Talk 4:
Introduction to ∞-categories

Julius Frank

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1 What's the point?

In the last two talks, Irakli introduced us to simplicial sets and model categories. It's been a while, so we recall things we need along the way.

Categories are often insufficient when doing homotopy theory: Homotopies are maps between maps, and that's not something a plain category can deal with. The point of ∞-categories is to give a generalization of categories where we can do homotopy theory nicely. Let's collect some properties that we want these to have:

Consider a map $f : X \to Y$ of spaces. We are interested in the fibre of this map, e.g., because it sometimes gives us a long exact sequence of homotopy groups. Now, a plain fibre of f would just be a pullback

$$\begin{tikzcd} f^{-1}y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ y \arrow[r] & Y. \end{tikzcd}$$

This pullback is classified by its universal property: A map $T \to f^{-1}y$ corresponds to a commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ y \arrow[r] & Y. \end{tikzcd}$$

But often, what we really want is the homotopy fibre: For example, this always gives you a long exact sequence! The homotopy fibre is a homotopy pullback

$$\begin{tikzcd} F_f \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ y \arrow[r] & Y. \end{tikzcd}$$

But now, a map $T \to F_f$ corresponds not just to a (homotopy) commutative diagram as above. Instead, it corresponds to a diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr] & X \arrow[d, "f"] \\ y \arrow[r] & Y, \end{tikzcd}$$

together with the data of homotopies making this diagram homotopy commutative. This is not a purely categorical construction anymore: A plain category cannot keep track of homotopy data.

First demand. Data is to ∞-categories as properties are to categories. Any homotopical category should be able to keep track of homotopies (and homotopies between homotopies,...)

For plain categories, things are often unique, or unique up to unique isomorphism. That doesn't happen in homotopy theory: For example, every isomorphism has a unique inverse. But a homotopy equivalence can have many inverses. Now, any two inverses will be homotopic. But there can be many homotopies between two inverses! Now, any two homotopies will be homotopic. But...

This is a lot of data to keep track of, just to say that something is essentially unique. Of course, there's a nicer way: We need to structure the data. Specifically, one should have a space of inverses, not just a set. Then homotopical uniqueness just means that this space is (weakly) contractible.

Second demand. Sets should be replaced by spaces. In particular, we want mapping spaces between objects. We consider these spaces only up to weak equivalence. One direct application: Uniqueness should be replaced by contractibility (i.e., "the set of choices is a singleton" becomes "the space of choices has the homotopy type of a singleton".)

An obvious solution at this point would be categories enriched over spaces (e.g., topological, or sSet, or Kan complexes). Also true, but harder to see: Even for model categories, one can construct mapping spaces! But while these do give valid models for ∞-categories, they are hard to work with:

Third demand: Statements from categories should carry over to ∞-categories, as long as one translates them in the spirit of the first two demands. For example, we would like analogues to the following statements:

- An essentially surjective, fully faithful functor is an equivalence of categories.
- A natural transformation is an isomorphism iff it is an object-wise isomorphism.

Neither of these work in these models. In fact, (the homotopical version of) essentially surjective fully faithful functors don't even necessarily have inverses there! For example, the inclusion

(finite cell complexes, cellular maps) $\hookrightarrow$ (topological spaces equivalent to finite cell complexes, continuous maps

can be realized a such a functor, but there is no reason to believe that one can go the other way.

There are at least two models that satisfy all of the above: Quasicategories and complete Segal spaces. We'll see both during this seminar, but we will start with quasicategories.

How to think about quasicategories : Every concept and statement from 1-categories can be easily translated, and is almost certainly true. This makes them very natural to work with, because you already understand how they work!

The disadvantage will be: Things are hard to prove, and hard to explicitly construct. The first one is not a huge problem, people (i.e., Jacob Lurie) have proved most statements that you'll want to use. The second part is a problem: For example, it is often very hard to explicitly construct functors.

2 Spaces and Categories

Let's take a step back.

Recall that we learned about simplicial sets last time. We defined the category Δ with objects the posets $[n] = \{1 \le \ldots \le n\}$, and morphisms as order-preserving maps. Then a simplicial set is a presheaf $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$. More concretely, this consists of the sets of n-simplices X_n, together with boundary and degeneracy maps:

$$\cdots \; X_2 \rightleftarrows X_1 \rightleftarrows X_0$$

An important example is the n-simplex $\Delta^n = \mathrm{Hom}(-,[n])$. [draw 2-simplex.] By the Yoneda lemma, we have $X_n = \mathrm{Hom}(\Delta^n, X)$—i.e., X_n really tells us about how n-simplices can fit into X.

We've also encountered horns: For $0 \le i \le n$, the simplicial set $\Lambda^n_i \hookrightarrow \Delta^n$ was obtained by deleting the n-simplex and its ith face. [draw 2-horns.] Moreover, we had *spines*: The n-spine $I^n \hookrightarrow \Delta^n$ consists of just the 1-simplices $(i, i+1)$. [draw 3-spine.]

Then Irakli introduced us to Kan complexes. To recall, a Kan complex was a simplicial set K that admits *horn fillers*, i.e.: For any $0 \le i \le n$, one can find a dotted arrow:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \\ \Delta^n. \arrow[ur, dotted] & \end{tikzcd}$$

We saw that for any topological space X, the singular complex $S(X)$ with $S_n(X) = \mathrm{Map}(|\Delta^n|, X)$ is a Kan complex, and this assembles into a functor that induces an equivalence of homotopy categories (more precisely: S is a Quillen equivalence between **Top** and **sSet**, and Kan complexes are the fibrant-cofibrant objects). Succinctly: Kan complexes are spaces.

As it turns out, one can do a similar thing for categories: Note that the poset $[n]$ can be viewed as a category:

$$\underline{n} = 0 \to 1 \to \cdots \to n.$$

Likewise, the maps $[n] \to [m]$ correspond precisely to functors between these categories.

Definition 2.1. Let $\mathcal{C}$ be an ordinary category. The *nerve* of $\mathcal{C}$ is the simplicial set $N\mathcal{C}$, where the n-simplices are

$$N\mathcal{C}_n = \mathrm{Fun}(\underline{n}, \mathcal{C}),$$

and the maps are induced by precomposition. In more concrete terms: A 0-simplex of $N\mathcal{C}$ is an object of $\mathcal{C}$, a 1-simplex is a morphism in $\mathcal{C}$, and an n-simplex of $N\mathcal{C}$ is a chain of n composable morphisms in $\mathcal{C}$:

$$c_0 \to c_1 \to \cdots \to c_n.$$

Faces are obtained by omitting and composing morphisms, and degeneracies by inserting identities. More intuitively, draw a simplex, and put the morphisms on the spine. Then triangles are filled by composition, and one can read off the faces and degeneracies:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "gf"] & & z. \end{tikzcd}$$

Now, as we will see later, this is a fully faithful functor: That is, one can do category theory loss-free by looking at nerves. Therefore it is useful to characterize those simplicial sets which arise as the nerve of a category. It turns out that the characteristic property is very similar to that of Kan complexes:

Theorem 2.2 (1.1.52)**.** *For a simplicial set $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$, the following are equivalent:*

(1) X is (isomorphic to) the nerve of some 1-category $\mathcal{C}$.

(2) X has unique spine fillers*, i.e.: For any spine $I^n \to X$, there is a unique extension*

$$\begin{tikzcd} I^n \arrow[r] \arrow[d, hook] & X \\ \Delta^n. \arrow[ur, dotted] & \end{tikzcd}$$

(3) X has unique inner horn fillers*, i.e.: For any inner horn $\Lambda^n_i \to X$, with $0 < i < n$, there is a unique extension*

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \\ \Delta^n. \arrow[ur, dotted] & \end{tikzcd}$$

Proof. We will only sketch a proof. First, it is rather straightforward to see that (1) $\Rightarrow$ (2):

- $(\mathbf{1}) \Rightarrow (\mathbf{2})$: A spine $I^n \to X$ is a string of "composable" 1-simplices:

  $$\mathrm{Hom}_{\mathbf{sSet}}(I^n, X) = X_1 \times_{X_0} \cdots \times_{X_0} X_1.$$

  For nerves, these are precisely the n-simplices:

  $$\mathrm{Hom}_{\mathbf{sSet}}(I^n, N\mathcal{C}) = N\mathcal{C}_1 \times_{N\mathcal{C}_0} \cdots \times_{N\mathcal{C}_0} N\mathcal{C}_1$$
  $$\cong \mathrm{mor}\mathcal{C} \times_{\mathrm{obj}\mathcal{C}} \cdots \times_{\mathrm{obj}\mathcal{C}} \mathrm{mor}\mathcal{C} \cong N\mathcal{C}_n \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, N\mathcal{C}).$$

  This tells us that any spine $I^n \to N\mathcal{C}$ is the restriction of a unique simplex $\Delta^n \to N\mathcal{C}$.

- $(\mathbf{2}) \Rightarrow (\mathbf{1})$: Given a simplicial set X with unique spine fillers, let us construct a category $\mathcal{C}$ s.t. $X \cong N\mathcal{C}$: We set $\mathrm{obj}\mathcal{C} = X_0$, and the morphisms as the 1-simplices:

  $$\mathrm{Hom}_{\mathcal{C}}(x, y) = \{f \in X_1 | d_1 f = x, d_0 f = y\}.$$

  The composition is defined through filling 2-spines, and the identities are degeneracies of the 0-simplices. Then one checks that this is a category (by filling the appropiate spines), and it is immediate that $X \cong N\mathcal{C}$. (If you're confused about this, it is very useful to spell this out right now!)

For (2) $\Leftrightarrow$ (3), one needs to get a bit technical. Let us denote for $S \subset [n]$ the face $\Delta^S \subset \Delta^n$ as the maximal subset on the 0-vertices in S. E.g., $d_j \Delta^n = \Delta^{[n]-\{j\}}$. Likewise for spines: I^S is the spine of Δ^S.

Also note that for $n = 2$, we have $I^2 = \Lambda^2_1$, so the only inner horn is also a spine, and the statement is trivial. We can therefore assume $n \geq 3$.

- $(\mathbf{2}) \Rightarrow (\mathbf{3})$: Consider an inner horn $\alpha : \Lambda^n_i \to X$. Then $I^n \hookrightarrow \Lambda^n_i$, so we can restrict α to the spine, and extend that uniquely to a simplex $\beta : \Delta^n \to X$. We only need to show that β restricted to the horn agrees with α. Check this for the faces $\Delta^{[n]-\{j\}}$ separately: For $j = 0, n$, this is immediate, as their spine is included in the larger spine: $I^{[n]-\{0\}} \subset I^n$. Hence the restrictions of α and β to that face are the unique extension, and agree. For other j, note that their spine has only one edge not in I^n: the one from $j-1$ to $j+1$. But as $n \geq 3$, this lies in either the 0th or the nth face, which agrees with α. So the same argument applies. In total, we get a unique extension to all faces—in particular, we get a unique extension to some inner horn, from which we can uniquely extend to the simplex.

- $(\mathbf{3}) \Rightarrow (\mathbf{2})$: We do an induction over the dimension n. For $n > 2$, and assume that we can lift all k-spines uniquely to k-simplices, for $k < n$. Consider a map $I^n \to X$, which we want to lift uniquely to Δ^n. The Idea is the following: First, we can look at the first $n-1$ segments of the spine: These make up the spine $I^{[n]-\{n\}}$ of the nth face $\Delta^{[n]-\{n\}}$, so we can uniquely fill that. Likewise, we can fill the 0th face, by filling in the last $n-1$ segments of the spine. On the middle $n-2$ segments, these must then agree, by the uniqueness condition.

  Now we have 2 faces filled in, and some more edges to chose from: In particular, having the 0th face guarantees that we have the spine for the 1st face, which we can therefore fill in. Iterating this leaves us with all faces of the simplex filled in–in particular, we have a unique extension to any inner horn, where we then have a unique extension to the full simplex by (1).

□

3 Quasicategories

The upshot is, spaces and categories are simplicial sets with very similar horn filling properties: Spaces admit all horn fillers, and categories admit unique inner horn fillers. A curious mind would now want to know what lies in between: First, let's look at what happens if one takes both conditions at once:

Remark 3.1. Let $\mathcal{C}$ be a category. Note that if $N\mathcal{C}$ is Kan, then $\mathcal{C}$ must be a groupoid: For any map f in $\mathcal{C}$, filling the outer 2-horns Λ^2_0 and Λ^2_2 with f as one leg, and the respective identities as the other, gives an inverse to f.

In fact, the converse is true: If $\mathcal{C}$ is a groupoid, then $N\mathcal{C}$ is Kan. For 2-horns, this is immediate. (Try it for 3-horns!) For $n \geq 4$, it is a technical fact that outer horns for nerves can *always* be filled! C.f. Land, 1.1.53)

So, this describes the intersection: Groupoids are precisely both categories and spaces. What about the common ground of the two horn-filling conditions?

Definition 3.2. A simplicial set $\mathcal{C}$ is called *quasicategory* (or ∞-category, $(\infty,1)$-category, weak Kan complex) if it admits (possibly non-unique) fillers for inner horns, i.e.: For $0 < i < n$, the dotted arrow exists:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n. \arrow[ur, dotted] & \end{tikzcd}$$

(Warning: Without uniqueness, this is not the same as just lifting spines! Look into Land for that, he calls simplicial sets with spine fillers "composers", and essentially concludes that they're not very useful.)

We then call the 0-simplices *objects*, the 1-simplices *morphisms* or *arrows*, and denote them by $f : x \to y$. A *functor* of quasicategories is simply a map of simplicial sets.

By construction, we have two classes of examples: Every category is an ∞-category, by considering its nerve; and every space is an ∞-category, by considering it as a Kan complex. Other examples are annoyingly difficult, see next talk!

For now, let's equip ourselves with some analogies between ∞-categories and 1-categories.

Definition 3.3. Let $f, g \in \mathcal{C}_1$ be two arrows in an ∞-category $\mathcal{C}$. They are called *composable* if $d_0 f = d_1 g$, and for any 2-simplex

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"] & & \bullet, \end{tikzcd}$$

h is called a *composite* of f, g. (But, note that this comes with the *data* of the composition through the 2-simplex!)

Remark 3.4. Note that these always exist, by filling the 2-horn, but needn't be unique! So there is generally no strict composition law in ∞-categories. (Later: In accordance with the second demand, there will be a contractible space of composites $\mathrm{comp}(f, g) \simeq *$.)

Remark 3.5. The identity on $x \in \mathcal{C}_0$ is $\mathrm{id}_x = s_0 x$. To see that this makes sense, take some arrow $f : x \to y$ in $\mathcal{C}$: Then the two degeneracies of f are 2-simplices

$$\begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"] & & y \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "s_0 x"] \arrow[rr, "f"] & & y. \end{tikzcd}$$

In other words, f is a composite of f and the respective degeneracies.

What about associativity? Well, we don't have strict composition, so the better question is: What would associativity be? Consider composable maps f, g, h. Then we can fill the following 3-simplex:

$$\begin{tikzcd} & y \arrow[rr] \arrow[dr, "g"] & & z \\ x \arrow[ur, "f"] \arrow[rr] \arrow[urrr, dotted] & & y. \arrow[ur, "h"] & \end{tikzcd}$$

The dotted line is then a composite of f, g, h—in whatever order we compose. (Again, the bigger picture is that there is also a space $\mathrm{comp}(f, g, h) \simeq *$.)

We want to do homotopy theory, so we need homotopies:

Definition 3.6. Let $f, g : x \to y$ be two parallel arrows in an ∞-category $\mathcal{C}$. They are called *homotopic* if there exists a 2-simplex

$$\begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"] & & y. \end{tikzcd}$$

It is a good exercise to check that this is indeed an equivalence relation, and that one could equivalently put the degeneracy to the other side. Another nice exercise: Show that any two composites of two composable maps are homotopic.

4 Homotopy Categories

If ∞-categories are a good model for homotopical categories, they need an associated homotopy category, i.e., a 1-category obtained by strictly inverting all equivalences. It makes sense to construct them for all simplicial sets, where they are a bit complicated, but they'll be much nicer for quasicategories!

Construction 4.1. Let X be any simplicial set. The *homotopy category* hX of X is obtained as follows: Objects of hX are the 0-simplices X_0. The morphisms are *generated* by the 1-simplices: For every $\Delta^1 \xrightarrow{f} X$, there exists a morphism f from $d_1 f$ to $d_0 f$; and those can be composed; up to some relations:

$$\mathrm{Hom}(x, y) = \left\{(f_n, \cdots, f_1) | d_1 f_1 = x, d_0 f_n = y, d_0 f_i = d_1 f_{i+1}\right\} / \sim .$$

As for relations, we have:

- The degeneracy $s_0(x)$ is the identity on x,
- For any 2-simplex

$$\begin{tikzcd} & \bullet \arrow[dr, "h"] & \\ \bullet \arrow[ur, "g"] \arrow[rr, "k"] & & \bullet, \end{tikzcd}$$

  we identify

$$(f_n, \cdots, h, g, \cdots f_1) \sim (f_n, \cdots, k, \cdots f_1).$$

  (Check that this is compatible with identities.)

This describes a functor $h : \mathbf{sSet} \to \mathbf{Cat}$.

Fact 4.2. If C is a quasicategory, then $h\mathcal{C}$ has a simpler description: The objects are still $\mathcal{C}_0$, and morphisms are now simply a quotient of $\mathcal{C}_1$ by homotopy; no need for free composites: That is,

$$h\mathcal{C}(x, y) = \left\{f \in \mathcal{C}_1 | d_1 f = x, d_0 f = y\right\} / \sim,$$

where $f \sim g$ whenever there is a homotopy between them:

$$\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"] & & y. \end{tikzcd}$$

It is a good (albeit lengthy) exercise to check that this works: You just fill horns all the way through. Note: Land calls this construction $\pi\mathcal{C}$, but it is naturally isomorphic to $h\mathcal{C}$.

Proposition 4.3. *There is an adjunction*

$$h : \mathbf{sSet} \Leftrightarrow \mathbf{Cat} : N,$$

i.e., for any simplicial set X and 1-category $\mathcal{C}$, we have

$$\mathrm{Fun}(hX, \mathcal{C}) \cong \mathrm{Hom}_{\mathbf{sSet}}(X, N\mathcal{C}).$$

Proof. Let us construct unit and counit:

The counit $hN\mathcal{C} \to \mathcal{C}$ is almost tautological: As $N\mathcal{C}$ is a quasicategory, its homotopy category allows for the easier description. Its objects are given by $N\mathcal{C}_0$, which are the objects of $\mathcal{C}$. The morphisms from x to y are arrows in $N\mathcal{C}$, i.e., morphisms in $\mathcal{C}$, modulo the homotopy relation:

$$\mathrm{Hom}_{hN\mathcal{C}}(x, y) = \{f \in N\mathcal{C}_1 | d_1 f = x, d_0 f = y\}/\sim = \mathrm{Hom}_{\mathcal{C}}(x, y)/\sim.$$

If $f, g : x \to y$ are homotopic, this is witnessed by a 2-simplex

$$\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"] & & y. \end{tikzcd}$$

But 2-simplexes in the nerve come from composition in $\mathcal{C}$, i.e., $g = \mathrm{id}_y \circ f = f$. So there is no relation, hence the morphisms in $hN\mathcal{C}$ and $\mathcal{C}$ also agree. Hence, $hN\mathcal{C}$ is naturally isomorphic(!) to $\mathcal{C}$.

For the unit $X \to NhX$, note there are canonical maps on 0- and 1-simplices: The 0-simplices are just the same, and there is a canonical generator map from X_1 to the morphisms of hX, which make the 1-simplices of NhX. For $n > 1$, we get a map

$$X_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) \xrightarrow{\mathrm{res}} \mathrm{Hom}_{\mathbf{sSet}}(I^n, X) \cong X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

$$\xrightarrow{h} \mathrm{mor}(hX) \times_{\mathrm{obj}(hX)} \cdots \times_{\mathrm{obj}(hX)} \mathrm{mor}(hX) \cong NhX_n.$$

Then one checks that these are natural maps, and satisfy the unit/counit relations. □

Corollary 4.4. *As the counit is a natural isomorphism, this also proves the earlier statement that N is fully faithful.*

Remark 4.5. This adjunction is a shadow of a general phenomenon: Let $\mathcal{C}$ be cocomplete, and consider a functor $F : \Delta \to \mathcal{C}$. Then F has a unique colimit-preserving extension $\hat{F} : \mathbf{sSet} \to \mathcal{C}$, which admits a right adjoint R satisfying

$$R(C)_n = \mathrm{Hom}_{\mathcal{C}}(F[n], C).$$

Examples of this are:

- $\Delta \to \mathbf{Top}, [n] \to |\Delta^n|$: This extends to the geometric realization, with right adjoint the singular complex $S(X)_n = \mathrm{Map}(|\Delta^n|, X)$.
- $\Delta \to \mathbf{Cat}, [n] \to \underline{n}$. This extends to the homotopy category functor, with right adjoint the nerve $N(\mathcal{C})_n = \mathrm{Fun}(\underline{n}, \mathcal{C})$.
- Next lecture, we'll see the simplicial nerve, which is obtained as a right adjoint from simplicially enriched categories to simplicial sets.

Even more so, there is nothing special here about Δ: This works for any (essentially small) category $\mathcal{D}$, it's really more about the magic of presheaves: The category of presheaves on $\mathcal{D}$, i.e., $\mathcal{P}(\mathcal{D}) = \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathbf{Set})$, should be thought of as "freely attaching all colimits to $\mathcal{D}$." This is one consequence of that.

5 Equivalences

Now, if quasicategories do homotopy theory, we need a notion of equivalences:

Definition 5.1. Let $\mathcal{C}$ be a quasicategory. An arrow $f : x \to y$ in $\mathcal{C}$ is an *equivalence* if it becomes an isomorphism in the homotopy category. Untangling definitions, this means that there exist 2-simplices

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_x"'] & & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "g'"] \arrow[rr, "\mathrm{id}_y"'] & & y. \end{tikzcd}$$

As mentioned initially, we should later expect a contractible space of inverses to f. In particular, we can later show that any choices for g and g' will be homotopic.

Earlier, we noticed that groupoids (i.e., categories where all morphisms are isomorphisms) are also Kan complexes. The obvious analogue for ∞-categories is the following:

Definition 5.2. An ∞-groupoid (anima, $(\infty, 0)$-category, surprisingly never quasigroupoid) is an ∞-category where all arrows are equivalences.

Proposition 5.3. *Every Kan complex is an anima.*

Proof. Let K be Kan. Then it is in particular a quasicategory. Consider any edge $f : x \to y$ in K. Then we can fill the outer horns

$$\begin{tikzcd} & y & \\ x \arrow[ur, "f"] \arrow[rr, "\mathrm{id}"'] & & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[rr, "\mathrm{id}"'] & & y. \end{tikzcd}$$

The resulting 2-simplices yield precisely our homotopy inverses. $\square$

Surprisingly, the converse holds! This is immensely useful, and was one of the initial motivations for developing the theory of ∞-categories:

Fact 5.4 (Grothendieck homotopy hypothesis)**.** Animae are precisely Kan complexes.

We'll see a (rather involved) proof in talk 7. Historically, this proof was the (first) big obstacle to making ∞-categories work—when Joyal managed this, many things fell into place quickly. In particular, all the "contractible spaces of choices" mentioned throughout will use some form of this statement.