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Reading Seminar 2021
Baez–Dolan Cobordism Hypothesis and ∞-Categories

Logo by Jarek Kedra

1. Overview

Topological quantum field theories (TQFTs) are a mathematical model of physics with zero Hamiltonian and in which spacetime has no metric structure (a toy model). Mathematically, TQFTs are used to formalize and construct invariants of manifolds (e.g., knot invariants [1]) and to organize the rules for computing these invariants by cutting-and-pasting laws.

In a way, the simplest TQFTs are fully extended, as these incorporate the largest amount of computational laws, allowing the decomposition of the manifold into pieces of arbitrary codimension all the way down to points. Keeping track of how these operations interact is a delicate matter that leads to complicated combinatorics. The development of rigorous foundations for fully extended TQFTs has been one of the driving forces of higher category theory. The cobordism hypothesis should be compared with the homotopy hypothesis, which is simpler and expresses the principle due to Grothendieck that topological spaces are equivalent to $(\infty, 0)$-categories (also called ∞-groupoids). As such, the cobordism hypothesis is a mixture of definition and conjecture that has informed the development of higher category theory.

To state it, recall that a classical result is that $1d$ TQFTs are classified by finite-dimensional vector spaces V, W equipped with duality pairings (V, W, ϵ, η). Similarly, a $2d$ TQFT is completely determined by a Frobenius algebra $(A, \nabla, \Delta, \epsilon, \eta)$, see [13]. The Baez–Dolan cobordism hypothesis [2] is a generalization of these results to higher dimensions and asserts that a fully extended TQFT is completely determined by a fairly small amount of algebraic data, its value on a point. The main difficulty in the proof of this conjecture [15] is to show that different decompositions of the same manifold, obtained using a generalization of Morse theory [12], result in the same operation in the TQFT. Here (∞, n)-categories are used in a profound way to exhibit these coherence properties.

The goal of the reading seminar is to understand this particular motivation for higher category theory, to learn the foundations of (∞, n)-categories following Land's book [14], and to give a sketch proof of the cobordism hypothesis. Examples will be emphasized throughout.

The schedule is not definite, and comments and suggestions are very welcome!

Time: Fridays 11:00 – 14:00 (lunch break 12:00-13:00)

2. Schedule of Talks

Classification of low-dimensional TQFTs.

1. (Nov. 15, Barthelemy Neyra) Topological quantum field theories. Classification of $1d$ TQFTs Introduce oriented bordisms between manifolds [13, §1.2.11], carefully discussing orientations and incoming and outgoing boundaries [13, §1.1.11]. Explain the classification of $1d$ TQFTs in terms of dualities (V, W, ϵ, δ) as in §4 of Freed [8] and prove [8, Thm. 4.9], filling in additional details. Explain some motivation for TQFTs from physics (e.g., from [1, §2]).

2. (Nov. 22, Peter Guthmann) Frobenius algebras and $2d$ TQFTs

Recall (symmetric) monoidal categories and functors [13, §3.2.13–§3.2.15, §3.2.17]. Give examples of both. Adapt the definition of TQFT in [13, 1.2.23] and instead define a TQFT as a monoidal functor as in [8, Def. 2.6].

Define (symmetric) Frobenius algebras [13, §2.2.1,§2.2.9] and give some examples, e.g. [13, §2.2.16, §2.2.18, §2.2.20, §2.2.23]. Carefully state the generators of the 2-dimensional bordism category following [13, §3.3.1] and sketch the proof of [13, Thm. 3.3.2].

Foundations of ∞-category theory.

3. (Dec. 6 & Dec. 13, Irakli Patchkoria) Model categories and simplicial sets

Give a brief overview of simplicial sets following [9, §I.1–§I.3, §I.5, §I.7]. Define model categories [7, §3], left and right homotopy classes [9, §II.1], derived functors. Give examples of model categories (only sketching proofs): simplicial sets and topological spaces [9, Thm. I.11.3] and chain complexes [9, p. 157]. Mention Kan complexes. Briefly point out the connection of the latter to homological algebra (resolutions, derived functors). Define Quillen equivalences and mention [9, Thm. I.11.4].

The original work by Quillen [17] is an additional reference on model categories. This talk should be given someone who has a good overview of the literature.

4. (Jan. 28, Julius Frank) Introduction to $(\infty, 1)$-categories

Follow Land's book [14] starting with [14, Thm. 1.1.52] and sketching its proof. Then prove [14, Lemma 1.1.54]. Then go through [14, §1.2] defining $(\infty, 1)$-categories and their homotopy category [14, Def. 1.2.5] and ∞-groupoids. Sketch the proof of associativity and other properties of the composition. Discuss the adjunction of [14, Prop. 1.2.18]. You can stop at page 35 after showing that any Kan complex is an ∞-groupoid.

5. (Feb. 4, Mark Grant) Examples of $(\infty, 1)$-categories I

This talk is about examples of $(\infty, 1)$ categories. The main source is simplicial categories. The models of $(\infty, 1)$-categories we looked at so far are also known as Joyal quasi-categories. Simplicial categories are another model for $(\infty, 1)$-categories. Introduce simplicial categories following [9, §II.3] or [14, Def. 1.2.39]. Give the construction of the simplicial nerve as in [14, Def. 1.2.56, Def. 1.2.63] and explain the low degree examples [14, Obs. 1.2.66] in detail. Prove [14, Lem. 1.2.64, Lem. 1.2.67] and sketch the proof that the simplicial nerve of a fibrant simplicial category is an $(\infty, 1)$-category as in [14, Lemma 1.2.70]. A good overview can be found in [10, §1.2].

6. (Feb. 11, Markus Upmeier) Joyal's lemma, functors, localizations and mapping spaces

Start by recalling [14, Thm. 1.4.23]. Then sketch the proof of [14, Thm 2.1.8] about lifting equivalences, and then get [14, Cor. 2.1.12] which finally tells us that ∞-groupoids are the same as Kan complexes. Discuss the notion of equivalence, natural transformations and fully faithful functors [14, 2.2-2.3]. Sketch the proof of [14, Thm. 2.3.20]. Prove the existence of localizations as in [14, Lem. 2.4.6]. Finally, discuss [14, Def. 2.5.28] and [14, Cor. 2.5.34].

7. (Feb. 25, Mar. 4, Irakli Patchkoria) The zoo of fibrations

Follow Land's book [14, Chapter 3].

7.5. (Mar. 11, Irakli Patchkoria) Straightening and unstraightening

Prove Theorem 8.3.8 in Haugseng [11, §8.3]. See also Higher topos theory [16, Chapters 2–3], Land's book [14, Chapter 3].

8. (Mar. 18, Richard Hepworth) Limits and colimits

Focus throughout on colimits. Define the Yoneda map [14, Def. 4.2.1] and build up to the embedding theorem [14, Prop. 4.2.11]. Define colimits as in [14, Def. 4.3.4] and state [14, Thm. 4.3.11] which compares these to the terminal objects of [14, Def. 4.1.1] and Lurie's definition. State and prove

[14, Lem. 4.3.13] on the uniqueness of limits. Define colimit preserving functors [14, Def. 4.3.30]. State [14, Prop. 4.3.28] and [14, Thm. 4.3.37] on (co)completeness.

9. (N.N.) Adjunctions and presentable $(\infty, 1)$-categories

Follow Land's book [14, Chapter 5].

10. (N.N.) Symmetric monoidal ∞-categories

TBD

Lurie's proof of the cobordism hypothesis.

11. (N.N.) Complete Segal spaces

Introduce the Reedy model structure following [5, §2.6] and state [5, Thm. 2.6.6] precisely. Define Segal spaces as in [5, §5.1] and explain how these form a model for $(\infty, 1)$-categories as in [5, §5.2]. Define complete Segal spaces and following [5, §5.3] or [3] and formulate [3, Thms. 7.1–7.5].

12. (N.N.) n-fold complete Segal sapces

Briefly give a reminder of complete Segal spaces and then introduce n-fold complete Segal spaces as in [6, §2]. Also, the overview given in Bergner [3, §10.4] and [4, §9] may be helpful.

13. (N.N.) Construction of the (∞, n)-category of bordisms I

Explain the toy case of an n-fold Segal space of closed intervals [6, §4].

14. (N.N.) Construction of the (∞, n)-category of bordisms II

Explain [6, §5] which makes Lurie's sketch [15, §2.2] precise (and adds a correction). This talk should be given by someone with experience in differential topology (explain role of Whitney embedding theorem). Briefly mention interpretation in terms of Morse theory and manifold with corners [6, §8].

15. (N.N.) Lurie's proof of the cobordism hypothesis

Given an overview of [15, §3].

References

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