Talk 10:
Symmetric monoidal ∞-categories

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1 Prelude: A Fibration

Before we start the actual talk, I want to revisit a 1-categorical example for a cocartesian fibration in some detail–I did this recently, and it really helped me to get a feeling for these. We'll also be able to extract some lessons for today's talk.

Consider the category CRing of commutative rings, and for each ring R, its module category Mod_R. (If you're feeling adventurous, you may also take associative rings.) There are several ways to make this functorial: If $f : R \to S$ is a ring map, we get adjoint functors

$$\begin{aligned} \mathrm{Mod}_R &\longleftrightarrow \mathrm{Mod}_S \\ f^*N &\longleftarrow\!\shortmid N \\ M &\longmapsto S \otimes_R M. \end{aligned}$$

The first one, restriction of scalars, is readily made into a functor $\mathrm{CRing}^{\mathrm{op}} \to \mathrm{Cat}$: The module f^*N has the same underlying abelian group as N, and $r \in R$ acts on n via $r.n = f(r)n$. This is compatible with composition and identities.

However, this does not quite work for the inductions: For example, the identity $R \to R$ gets mapped to $R \otimes_R M \to M$, which is a natural isomorphism, but not the identity! (This depends on how you model your tensor products, but morally, they only exist up to natural isomorphism.) Same for composition.

This is a natural example of a *pseudofunctor*: What's happening here is that Cat is actually a 2-category (categories, functors, natural transformations), and functors into Cat often want to be 2-functors. In this specific setting, we map from a 1-category, but composition and identity only hold up to natural isomorphism, making this into a $(2,1)$-functor / weak 2-functor / pseudofunctor.

Now, Irakli explained to us that a pseudofunctor $F : \mathcal{C} \to \mathrm{Cat}$ is the same data as a cocartesian fibration (classically: Grothendieck opfibration) $\int F \to \mathcal{C}$. The reason we care about this quite technical notion is that it turns the coherence *data* of the pseudofunctor into a *property* of the fibration, which is easier to construct and check!

Let us consider the cocartesian fibration for our example: By construction, $\int \mathrm{Mod}$ is the 1-category with objects (R,M) where R is a commutative ring, $M \in \mathrm{Mod}_R$, and a morphism from (R,M) to (S,N) consisting of

$$f : R \longrightarrow S$$
$$\phi : S \otimes_R M \longrightarrow N.$$

There is a natural projection $p : \int \mathrm{Mod} \to \mathrm{CRing}$ that forgets the modules. Let us unravel the technical notions in this setting:

Recall that a morphism $(f,\phi) : (R,M) \to (S,N)$ is *cocartesian* if for all diagrams

$$\begin{tikzcd} (R,M) \arrow[rr, "(f{,}\phi)"] \arrow[dr, "(g{,}\psi)"'] & & (S,N) \\ & (Q,L) & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[rr, "f"] \arrow[dr, "g"'] & & S \arrow[dl, "h"] \\ & Q, & \end{tikzcd}$$

there is a unique lift of $h : S \to Q$ to a $(h,\xi) : (S,N) \to (Q,L)$. In other words, for such a setting we need a unique $\xi : Q \otimes_S N \to L$ such that $\xi \circ (Q \otimes_S \phi) = \psi$:

$$\begin{tikzcd} Q \otimes_R M \cong Q \otimes_S (S \otimes_R M) \arrow[rr, "Q\otimes_S\phi"] \arrow[dr, "\psi"'] & & Q \otimes_S N \arrow[dl, "\xi"] \\ & L. & \end{tikzcd}$$

It turns out that we can give a nice explicit description of the cocartesian edges:

Proposition 1.1. *A morphism $(f,\phi) : (R,M) \to (S,N)$ is cocartesian if and only if $\phi : S \otimes_R M \to N$ is an isomorphism.*

Proof. If ϕ is an isomorphism, so is $Q \otimes_S \phi$, and ξ must be $\psi \circ (Q \otimes_S \phi)^{-1}$.

Conversely, let (f,ϕ) be cocartesian. To see that ϕ is an isomorphism, we take good choices for (Q,L): First, consider the setting

$$\begin{tikzcd} (R,M) \arrow[rr, "(f{,}\phi)"] \arrow[dr, "(f{,}0)"'] & & (S,N) \\ & (S, \mathrm{coker}\phi) & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[rr, "f"] \arrow[dr, "f"'] & & S \arrow[dl, "\mathrm{id}"] \\ & S, & \end{tikzcd}$$

Then both 0 the canonical projection $S \otimes_S N \cong N \to \mathrm{coker}\phi$ are lifts of id_S, so they must be equal–hence ϕ is surjective.

Likewise, in

$$\begin{tikzcd} (R,M) \arrow[rr, "(f{,}\phi)"] \arrow[dr, "(f{,}\mathrm{id})"'] & & (S,N) \\ & (S, S \otimes_R M) & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[rr, "f"] \arrow[dr, "f"'] & & S \arrow[dl, "\mathrm{id}"] \\ & S, & \end{tikzcd}$$

the existence of a lift of id_S gives us a left inverse for ϕ, hence ϕ is injective. $\square$

It is now easy to see that $p : \int \mathrm{Mod} \to \mathrm{CRing}$ is indeed a cocartesian fibration: For any ring map $f : R \to S$ and object (R, M), there exists a cocartesian lift $(f, \mathrm{id}) : (R, M) \to (S, S \otimes_R M)$.

If we start with any morphism $(f, \phi) : (R, M) \to (S, N)$, we can factor it as

$$(R, M) \xrightarrow{(f,\mathrm{id})} (S, S \otimes_R M) \xrightarrow{(\mathrm{id},\phi)} (S, N),$$

i.e., into a cocartesian map, and a map in the fibre $p^{-1}(S)$. This works for any cocartesian fibration: The first map is a cocartesian lift of f, the second map exists uniquely by the cocartesianness. This suggests to think of $\int \mathrm{Mod}$ (or any cocartesian fibration) as consisting of the categories sitting in the fibres, and cocartesian morphisms between the fibres that assemble into functors between the categories. All other morphisms are then generated from these.

Remark 1.2. There is another nice thing hidden in this example: Note that by the adjunction, a map $\phi : S \otimes_R M \to N$ is the same as a map $\hat{\phi} : M \to f^* N$. Then one can show dually to the above that the *cartesian* edges are precisely those where $\hat{\phi}$ is an isomorphism, this gives us the other (contravariant) functor into Cat. And since this means that p is both cartesian and cocartesian, this gives us the adjunctions back! For any $f : R \to S$, the pullback of p along $\Delta^1 \xrightarrow{f} \mathrm{CRing}$ yields precisely the cartesian and cocartesian fibration over Δ^1 corresponding to the adjunction.

There are two lessons to be taken away here:

1. ∞-categories are just fancy 1-categories, and it is always necessary to understand these, first.

2. (Co)cartesian fibrations are a useful tool to handle (some) 2-categorical issues in a 1-categorical way.

2 Monoidal 1-categories

According to the first lesson, we need to properly understand symmetric monoidal 1-categories first: The definition of symmetric monoidal ∞-categories will be *just the same*, as long as we've put it in the right language in the 1-category setting first. Let's start with the classical definition:

Definition 2.1. A monoidal 1-category consists of the following data:

- A 1-category $\mathcal{C}$,
- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- an object $1 \in \mathcal{C}$ (considered as a functor $* \to \mathcal{C}$),
- a natural isomorphism $(x \otimes y) \otimes z \xrightarrow{\alpha} x \otimes (y \otimes z)$,
- natural isomorphisms $1 \otimes x \xrightarrow{\eta_L} x \xleftarrow{\eta_R} x \otimes 1$,

satisfying the following coherence axioms:

$$\begin{tikzcd}
(x\otimes 1)\otimes y \arrow[rr] \arrow[dr] & & x\otimes(1\otimes y) \arrow[dl] \\
& x\otimes y &
\end{tikzcd}$$

[and the pentagon of different bracketings of $x\otimes y\otimes z\otimes w$. I won't tex that.]

This is, of course, a failure in mathematical communication: We don't intrinsically care about the pentagon, it's just another level of coherence: The associator gives us coherent 3-fold tensors, and the pentagon 4-fold tensors. Why stop there? The answer is the following theorem, which should be the definition:

Theorem 2.2 (Mac Lane)**.** *In a monoidal* 1*-category* $\mathcal{C}$*, there is a canonical isomorphism between any two bracketings of* $x_1\otimes\cdots\otimes x_n$*, obtained by any possible combination of associators and inserting units.*

This allows us to denote $x_1\otimes\cdots\otimes x_n$ like that, without brackets. The classical definition is really a theorem, namely that this property can be checked on the pentagon.

It also gives us the following:

Corollary 2.3. *Any monoidal* 1*-category can be strictified, i.e., it is monoidally equivalent to a strict monoidal* 1*-category where associator and unitors are the identity.*

It is useful to have both tools at hand: Strict monoidal categories rarely come up in nature, but come with less notational overhang. Note that this distinction will vanish in the homotopy-coherent world: In the non-strict monoidal category, the coherence theorem gives us a contractible groupoid between all the bracketings, which we cannot distinguish from a point.

Let us now take symmetry into the mix:

Definition 2.4. A *symmetric monoidal category* is a monoidal category $\mathcal{C}$ with a natural isomorphism

$$x\otimes y \xrightarrow{s} y\otimes x,$$

satisfying hexagon coherence diagrams with the associator (the two ways to turn $(x\otimes y)\otimes z$ into $y\otimes(z\otimes x)$, and the same with swapped bracketing), and is involutive:

$$s^2=\mathrm{id}: x\otimes y\longrightarrow x\otimes y.$$

(Without the latter, we'd end up with a braided monoidal category.)

Again, this is the wrong definition: Morally, a symmetric monoidal category is a category with tensor product where we don't have to care about bracketing *and order* of the factors.

Now, in order to transfer these definitions to ∞-categories, we need to make them more coherent and less "generators and relations"-y. Let's start with the strict version.

Construction 2.5. Consider the category Fin_* of pointed finite sets. We identify the objects $\langle n\rangle=\{0,\dots,n\}$ with 0 the basepoint.

Let $\mathcal{C}$ be a strict symmetric monoidal category. We construct a functor

$$\begin{aligned}\underline{\mathcal{C}}:\mathrm{Fin}_*&\longrightarrow \mathrm{Cat}\\ \langle n\rangle&\longmapsto \mathcal{C}^{\times n}\end{aligned}$$

as follows: Any map $f : \langle n\rangle \to \langle m\rangle$ decomposes uniquely (up to isomorphism) as

$$\langle n\rangle \xrightarrow{i} \langle n'\rangle \xrightarrow{g_+} \langle m\rangle,$$

where i is inert, i.e., it sends some points to the basepoint, and is isomorphic everywhere else; and g_+ sends only 0 to 0. Then the induced map $f : \mathcal{C}^{\times n} \to \mathcal{C}^{\times m}$ is given as

$$\mathcal{C}^n \xrightarrow{i_*} \mathcal{C}^{n'} \xrightarrow{g_*} \mathcal{C}^m$$

where i_* forgets all factors that i sends to 0, and g_* is given on the kth factor of $\mathcal{C}^m$ as the tensor product

$$\mathcal{C}^{\times n'} \xrightarrow{\text{proj}} \prod_{j\in g^{-1}k} \mathcal{C} \xrightarrow{\otimes} \mathcal{C}.$$

[Draw some trees.] In words: We multiply those $\mathcal{C}$-factors which are identified by f, and forget all the factors that are sent to 0.

It turns out that this really gives us an equivalent notion of strict symmetric monoidal categories:

For any $1 \le i \le n$, we define $p_i : \langle n\rangle \to \langle 1\rangle$ sending i to 1, and all else to 0.

Theorem 2.6. *The assignment $\mathcal{C} \mapsto \underline{\mathcal{C}}$ identifies strict symmetric monoidal categories with those functors $F : \mathrm{Fin}_* \to \mathrm{Cat}$ for which the maps $p_i : \langle n\rangle \to \langle 1\rangle$, $1 \le i \le n$ assemble into an isomorphism*

$$F\langle n\rangle \xrightarrow{\sim} F\langle 1\rangle^n$$

This new perspective on symmetric monoidal categories is the right one for us to consider: It encodes all the ways of writing n factors into a product, and ensures that all ways of multiplying them are equal. (It is a nice exercise to extract the symmetric monoidal category from such a functor!)

For a general symmetric monoidal category, we want associativity and unitality only up to coherent isomorphisms. It turns out that translating this to this functor setting means that we should consider *pseudofunctors* $F : \mathrm{Fin}_* \to \mathrm{Cat}$. And we have a nicer model for these: Cocartesian fibrations $\int F \to \mathrm{Fin}_*$ are precisely the data of a symmetric monoidal category! (Again, it is very instructive to work out how one recovers a symmetric monoidal category from this.)

3 Monoidal functors

Before we go *homotopical*, we still need to talk about morphisms of symmetric monoidal categories. There is more than one useful notion here:

Definition 3.1. Let $\mathcal{C}$, $\mathcal{D}$ be symmetric monoidal categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called

1. *lax symmetric monoidal* if it comes with natural transformations

$$F(x) \otimes F(y) \longrightarrow F(x \otimes y)$$

$$1 \longrightarrow F(1),$$

such that these are suitably coherent with associators (get a diagram for the two ways of going from $(Fx\otimes Fy)\otimes Fz$ to $F(x\otimes(y\otimes z))$), units, and commute with the symmetry;

2. *(strong) symmetric monoidal* if it is lax and these transformations are isomorphisms, and

3. *strict symmetric monoidal* if it is lax and these transformations are identities.

Most examples we encounter will be strong, e.g., TQFTs $\mathrm{Bord}_{\langle n-1,n\rangle} \to \mathrm{Vect}_{\mathbb{C}}$. But every so often, lax functors come up. To have one example, the forgetful $\mathrm{Vect}_{\mathbb{C}} \to$ **Set** is lax monoidal: There is a natural transformation of sets $V \times W \to V \otimes W$, sending (v, w) to $v \otimes w$. But this is not a bijection. Note that like adjunctions, lax functors are 2-categorical, so we'll have to employ some trickery to formulate them in $(\infty, 1)$-categories. Strong functors are $(2, 1)$-categorical, so they will be more straightforward. Let's see this in our discrete setting, first:

Proposition 3.2. *A strict monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is the same data as a natural transformation $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ of the associated functors* $\mathrm{Fin}_* \to \mathrm{Cat}$.

Proof. We'll only sketch how $\underline{F}$ is constructed: Given F, $\underline{F}$ is given by

$$\underline{F}\langle n\rangle : \underline{\mathcal{C}}\langle n\rangle = \mathcal{C}^n \xrightarrow{F^n} \mathcal{D}^n = \underline{\mathcal{D}}\langle n\rangle.$$

Then the symmetric monoidal structure translates precisely to this being natural, e.g., the multiplication map $m : \langle 2\rangle \to \langle 1\rangle$ inducing the product yields a square

$$\begin{tikzcd} \mathcal{C}^2 \arrow[r, "F"] \arrow[d, "m"] & \mathcal{D}^2 \arrow[d, "m"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

the commutativity then means precisely that $F(x \otimes y) = F(x) \otimes F(y)$. □

Likewise, a strong monoidal functor should be a transformation of pseudofunctors. Instead of annoying ourselves with collecting the required coherence data, let's immediately consider cocartesian fibrations:

Definition 3.3. Let $p : \mathcal{E} \to \mathcal{C}$ and $p' : \mathcal{E}' \to \mathcal{C}$ be two cocartesian fibrations. A *morphism of cocartesian fibrations* from p to q is a commuting diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "F"] \arrow[dr, "p"] & & \mathcal{E}' \arrow[dl, "p'"] \\ & \mathcal{C} & \end{tikzcd}$$

such that F sends cocartesian edges to cocartesian edges.

Then the correspondence between cocartesian fibrations and pseudofunctors becomes an equivalence of categories, i.e., these give precisely the right notion of transformations of pseudofunctors.

Proposition 3.4. *Under the identification of symmetric monoidal categories and cocartesian fibrations over* Fin_**, the strong symmetric monoidal functors correspond to morphisms of cocartesian fibrations.*

Proof. This time, we'll only sketch how one recovers F from $F^{\otimes}$:

Let $\mathcal{C}$, $\mathcal{D}$ be symmetric monoidal categories, and $F^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a morphism of their associated cocartesian fibrations. Then one recovers a symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ as follows:

- As a functor, F is obtained by pulling back $F^{\otimes}$ over the object $\langle 1\rangle$, we obtain
$$F = F^{\otimes}_{\langle 1\rangle} : \mathcal{C} = \mathcal{C}^{\otimes}_{\langle 1\rangle} \to \mathcal{D}^{\otimes}_{\langle 1\rangle} = \mathcal{D}.$$
- Likewise, we obtain $F^{\otimes}_{\langle 2\rangle} : \mathcal{C}^{\otimes}_{\langle 2\rangle} \to \mathcal{D}^{\otimes}_{\langle 2\rangle}$. First, we need to see that under the identifications $\mathcal{C}^2 = \mathcal{C}^{\otimes}_{\langle 2\rangle}$ and $\mathcal{D}^{\otimes}_{\langle 2\rangle} = \mathcal{D}^2$, $F^{\otimes}_{\langle 2\rangle}$ corresponds to F^2. For this, consider an object $(c_1, c_2) \in \mathcal{C}^2$ and the projections $\langle 1\rangle \xleftarrow{p_1} \langle 2\rangle \xrightarrow{p_2} \langle 1\rangle$ in Fin_* that send 2 resp. 1 to the basepoint. Then there are cocartesian lifts given as $c_1 \mapsfrom (c_1, c_2) \mapsto c_2$. Applying $F^{\otimes}$ then gives us cocartesian arrows
$$F(c_1) \mapsfrom F^{\otimes}_{\langle 2\rangle}(c_1, c_2) \mapsto F(c_2).$$
But after identifying $\mathcal{D}^{\otimes}_{\langle 2\rangle} = D^2$, this means that the projections of $F^{\otimes}_{\langle 2\rangle}(c_1, c_2)$ to the factors are $F(c_1)$ and $F(c_2)$, i.e., $F^{\otimes}_{\langle 2\rangle}(c_1, c_2) = (F(c_1), F(c_2))$. (The same argument works for higher $\langle n\rangle$.)
- To see how $F(c_1 \otimes c_2)$ relates to $F(c_1) \otimes F(c_2)$, consider again the multiplication map $m : \langle 2\rangle \to \langle 1\rangle$ that sends 1 and 2 to 1. Then this lifts to a cocartesian arrow $(c_1, c_2) \to c$, where cocartesianness here translates to the fact that $c_1 \otimes c_2 \cong c$. Applying F then gives us a cocartesian arrow
$$(F(c_1), F(c_2)) = F^{\otimes}_{\langle 2\rangle}(c_1, c_2) \to F(c),$$
where the fact that this is cocartesian translates to an isomorphism $F(c_1) \otimes F(c_2) \to F(c) \cong F(c_1 \otimes c_2)$.

□

Now, to define a *lax* monoidal functor, we need a setting where the first two arguments of the previous proof work, but the third one doesn't:

Proposition 3.5. *A lax monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is the same as a commuting diagram*

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F"] \arrow[dr] & & \mathcal{D}^{\otimes} \arrow[dl] \\ & \mathrm{Fin}_* & \end{tikzcd}$$

such that F preserves cocartesian arrows if they are lying over inert maps $\langle m\rangle \to \langle n\rangle$.

Proof. The projections p_1, p_2 are inert, besides the basepoints they are bijections of singletons. The multiplication map is not inert. However, one still obtains a map
$$(F(c_1), F(c_2)) = F^{\otimes}_{\langle 2\rangle}(c_1, c_2) \to F(c).$$
To see this, note that there is a cocartesian lift $(F(c_1), F(c_2)) \to F(c_1) \otimes F(c_2)$ of m. There is also the map $F(m) : (F(c_1), F(c_2)) \to F(c) \cong F(c_1 \otimes c_2)$. The cocartesianness of the lift then precisely gives us a unique map $F(c_1) \otimes F(c_2) \to F(c_1 \otimes c_2)$. The only thing from the previous proof that fails is that this map needn't be an isomorphism. (Of course, one has to show similar things for unity and coherences.) □

4 Infinity

This will now be uninteresting: Everything is just the same.

Remark 4.1. Just because it hasn't been said before: Cat_∞ wants to be an $(\infty, 2)$-category. In order to get the $(\infty, 1)$-category, we had to forget all natural transformations that were not natural equivalences. In other words,

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}.$$

This raises the same complications as in the discrete setting, and is the whole reason we need to talk about these fibrations.

Definition 4.2.

- A *symmetric monoidal ∞-category* is a functor $\underline{\mathcal{C}} : N\mathrm{Fin}_* \to \mathrm{Cat}_\infty$, or, equivalently, a cartesian fibration $\mathcal{C}^\otimes \to N\mathrm{Fin}_*$.
- A *strong monoidal functor* is a natural transformation $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, or, equivalently, a (homotopy) commutative diagram

$$\begin{tikzcd} \mathcal{C}^\otimes \arrow[rr, "F^\otimes"] \arrow[dr] & & \mathcal{D}^\otimes \arrow[dl] \\ & N\mathrm{Fin}_* & \end{tikzcd}$$

  s.t. $F^\otimes$ preserves cocartesian edges.

- A *lax symmetric monoidal* functor is a (homotopy) commutative diagram

$$\begin{tikzcd} \mathcal{C}^\otimes \arrow[rr, "F^\otimes"] \arrow[dr] & & \mathcal{D}^\otimes \arrow[dl] \\ & N\mathrm{Fin}_* & \end{tikzcd}$$

  s.t. $F^\otimes$ preserves cocartesian edges over inert morphisms.

Let's collect examples:

Example 4.3.

1. If $\mathcal{C}$ is a symmetric monoidal 1-category, $N\mathcal{C}$ is a symmetric monoidal ∞-category, e.g. through $N\mathrm{Fin}_* \xrightarrow{N\underline{\mathcal{C}}} N\mathrm{Cat} \hookrightarrow \mathrm{Cat}_\infty$.
2. There is a notion of monoidal model categories, their underlying ∞-categories then become monoidal, too.
3. If $\mathcal{C}$ has all coproducts or products, these give a symmetric monoidal structure.
4. To give a non-trivial example: The ∞-category **Sp** of spectra has a symmetric monoidal structure through the smash product. This was classically very hard to describe, now you can define it in one line through its universal property. (It's still complicated, though–nothing is ever free.)

5. The disjoint union makes the (upwards-extended) bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}$ into a symmetric monoidal ∞-category. This is where TQFTs live! (Likewise, the fully extended Bord_n is a symmetric monoidal $(\infty,1)$-category, but we don't know yet what that means.)

Remark 4.4. In case you wondered about non-symmetric monoidal categories: Essentially the same thing works, but we need to inflate Fin_* in order to remember the ordering of tensor products. One way of doing this is to take the same objects, but to equip morphisms with a total ordering on their fibres–it's relatively straightforward to trace this ordering through all our considerations above.

Surprisingly (confusingly?), there is another way of doing this: One can instead take the category Δ^{op}: It is a nice exercise to translate this to our setting. Note that this category is not equivalent to the previous one! Essentially, Δ^{op} doesn't allow permutations of the factors, while the previous category allows it and has to keep track of it.