathcal{S} \in \mathcal{F}\}$. It is said that two sets of formulae $\mathcal{S}$ and $\mathcal{T}$ are *syntactically disjoint* if and only if $\text{atoms}(\mathcal{S}) \cap \text{atoms}(\mathcal{T}) = \emptyset$.

Definition 12. *Let $\mid\!\sim$ be an entailment relation for $\mathcal{L}$. A set $\mathcal{S}'$ of $\mathcal{L}$-formulae is called* contaminating *(with respect to $\mid\!\sim$) if for any set $\mathcal{S}^* \subseteq \mathcal{L}$, such that $\mathcal{S}'$ and $\mathcal{S}^*$ are syntactically disjoint, and for every $\mathcal{L}$-formula ϕ, it holds that $\mathcal{S}' \mid\!\sim \phi$ if and only if $\mathcal{S}' \cup \mathcal{S}^* \mid\!\sim \phi$.*

Definition 13. *A logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ is called* non-trivial*, if for every non-empty set $\mathcal{AT} \subseteq atoms(\mathcal{L})$ there are sets $\mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{L}$-formulae such that $atoms(\mathcal{S}_1) = atoms(\mathcal{S}_2) = \mathcal{AT}$ but $\mathsf{CN}_{\mathsf{L}}(\mathcal{S}_1)_{|\mathcal{AT}} \neq \mathsf{CN}_{\mathsf{L}}(\mathcal{S}_2)_{|\mathcal{AT}}$.*

The two postulates from [14], in hypersequent-based notation, are then:

Definition 14. *Let $\mathcal{L}$ be a language and $\mid\!\sim \subseteq \wp(\mathcal{L}) \times \mathcal{L}$ a consequence relation. Then $\mid\!\sim$ satisfies:*

– non-interference *if and only if for every syntactically disjoint sets of $\mathcal{L}$-formulae $\mathcal{S}_1$, $\mathcal{S}_2$, and any $\mathcal{L}$-formula ϕ such that atoms$(\phi) \subseteq$ atoms$(\mathcal{S}_1)$, $\mathcal{S}_1 \mathrel{|\!\sim} \phi$ if and only if $\mathcal{S}_1 \cup \mathcal{S}_2 \mathrel{|\!\sim} \phi$;*
– crash-resistance *if and only if there is no set $\mathcal{S}$ of $\mathcal{L}$-formulae that is contaminating w.r.t. $\mathrel{|\!\sim}$.*

For proving the above postulates, we need the next lemma.

Lemma 1. *Let $\mathcal{AF}_{\mathsf{RM}}(\mathcal{S})$ be as in Def. 8. The following are equivalent: (a) $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{RM}}(\mathcal{S}))$; (b) $\mathcal{E} \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{RM}}(\mathcal{S}))$; (c) $\mathcal{E} = Arg_{\mathsf{RM}}(\mathcal{S}')$, where $\mathcal{S}'$ is a $\subseteq$-maximally consistent subset of $\mathcal{S}$ relative to RM.*[14]

Theorem 3. *Let RM be the core logic, $\pi \in \{\cap, \cup\}$ and* sem *a completeness-based semantics, then $\mathrel{|\!\sim}^{\pi}_{H,\mathsf{sem}}$ satisfies non-interference.*[15]

Proof (outline). Let $\mathcal{AF}_{\mathsf{RM}}(\mathcal{S})$ be some hypersequent-based argumentation framework for the logic RM, with the attack rules Def_H and/or Ucut_H and set of RM-formulae $\mathcal{S}$. Consider two syntactically disjoint sets $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}$ and let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. For any $\mathsf{S} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$, let $\mathcal{D}_{\mathcal{AF}_{\mathsf{L}}}(\mathsf{S}) = \{\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}) \mid \mathsf{S} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ defends $\mathcal{H}\}$. Then, by Lemma 1 and the fact that RM satisfies the basic relevance criterion [9], the following points can be shown for complete, preferred and stable semantics (proofs are omitted due to space resitrictions):
1. if $\mathcal{E} \in \mathsf{Ext}(\mathcal{AF}_{\mathsf{RM}}(\mathcal{S}))$, then $\mathcal{E} \cap \mathrm{Arg}_{\mathsf{RM}}(\mathcal{S}_1) \in \mathsf{Ext}(\mathcal{AF}_{\mathsf{RM}}(\mathcal{S}_1))$;
2. let $\mathcal{E}^1 \in \mathsf{Ext}(\mathcal{AF}_{\mathsf{RM}}(\mathcal{S}_1))$, $\mathcal{E}^2 \in \mathsf{Ext}(\mathcal{AF}_{\mathsf{RM}}(\mathcal{S}_2))$, then $\mathcal{D}_{\mathcal{AF}_{\mathsf{RM}}(\mathcal{S})}(\mathcal{E}^1 \cup \mathcal{E}^2) \in \mathsf{Ext}(\mathcal{AF}_{\mathsf{RM}}(\mathcal{S}))$.

Let ϕ be an RM-formula with atoms$(\phi) \subseteq$ atoms$(\mathcal{S}_1)$. We show that $\mathcal{S}_1 \mathrel{|\!\sim} \phi$ if and only if $\mathcal{S} \mathrel{|\!\sim} \phi$.

⇒ Assume that $\mathcal{S}_1 \mathrel{|\!\sim}^{\cap} \phi$ but $\mathcal{S} \mathrel{|\!\not\sim}^{\cap} \phi$. Thus, there is some $\mathcal{E} \in \mathsf{Ext}(\mathcal{AF}_{\mathsf{RM}}(\mathcal{S}))$, such that there is no argument $\mathcal{H} \in \mathcal{E}$ with $\mathsf{Conc}(\mathcal{H}) = \phi$. By Item 1 above $\mathcal{E} \cap \mathrm{Arg}_{\mathsf{RM}}(\mathcal{S}_1) \in \mathsf{Ext}(\mathcal{AF}_{\mathsf{RM}}(\mathcal{S}_1))$, a contradiction to $\mathcal{S}_1 \mathrel{|\!\sim}^{\cap} \phi$.

⇐ Assume that $\mathcal{S} \mathrel{|\!\sim}^{\cap} \phi$ but $\mathcal{S}_1 \mathrel{|\!\not\sim}^{\cap} \phi$. Thus, there is some $\mathcal{E} \in \mathsf{Ext}(\mathcal{AF}_{\mathsf{RM}}(\mathcal{S}_1))$ such that there is no argument $\mathcal{H} \in \mathcal{E}$ with $\mathsf{Conc}(\mathcal{H}) = \phi$. For any $\mathcal{E}' \in \mathsf{Ext}(\mathcal{AF}_{\mathsf{RM}}(\mathcal{S}_2))$ (note that by [16] there is at least one such extension), there is no argument $\mathcal{H} \in \mathcal{E}'$ with $\mathsf{Conc}(\mathcal{H}) = \phi$ either. This follows because of the basic relevance criterion [3]. Thus, by Item 2 above, $\mathcal{D}_{\mathcal{AF}_{\mathsf{RM}}(\mathcal{S})}(\mathcal{E} \cup \mathcal{E}') \in \mathsf{Ext}(\mathcal{AF}_{\mathsf{RM}}(\mathcal{S}))$. By definition of $\mathcal{D}_{\mathcal{AF}_{\mathsf{L}}}$, there is no argument $\mathcal{H} \in \mathcal{D}_{\mathcal{AF}_{\mathsf{RM}}(\mathcal{S})}(\mathcal{E} \cup \mathcal{E}')$ with $\mathsf{Conc}(\mathcal{H}) = \phi$, a contradiction to $\mathcal{S} \mathrel{|\!\sim}^{\cap} \phi$.

The case for $\mathrel{|\!\sim}^{\cup}$ is similar and left to the reader. It follows that $\mathrel{|\!\sim}^{\cup}_{H,\mathsf{sem}}$ and $\mathrel{|\!\sim}^{\cap}_{H,\mathsf{sem}}$, for $\mathsf{sem} \in \{\mathsf{gr}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$, satisfy non-interference. □

Theorem 4. *Let RM be the core logic, $\pi \in \{\cap, \cup\}$ and* sem *a completeness-based semantics, then $\mathrel{|\!\sim}^{\pi}_{H,\mathsf{sem}}$ satisfies crash-resistance.*

[14] The proof is partially based on [5, Lemma 1] and [16, Lemma 15], but omitted due to space restrictions.
[15] The structure of the proof is roughly based on the proofs in [30].

Proof (sketch). Note that RM is non-trivial, therefore, by [14, Thm. 1] and Thm. 3, $\mathrel{|\!\sim}^{\cap}_{H,\mathsf{sem}}$ and $\mathrel{|\!\sim}^{\cup}_{H,\mathsf{sem}}$, for $\mathsf{sem} \in \{\mathsf{gr}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$, satisfy crash-resistance. □

6 Conclusion

Hypersequent-based argumentation, like sequent-based argumentation, avoids some limitations of other approaches to logic-based argumentation (e.g., [12]), where the support set of an argument has to be consistent and minimal. Furthermore, it incorporates any logic with a corresponding sound and complete (hyper)sequent calculus, and allows a great flexibility in the specification of the attack rules.

In this paper we have examined more specifically hypersequent frameworks that are based on the logic RM and with defeat and/or undercut as the attack rule. It was shown that such frameworks satisfy the logic-based rationality postulates from [1, 13] and non-interference and crash-resistance from [14]. Moreover, a problem raised in [15] (and further discussed in [2]), in which complete extensions may not be consistent, is avoided.

A comparison to related literature has to be postponed. However, it is worth noting that our non-interference result is more general than the one in [30], where this is only proven for frameworks under complete semantics.

Future research directions include the extension of dynamic proof theory [6] from sequent-based frameworks to hypersequent-based ones. Moreover, we plan to investigate the integration of priorities among arguments and the use of assumptions, such as default assumptions [20] and assumptions taken in adaptive logics [10, 28], for further extending the expressive power of hypersequent-based argumentation.

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