BORDISM OF IMMERSIONS

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Contents

Abstract

This thesis describes several aspects of the functor given by bordism of immersions of closed manifolds, from a geometrical viewpoint. There are two new results.

The first is a generalisation of Herbert's Theorem [15] relating the homology classes represented by multiple points of a self-transverse immersion. Whilst admitting a much simpler proof, Herbert's Theorem in the bordism of immersions also implies analogues in other generalised cohomology theories.

The second result, again geometrical in nature, shows a relationship between the self-intersection operations in the bordism of immersions, and the generalised Steenrod operations of tom Dieck [37]. As a Corollary, we obtain a new construction of the Steenrod squares on those $\mathbb{Z}_2$-cohomology classes of a manifold whose Poincaré dual homology class contains an immersion.

The main technique employed is that of extending an immersion to an immersion of its normal disc bundle, to avoid transversality issues. Such 'spreadings' were studied extensively by Vogel [38], whose proof of the classification result (relating bordism of immersions to the stable homotopy of Thom spaces) is reproduced here.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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Chapter 1

Introduction

Here we give the basic definitions of differential topology which we shall need, and then briefly summarise the contents of each Chapter. References can be found in the main text.

In this thesis, the word 'manifold' will mean 'smooth, Hausdorff manifold (differentiable of class C^∞)'. Superscripts without parentheses will be used to denote dimension, so for example, N^n may denote a manifold of dimension n. A map $f\colon M^{n-k} \to N^n$ between manifolds will be assumed smooth unless otherwise stated. The integer k is called the *codimension* of the map f. Recall that such a map f induces a map $df\colon TM \to TN$ of tangent bundles, called the *derivative* of f.

Definition 1.1. *A map $f\colon M^{n-k} \to N^n$ of non-negative codimension k is called an* immersion *if for every point $x \in M$, the linear map of tangent spaces $df_x\colon TM_x \to TN_{f(x)}$ has rank $(n-k)$. An immersion will be denoted by $f\colon M^{n-k} \looparrowright N^n$.*

Note that although the derivative df_x of an immersion $f\colon M^{n-k} \looparrowright N^n$ is injective at each point $x \in M$, the map f may not itself be injective. We call the points of M at which injectivity fails the *multiple points* of f. The image of the multiple points in N will be called *self-intersections* of f.

Definition 1.2. *An immersion $f\colon M^{n-k} \looparrowright N^n$ which is homeomorphic onto its image $f(M) \subseteq N$ is called an* embedding*, and is written $f\colon M^{n-k} \hookrightarrow N^n$.*

Note that an embedding is an immersion without self-intersections, but an injective immersion may not be an embedding. For example, there is an injective immersion $f\colon \mathbb{R}^1 \looparrowright \mathbb{R}^2$ with image a figure eight.

The correct notions of homotopies of immersions and embeddings are as follows. Let $I = [0,1]$ be the closed unit interval.

Definition 1.3. *Two immersions $f_0, f_1\colon M^{n-k} \looparrowright N^n$ are said to be* regularly homotopic *if there is a smooth homotopy $F\colon M \times I \to N$ from f_0 to f_1 such that at each stage $t \in I$ the map $F(-,t)\colon M \to N$ is an immersion.*

Definition 1.4. *Two embeddings $f_0, f_1\colon M^{n-k} \hookrightarrow N^n$ are said to be* isotopic *if there is a smooth homotopy $F\colon M \times I \to N$ from f_0 to f_1 such that at each stage $t \in I$ the map $F(-,t)\colon M \to N$ is an embedding.*

For an immersion $f\colon M^{n-k} \looparrowright N^n$, we may regard TM as a sub-bundle of the pullback bundle f^*TN. This is done by identifying TM with the isomorphic bundle $df(TM)$, whose fibre over $x \in M$ is the $(n-k)$-dimensional vector subspace $df_x(TM_x) \leq TN_{f(x)}$.

Definition 1.5. *The quotient bundle $\nu(f) := f^*TN/df(TM)$ is a k-dimensional bundle over M called the* normal bundle *of f.*

The normal bundle to an immersion $f\colon M^{n-k} \looparrowright N^n$ may be given the structure of a *smooth vector bundle*. This means that we may give the total space $\nu(f)$ the structure of a smooth manifold of dimension n, such that the bundle projection $p\colon \nu(f) \to M$ is a smooth map. It is a well known fact (see [6] Remark 4.12, for example) that any smooth vector bundle may be given a smooth Riemannian metric. A vector bundle with a Riemannian metric will be called a *Riemannian vector bundle*. Note that a bundle map $v\colon \zeta \to \xi$ between Riemannian vector bundles which is isometric on fibres induces a map of unit disk and unit sphere bundles, and hence a map $Tv\colon T\zeta \to T\xi$ of Thom spaces. We henceforth assume $\nu(f)$ to be smooth and Riemannian.

Definition 1.6. *Let $f\colon M^{n-k} \looparrowright N^n$ be an immersion, and let ζ be a Riemannian vector bundle of dimension k. A ζ-*structure *for f is a bundle map $v\colon \nu(f) \to \zeta$ which is isometric on fibres.*

We now come to our main objects of study. For a fixed manifold N^n with empty boundary and a Riemannian vector bundle ζ of dimension k, consider the set $\mathrm{Imm}(N;\zeta)$ of all immersions $f\colon M^{n-k} \looparrowright N^n$ with ζ-structure, where M is closed (compact and with empty boundary). This set may be very large; a classification tool is provided by bordism of immersions.

In Chapter 2 we define the equivalence relation on $\mathrm{Imm}(N;\zeta)$ given by bordism of immersions, and the resulting commutative monoid $\mathcal{I}(N;\zeta)$. These monoids are functorial in both arguments N and ζ, and between them are external and internal product pairings. Additional structure on the functor $\mathcal{I}(-;-)$ is provided by the *self-intersection operations*, which are functions $\psi_r\colon \mathcal{I}(N;\zeta) \to \mathcal{I}(N;\mathcal{S}_r\zeta)$ natural in N, where the bundle $\mathcal{S}_r\zeta$ is the *r-th extended power* of ζ. For a generic immersion $f\colon M^{n-k} \looparrowright N^n$ the set of r-fold self-intersections is itself the image of an immersion $\psi_r(f)\colon \Delta_r(f) \looparrowright N$ with $\mathcal{S}_r\zeta$-structure, leading to the definition of ψ_r. We may also extend the definition of $\mathcal{I}(-;-)$ to allow manifolds with boundary.

The main result of Chapter 3 is a new proof of Herbert's Theorem in the setting of bordism of immersions. Herbert originally gave a recursive formula relating the homology classes in M represented by the multiple points of an immersion $f\colon M^{n-k} \looparrowright N^n$, but his proof was encumbered by the fact that homology classes are represented by singular simplices rather than immersions. Our proof seems simpler as it is in some sense closer to the geometry.

Chapter 4 is a review of cobordism theories. We adopt Quillen's approach and think of a cobordism class of N^n in degree k as being represented by a codimension k proper map to N with some prescribed orientation. This allows us to relate the cobordism functors to the functor $\mathcal{I}(-;-)$, and hence obtain Herbert's Theorem in any cohomology theory where the constituent normal bundles are orientable. Finally we discuss Steenrod operations in generalised cohomology theories, give their construction in cobordism theory due to tom Dieck, and describe their geometric interpretation in terms of proper maps.

A useful tool in studying immersions up to bordism is the functor $\mathcal{J}(-;-)$ on pairs of pointed spaces given by bordism of spreadings, as defined by Pierre Vogel with the French name 'étalements'. The close relationship between the two functors $\mathcal{I}(-,-)$ and $\mathcal{J}(-;-)$, described in Chapter 5, means that the bordism class of an immersion $f\colon M^{n-k} \looparrowright N^n$ with ζ-structure is essentially determined by an extension of f to an immersion of its normal unit disc bundle, and the behaviour of the map $v\colon \nu(f) \to \zeta$ near the zero section. Since such a 'spreading' of f has zero codimension, many transversality issues may be avoided by taking this approach.

As an example, in Chapter 6 we apply the results of Chapter 5 to obtain a formula relating the double point operation ψ_2 and the Steenrod operations of $\mathbb{Z}_2$ type in unoriented cobordism and $\mathbb{Z}_2$-cohomology. This is our main original result, but in some sense it raises more questions than it answers; some of these are discussed in Section 6.3.

In Chapter 7 we prove the homotopy classification result which relates bordism of immersions to the stable homotopy of Thom spaces. The proof we give is due to Vogel. We also outline the connection between self-intersection operations and the James-Hopf maps from stable homotopy theory. None of the material in this Chapter is new, but it is included here for completeness.

Chapter 2

Bordism of Immersions

We wish to study the set of all immersions of compact manifolds into a given manifold, with a given structure on their normal bundles. This can be made precise as follows. Let N^n be a connected manifold with empty boundary, and let ζ be a k-dimensional real Riemannian vector bundle over a space X having the homotopy type of a connected manifold. The data we wish to consider are triples (M^{n-k}, f, v) where M^{n-k} is a closed manifold, $f\colon M \looparrowright N$ is an immersion with normal bundle $\nu(f)$, and $v\colon \nu(f) \to \zeta$ is a ζ-structure for f. Here the word 'closed' means 'compact with empty boundary'. We shall denote the set of all such triples $\mathrm{Imm}(N; \zeta)$. We must first partition this set using a suitable equivalence relation. To begin with, we could consider immersions equivalent if they differ only by a diffeomorphism of the source manifold.

Definition 2.1. *Two triples* (M, f, v) *and* (M', f', v') *in* $\mathrm{Imm}(N; \zeta)$ *are* diffeomorphic *if there is a diffeomorphism* $g\colon M \to M'$ *such that* $f = f' \circ g$*, and* $v = v' \circ \overline{g}$ *as bundle maps, where the bundle map* $\overline{g}\colon \nu(f) \to \nu(f')$ *comes from the isomorphism* $\nu(f) \cong g^*\nu(f')$*.*

However the resulting set of equivalence classes is still too large to be tractable in general. We need a coarser equivalence relation; this is provided by bordism of immersions.

2.1 The Monoid $\mathcal{I}(N;\zeta)$

Definition 2.2. *Two triples (M_0, f_0, v_0) and (M_1, f_1, v_1) in* $\mathrm{Imm}(N;\zeta)$ *are* bordant, *written $(M_0, f_0, v_0) \sim (M_1, f_1, v_1)$, if there is a triple (W^{n-k+1}, F, V) consisting of a compact manifold W with boundary ∂W, an immersion $F\colon W \looparrowright N \times I$ transverse to $N \times \partial I$ such that $F^{-1}(N \times \partial I) = \partial W$, and a ζ-structure $V\colon \nu(F) \to \zeta$ for F; and such that the triple $(\partial W, F|_{\partial W}, V|_{\partial W})$ is diffeomorphic to the triple $(M_0 \sqcup M_1, (f_0, 0) \sqcup (f_1, 1), v_0 \sqcup v_1)$.*

Here the symbol $\sqcup$ denotes disjoint union. One may show that bordism as defined is an equivalence relation. We denote the resulting set $\mathrm{Imm}(N;\zeta)/\sim$ of bordism classes by $\mathcal{I}(N;\zeta)$. The bordism class of a triple (M, f, v) will be denoted $[M, f, v]$, or simply by $[f]$ when we wish to suppress notation.

Lemma 2.3. *Diffeomorphic triples are bordant.*

Proof. The triple $(M \times I, f \times \mathbf{1}_I, v \circ \pi)$ is readily seen to give a bordism between diffeomorphic triples (M, f, v) and (M', f', v'), where $\pi\colon \nu(f \times \mathbf{1}_I) \cong \nu(f) \times I \to \nu(f)$ is the projection. □

Proposition 2.4. *$\mathcal{I}(N;\zeta)$ has the structure of a commutative monoid.*

Proof. The addition operation is given by disjoint union of representatives, so

$$[M, f, v] + [M', f', v'] = [M \sqcup M', f \sqcup f', v \sqcup v'].$$

This is easily checked as being well-defined. The unit is given by the empty immersion. Commutativity and associativity follow from the diffeomorphisms $M \sqcup M' \approx M' \sqcup M$ and $(M \sqcup M') \sqcup M'' \approx M \sqcup (M' \sqcup M'')$ and Lemma 2.3. □

We also record the following simple fact.

Proposition 2.5. *Let $f_0, f_1\colon M^{n-k} \looparrowright N^n$ be regularly homotopic immersions, and let $v_0\colon \nu(f_0) \to \zeta$ be a ζ-structure for f_0. Then f_1 may be given a ζ-structure v_1 such that the triples (M, f_0, v_0) and (M, f_1, v_1) are bordant.*

Proof. Let $F\colon M \times I \to N$ be a regular homotopy from f_0 to f_1. Then the map $\overline{F}\colon M \times I \to N \times I$ given by $\overline{F}(m,t) = (F(m,t),t)$ is an immersion, whose normal bundle $\nu(\overline{F})$ restricts to $\nu(f_i)$ over $M \times \{i\}$ for $i = 0,1$. The Proposition will be proved if we can find a compatible ζ-structure for $\overline{F}$. Note that $\nu(\overline{F}) \cong \pi^*\nu(f_0)$, where $\pi\colon M \times I \to M \times \{0\}$ is the projection, and composing this isomorphism with the bundle map $\overline{\pi}\colon \pi^*\nu(f_0) \to \nu(f_0)$ gives a bundle map from $\nu(\overline{F})$ to $\nu(f_0)$. Composing this with $v_0\colon \nu(f_0) \to \zeta$ gives the required ζ-structure. □

2.2 Functoriality

In fact the pairing $\mathcal{I}(-;-)$ is functorial in both variables, N and ζ. We now make this statement precise. Let $\mathcal{D}_0$ be the category whose objects are finite dimensional manifolds with empty boundary, and whose morphisms are the proper immersions (a map between topological spaces is called *proper* if the inverse image of any compact set is compact). Let Vect denote the category whose objects are real Riemannian vector bundles over spaces having the homotopy type of connected manifolds, and whose morphisms are the bundle maps between bundles of the same dimension which are isometric on fibres (our reasons for using this restricted category of vector bundles will become clear in Chapter 5, when we introduce the Thomification functor $T\colon \mathsf{Vect} \to \mathscr{T}_\bullet$ to the category of pointed spaces and maps). Let CMon be the category of commutative monoids and monoid maps. Finally, for a category $\mathcal{C}$, let $\mathcal{C}^{\mathsf{op}}$ denote its opposite category.

Proposition 2.6. *Bordism of immersions is a homotopy bifunctor*

$$\mathcal{I}(-;-)\colon \mathcal{D}_0^{\mathsf{op}} \times \mathsf{Vect} \to \mathsf{CMon}.$$

Proof. We first show that $\mathcal{I}(N;-)$ is a covariant homotopy functor. It is clear that a morphism $\eta\colon \zeta \to \xi$ in Vect induces a well-defined map of monoids $\eta_*\colon \mathcal{I}(N;\zeta) \to \mathcal{I}(N;\xi)$ sending $[M,f,v]$ to $[M,f,\eta \circ v]$, and that this assignment is functorial. To show that homotopic bundle maps $\eta \simeq \eta'\colon \zeta \to \xi$ induce the same map, we must show that $(M, f, \eta \circ v) \sim (M, f, \eta' \circ v)$. So let $H\colon \nu(f) \times I \to \zeta$ be a homotopy from $\eta \circ v$ to $\eta' \circ v$. Then the triple $(M \times I, f \times I, H)$ gives the desired bordism.

Now we fix ζ and show that $\mathcal{I}(-;\zeta)$ is a contravariant homotopy functor. Let $g\colon Q^{n-l} \looparrowright N^n$ be a proper immersion, and let $[M, f, v]$ be in $\mathcal{I}(N;\zeta)$; we must describe $g^*[M, f, v] \in \mathcal{I}(Q;\zeta)$. By Proposition A.5 in the Appendix and Proposition 2.5, we may find a representative f' of $[f]$ which is regularly homotopic to f and transverse to g as a map to N. We then form the pullback square.

$$\begin{tikzcd}
Q \times_N M \arrow[r, "\rho"] \arrow[d, "\delta"'] & M \arrow[d, "f'"] \\
Q \arrow[r, "g"] & N
\end{tikzcd}$$

The manifold

$$Q \times_N M = \{(q, m) \in Q \times M \mid g(q) = f'(m)\}$$

is compact since M is compact and g is proper. The map δ is an immersion with normal bundle isomorphic to $\rho^*\nu(f') \cong \rho^*\nu(f)$, hence admits a bundle map $\overline{\rho}\colon \nu(\delta) \to \nu(f)$. Then we may set

$$g^*[f] = [Q \times_N M, \delta, v \circ \overline{\rho}] \in \mathcal{I}(Q;\zeta).$$

Note that this construction is well-defined and functorial by Proposition A.6**(a)** in the Appendix, and that regularly homotopic immersions of Q in N give the same map $\mathcal{I}(N;\zeta) \to \mathcal{I}(Q;\zeta)$. □

2.3 Products

Given pairs of objects $N, N' \in Ob(\mathcal{D}_0)$ and $\zeta, \zeta' \in Ob(\mathsf{Vect})$ we may define the Cartesian product objects $N \times N' \in Ob(\mathcal{D}_0)$ and $\zeta \times \zeta' \in Ob(\mathsf{Vect})$. This allows the following definitions of products in the bordism of immersions.

Definition 2.7. *The* external *or* Cartesian product *is the binary operation*

$$\mathcal{I}(N;\zeta) \times \mathcal{I}(N';\zeta') \xrightarrow{\times} \mathcal{I}(N \times N';\zeta \times \zeta')$$

given by Cartesian product of representatives, so

$$[M, f, v] \times [M', f', v'] = [M \times M', f \times f', v \times v'].$$

Definition 2.8. *The* internal *or* cup product *is the binary operation*

$$\mathcal{I}(N; \zeta) \times \mathcal{I}(N; \zeta') \xrightarrow{\cup} \mathcal{I}(N; \zeta \times \zeta')$$

given by

$$[f_0] \cup [f_1] = \triangle^*[f_0] \times [f_1],$$

where $\triangle^*$ *denotes pullback by the diagonal embedding* $\triangle \colon N \hookrightarrow N \times N$ *(given by* $n \mapsto (n, n)$*) as in Proposition 2.6.*

That these operations are well-defined is immediate. Note that both are distributive over the addition. The diffeomorphism $(M \sqcup M') \times M'' \approx (M \times M'') \sqcup (M' \times M'')$ along with Lemma 2.3 yields this fact in the case of the $\times$-product, and the $\cup$-product case follows since $\triangle^*$ is a monoid homomorphism.

2.4 The Self-intersection Operations ψ_r

Thus the theory of bordism of immersions has a rich structure. It is enriched further by certain natural operations between bordism functors, constructed from the self-intersections of immersions. The idea is that if an immersion $f \colon M^{n-k} \looparrowright N^n$ satisfies a certain generic property, then the set of points in N whose pre-image under f consists of at least r distinct points of M is itself the image of an immersion $\psi_r(f) \colon \Delta_r(f) \looparrowright N$ of codimension rk. This *r-fold self-intersection immersion* has certain extra structure on its normal bundle.

The study of self-intersections of immersions began long ago with such authors as H. Whitney [41] and R.K. Lashof and S. Smale [19]. However, it was U. Koschorke and B. Sanderson, in their pioneering work [18], who first applied the theory to define operations between bordism monoids. These operations have since been studied extensively by P. J. Eccles [12], M. A. Asadi-Golmankhaneh and Eccles [1] and others, and will be the subject of much of this thesis.

We now describe the 'generic property' of immersions mentioned above, and in order to do so must introduce the following notations. Superscripts in parentheses will denote iterated products, so that if $r \geq 1$ is an integer, $N^{(r)}$ is the r-fold Cartesian product of N with itself. Let $\triangle\colon N \hookrightarrow N^{(r)}$ be the diagonal embedding $n \mapsto (n, \ldots, n)$. The *r-th configuration space of M* is the open submanifold

$$\mathcal{F}(M;r) = \{(m_1, \ldots, m_r) \in M^{(r)} \mid i \neq j \Rightarrow m_i \neq m_j\} \subseteq M^{(r)},$$

consisting of pairwise distinct r-tuples of points in M. Given $f\colon M^{n-k} \looparrowright N^n$, the restriction of the product immersion $f^{(r)}\colon M^{(r)} \looparrowright N^{(r)}$ to $\mathcal{F}(M;r)$ shall also be denoted $f^{(r)}$.

Definition 2.9. *An immersion $f\colon M^{n-k} \looparrowright N^n$ is* self-transverse *if for every $r \geq 1$ the immersion $f^{(r)}\colon \mathcal{F}(M;r) \looparrowright N^{(r)}$ is transverse to $\triangle\colon N \hookrightarrow N^{(r)}$.*

Given an integer $r \geq 1$ and a self-transverse immersion $f\colon M^{n-k} \looparrowright N^n$ of a closed manifold to a manifold without boundary, we now construct the r-fold self-intersection immersion $\psi_r(f)$ of f. By Definition 2.9, on forming the pull-back square,

$$\begin{tikzcd}
\overline{\Delta}_r(f) \arrow[r] \arrow[d, "\overline{\psi}_r(f)"'] & \mathcal{F}(M;r) \arrow[d, "f^{(r)}"] \\
N \arrow[r, "\triangle"] & N^{(r)}
\end{tikzcd}$$

the subspace

$$\overline{\Delta}_r(f) = \{(m_1, \ldots, m_r) \in \mathcal{F}(M;r) \mid f(m_1) = \ldots = f(m_r)\} \subseteq \mathcal{F}(M;r)$$

is a submanifold, whose dimension is computed as follows:

$$\begin{aligned}
\dim\overline{\Delta}_r(f) &= \dim\mathcal{F}(M;r) - \mathrm{codim}(\overline{\Delta}_r(f) \hookrightarrow \mathcal{F}(M;r)) \\
&= r(n-k) - \mathrm{codim}\triangle \\
&= r(n-k) - (r-1)n = n - rk.
\end{aligned}$$

Thus the map $\overline{\psi}_r(f)\colon \overline{\Delta}_r(f) \to N$ has codimension rk, and is an immersion since $\triangle \circ \overline{\psi}_r(f) = f^{(r)}|_{\overline{\Delta}_r(f)}$.

In fact, $\overline{\Delta}_r(f)$ is a closed manifold. It clearly has empty boundary; we must show why it is compact. Certainly, since $\overline{\Delta}_r(f) = (f^{(r)})^{-1}\triangle(N)$ it is a closed subset of $\mathcal{F}(M;r)$. Define a subspace $\underline{\triangle}M^{(r)} \subseteq M^{(r)}$, the *fat diagonal*, by

$$\underline{\triangle}M^{(r)} = \{(m_1, \ldots, m_r) \in M^{(r)} \mid \exists i \neq j \text{ with } m_i = m_j\}.$$

Note that $\mathcal{F}(M;r) = M^{(r)} - \underline{\triangle}M^{(r)}$. Using the fact that f is an immersion and therefore locally injective, one can show that

$$\overline{\Delta}_r(f) \subseteq M^{(r)} - U \subseteq \mathcal{F}(M;r),$$

where U is an open neighbourhood of $\underline{\triangle}M^{(r)}$ in $M^{(r)}$. The space $M^{(r)} - U$ is compact since it is a closed subspace of $M^{(r)}$. Hence $\overline{\Delta}_r(f)$ is compact.

Now let S_r denote the symmetric group on r elements. Note that S_r acts freely on $\overline{\Delta}_r(f)$ by permutation of factors, and acts trivially on N. The immersion $\overline{\psi}_r(f)$ is equivariant with respect to these actions. Hence, on factoring out by S_r we obtain an immersion of a closed $(n-rk)$-manifold,

$$\psi_r(f)\colon\ \Delta_r(f) \looparrowright N,$$

$$[m_1, \ldots, m_r] \mapsto f(m_1) = \ldots = f(m_r),$$

the so-called *r-fold self-intersection immersion* of f.

In the next chapter we shall reprove a result of R.J. Herbert [15], concerning homology classes in the source manifold M represented by the multiple points of a self-transverse immersion $f\colon\ M^{n-k} \looparrowright N^n$. By an *$r$-fold multiple point* of f we mean a point $m \in M$ such that $f^{-1}(f(m)) \subseteq M$ consists of at least r distinct points. Using the constructions of this section, it is easy to describe an immersion $\mu_r(f)\colon\ \widetilde{\Delta}_r(f)^{n-rk} \looparrowright M^{n-k}$ whose image is exactly the r-fold multiple points of f.

Note that the symmetric group S_{r-1} also acts freely on $\overline{\Delta}_r(f)$ by permuting the last $r-1$ factors, keeping the first fixed. The resulting quotient manifold $\widetilde{\Delta}_r(f) = \overline{\Delta}_r(f)/S_{r-1}$ is again a closed $(n-rk)$-manifold. The map

$$\mu_r(f)\colon\ \widetilde{\Delta}_r(f)^{n-rk} \to M^{n-k},$$

$$(m_1, [m_2, \ldots, m_r]) \mapsto m_1$$

given by projection to the first factor is an immersion, since all other maps in the equality $f \circ \mu_r(f) = \psi_r(f) \circ \rho$ are immersions, where $\rho\colon \widetilde{\Delta}_r(f) \to \Delta_r(f)$ is the obvious r-fold covering. This codimension $(r-1)k$ immersion $\mu_r(f)$ is the *r-fold multiple point immersion* of f.

We are almost ready to define some operations. First we need a lemma concerning the structure of the normal bundles of the immersions $\psi_r(f)$ and $\mu_r(f)$, which requires some preliminary bundle constructions.

Let G be a group. Given a right G-space X and a left G-space Y, the *diagonal action* of G on $X \times Y$ is the action given by $g(x, y) = (xg^{-1}, gy)$ for $g \in G$, $x \in X$ and $y \in Y$. Define $X \times_G Y$ to be the orbit space of $X \times Y$ under this action.

Now let G be a subgroup of the symmetric group S_r. If EG is a contractible space with a free right action of G, then it is the total space of a universal principal G-bundle. Let ζ be a k-dimensional bundle with projection $p\colon E(\zeta) \to X$. The product map $p^{(r)}\colon E(\zeta)^{(r)} \to X^{(r)}$ is an equivariant map of left G-spaces, whose G-actions are given by permutations.

Definition 2.10. *The rk-dimensional vector bundle*

$$1 \times_G p^{(r)}\colon EG \times_G E(\zeta)^{(r)} \to EG \times_G X^{(r)}$$

*will be denoted by $\mathcal{S}_G\zeta$, and the bundle $\mathcal{S}_{S_r}\zeta = \mathcal{S}_r\zeta$ will be called the r-*th extended power of *ζ.*

Note that $\mathcal{S}_r\zeta$ is only defined up to homotopy equivalence, and as such $\mathcal{S}_1\zeta = \zeta$. We define $\mathcal{S}_0\zeta$ to be the point bundle over a point, denoted by $\star$.

Lemma 2.11. *Suppose the self-transverse immersion $f\colon M^{n-k} \looparrowright N^n$ has a ζ-structure. Then the immersions $\psi_r(f)$ and $\mu_{r+1}(f)$ have a $\mathcal{S}_r\zeta$-structure, for $r \geq 1$.*

Proof. With a little thought, we can identify the relevant normal bundles as quotient bundles,

$$\nu(\psi_r(f)) \cong \nu(f)^{(r)}|_{\overline{\Delta}_r(f)}/S_r,$$

$$\nu(\mu_{r+1}(f)) \cong \varepsilon^0 \times \nu(f)^{(r)}|_{\overline{\Delta}_{r+1}(f)}/S_r,$$

where ε^0 is the 0-dimensional bundle over M. Let $v\colon \nu(f) \to \zeta$ be a ζ-structure for f.

The standard model for ES_r is the configuration space $\mathcal{F}(\mathbb{R}^\infty; r)$, where $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$ is given the direct limit topology and S_r acts by permutations.

By Whitney's Theorem A.1 in the Appendix, we can find an embedding $\lambda\colon \nu(f) \hookrightarrow \mathbb{R}^\infty$. An $\mathcal{S}_r\zeta$-structure for $\psi_r(f)$ is given by the bundle map

$$\mathcal{S}_r(v)\colon \nu(\psi_r(f)) \to \mathcal{S}_r\zeta$$

$$[x_1, \dots, x_r] \mapsto [(\lambda(x_1), \dots, \lambda(x_r)), v(x_1), \dots, v(x_r)],$$

where $x_i \in \nu(f)_{m_i}$ and $(\lambda(x_1), \dots, \lambda(x_r)) \in \mathcal{F}(\mathbb{R}^\infty; r)$. An $\mathcal{S}_r\zeta$-structure for $\mu_{r+1}(f)$ is given by

$$\widetilde{\mathcal{S}_{r+1}}(v)\colon \nu(\mu_{r+1}(f)) \to \mathcal{S}_r\zeta$$

$$(*, [x_1, \dots, x_r]) \mapsto [(\lambda(x_1), \dots, \lambda(x_r)), v(x_1), \dots, v(x_r)].$$

These structures are uniquely defined up to bundle homotopy, since λ is unique up to isotopy by Theorem A.1. □

Proposition (Koschorke and Sanderson) 2.12. *There exist operations*

$$\psi_r\colon \mathcal{I}(-;\zeta) \to \mathcal{I}(-;\mathcal{S}_r\zeta),$$

defined for $r \geq 0$, which satisfy the following properties. Let $[f], [g] \in \mathcal{I}(N;\zeta)$.

(i) $\psi_0[f] = [\mathbf{1}_N\colon N \looparrowright N] \in \mathcal{I}(N;\star)$.

(ii) $\psi_1[f] = [f]$, *and* $\psi_r[f] = 0$ *for* $r > 1$ *if* $[f]$ *can be represented by an embedding.*

(iii) *(Naturality) If $h\colon Q \looparrowright N$ is a proper immersion, then*

$$h^*\psi_r[f] = \psi_r h^*[f] \in \mathcal{I}(Q;\mathcal{S}_r\zeta).$$

(iv) *(Cartan formula)*

$$\psi_r([f] + [g]) = \sum_{i=0}^{r} \psi_{r-i}[f] \cup \psi_i[g] \in \mathcal{I}(N;\mathcal{S}_r\zeta).$$

(v) *If f is self-transverse,*

$$\psi_r[\mu_2(f)] = [\mu_{r+1}(f)] \in \mathcal{I}(M;\mathcal{S}_r\zeta).$$

Proof. We may take property **(i)** as the definition of ψ_0. To define $\psi_r[f]$ for $r \geq 1$, note that by Proposition A.5 in the Appendix, we may choose a self-transverse representative f' of $[f]$. Then

$$\psi_r[M, f, v] := [\Delta_r(f'), \psi_r(f'), \mathcal{S}_r(v')].$$

This is well defined by Proposition A.6**(b)**. Properties **(ii)**, **(iii)** and **(v)** follow directly from the definitions (for **(iii)** use the fact that $\triangle_N \circ h = h^{(r)} \circ \triangle_Q \colon Q \to N^{(r)}$). Property **(iv)** is slightly less obvious, but the moral is that "If $f \sqcup g$ is self-transverse then an r-fold self-intersection of $f \sqcup g$ is the intersection of an $(r-i)$-fold self-intersection of f with an i-fold self-intersection of g". Notice we have applied homomorphisms induced by bundle maps $\mathcal{S}_{r-i}\zeta \times \mathcal{S}_i\zeta \to \mathcal{S}_r\zeta$, so that the formula ends up in $\mathcal{I}(N; \mathcal{S}_r\zeta)$. These exist thanks to the product $S_{r-i} \times S_i \to S_r$ coming from concatenation of permutations, which induces a map $ES_{r-i} \times ES_i \to ES_r$. □

2.5 The Relative Monoid $\mathcal{I}(N, \partial N; \zeta)$

In this section we will extend the definition of the contravariant functor $\mathcal{I}(-;\zeta)$ to manifolds with boundary.

Let N^n be a connected manifold with boundary ∂N, and let ζ be a k-dimensional real Riemannian bundle over a space X having the homotopy type of a connected manifold. We wish to classify all ζ-structured immersions of manifolds with boundary into N which preserve boundaries. So the data set $\mathrm{Imm}(N, \partial N; \zeta)$ consists of all triples $((M, \partial M), f, v)$, where M^{n-k} is a compact manifold with ∂M its boundary, $f\colon M^{n-k} \looparrowright N^n$ is an immersion transverse to ∂N such that $f^{-1}(\partial N) = \partial M$, and $v\colon \nu(f) \to \zeta$ is a ζ-structure for f. We remark that Definition 2.1 extends in an obvious way to a definition of diffeomorphism of such triples.

Just as a bordism between closed manifolds is given by a manifold with boundary, defining a bordism between compact manifolds with boundary requires an object with one added level of complexity to its set of non-interior points. This is provided by the concept of a manifold with corners, or more specifically a $\langle 2 \rangle$-manifold, as introduced by K. Jänich in the paper [17].

Definition 2.13. *A topological manifold with boundary, W^n, is a* manifold with corners *if it has a C^∞-structure with corners. That is, W has a maximal atlas of smoothly compatible charts*

$$\phi_i \colon U_i \to \mathbb{R}^n_+ = [0, \infty)^n$$

which are homeomorphisms onto open subsets of $\mathbb{R}^n_+$.

For any point $x \in W$, let $c(x)$ be the number of zeroes in $\phi(x)$, where (U, ϕ) is a chart around x. This number does not depend on the choice of chart. Clearly a smooth manifold without boundary is a manifold with corners W having $c(x) = 0$ for all $x \in W$, and a manifold with boundary has $c(x) = 0$ or 1 according to whether x is an interior or boundary point.

A *connected face* of a manifold with corners W is the closure of a component of $\{x \in W \mid c(x) = 1\}$; a *face* is a disjoint union of connected faces. A manifold with corners W is a *manifold with faces* if each $x \in W$ belongs to exactly $c(x)$ different connected faces. Note that a face of a manifold with faces is again a manifold with faces.

Definition 2.14. *A* $\langle 2 \rangle$-manifold *is a manifold with faces W^n together with an ordered pair of faces $(\partial_0 W, \partial_1 W)$ of W, satisfying:*

(i) *$\partial_0 W \cup \partial_1 W = \partial W$,*

(ii) *$\partial_0 W \cap \partial_1 W$ is a face of both $\partial_0 W$ and $\partial_1 W$.*

Example 2.15. *Let M be a manifold with boundary ∂M. Then $M \times I$ is a $\langle 2 \rangle$-manifold with faces $(M \times \partial I, \partial M \times I)$.*

We are now ready to define a bordism relation on triples in $\mathrm{Imm}(N, \partial N; \zeta)$.

Definition 2.16. *Two triples $((M_i, \partial M_i), f_i, v_i)$, $i = 0, 1$, are* bordant *if there is a triple (W^{n-k+1}, F, V) satisfying the following conditions:*

(1) *W^{n-k+1} is a $\langle 2 \rangle$-manifold with faces $(\partial_0 W, \partial_1 W)$;*

(2) *$F \colon W \looparrowright N \times I$ is an immersion transverse to $\partial(N \times I)$ such that $F^{-1}(N \times \partial I) =$*

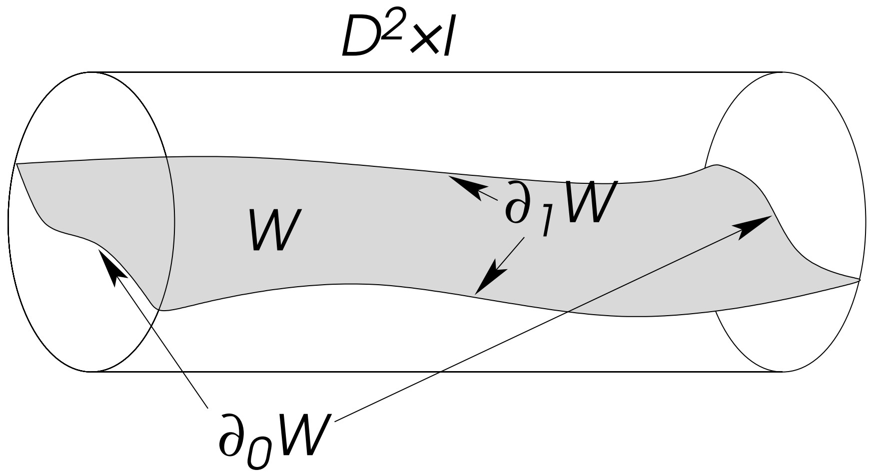

Figure 2.1: A typical bordism between two immersions $I \looparrowright D^2$.

$\partial_0 W$ and $F^{-1}(\partial N \times I) = \partial_1 W$;

(3) *$V\colon \nu(F) \to \zeta$ is a ζ-structure for F;*

(4) *The following triples are diffeomorphic: $((\partial_0 W, \partial_0 W \cap \partial_1 W), F|_{\partial_0 W}, V|_{\partial_0 W})$ and $((M_0 \sqcup M_1, \partial M_0 \sqcup \partial M_1), (f_0, 0) \sqcup (f_1, 1), v_0 \sqcup v_1)$;*

(5) *The triple $(\partial_1 W, F|_{\partial_1 W}, V|_{\partial_1 W})$ gives a bordism between $(\partial M_0, f_0|_{\partial M_0}, v_0|_{\partial M_0})$ and $(\partial M_1, f_1|_{\partial M_1}, v_1|_{\partial M_1})$ in* $\mathrm{Imm}(\partial N; \zeta)$.

This definition is illustrated in Figure 2.1, which shows the image of a bordism between two immersions (in fact embeddings) of the interval into a disc. As before bordism is an equivalence relation, and we denote the resulting set of equivalence classes by $\mathcal{I}(N, \partial N; \zeta)$. This is in fact a commutative monoid (compare Proposition 2.4).

To describe the functoriality, we introduce a new category of manifolds. The category $\mathcal{D}$ has as objects the finite dimensional manifolds with boundary, and as morphisms those proper immersions which do not take interior points to boundary points. Thus a proper immersion $g\colon Q \looparrowright N$ is in $\mathcal{D}$ if and only if $g^{-1}(\partial N) \subseteq \partial Q$. *Note that we do not require $g(\partial Q) \subseteq \partial N$,* since we wish an extension of a given immersion to an immersion of its unit disc normal bundle, for instance, to be a morphism in $\mathcal{D}$. Note also that $\mathcal{D}_0$ is a full subcategory of $\mathcal{D}$, and that for a manifold without boundary N we have $\mathcal{I}(N; \zeta) = \mathcal{I}(N, \emptyset; \zeta)$. The following Proposition generalises

Proposition 2.6, and is proved similarly.

Proposition 2.17. *Bordism of immersions with boundary is a homotopy bifunctor*

$$\mathcal{I}(-;-)\colon\ \mathcal{D}^{\mathsf{op}} \times \mathsf{Vect} \to \mathsf{CMon}.$$

Chapter 3

Herbert's Theorem

Let $f\colon M^{n-k} \looparrowright N^n$ be an immersion of closed manifolds. The manifold M has a unique *fundamental* $\mathbb{Z}_2$*-homology class* $[M]_2 \in H_{n-k}(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$, given by the non-zero element. If M is orientable, an *orientation* or *fundamental homology class* of M is a choice of generator $[M] \in H_{n-k}(M;\mathbb{Z}) \cong \mathbb{Z}$.

Definition 3.1. *We say that $f\colon M^{n-k} \looparrowright N^n$* represents *both the homology class $f_*[M]_2 \in H_{n-k}(N;\mathbb{Z}_2)$ and the homology class $f_*[M] \in H_{n-k}(N;\mathbb{Z})$ (when M is oriented).*

Now suppose that $f\colon M^{n-k} \looparrowright N^n$ is self-transverse (see Definition 2.9). Then the r-fold multiple point and self-intersection immersions of f,

$$\mu_r(f)\colon \widetilde{\Delta}_r(f)^{n-rk} \looparrowright M^{n-k}$$

$$\psi_r(f)\colon \Delta_r(f)^{n-rk} \looparrowright N^n$$

defined in Section 2.4, represent homology classes $m_r := \mu_r(f)_*[\widetilde{\Delta}_r(f)] \in H_{n-rk}(M;\mathbb{Z}_2)$ and $n_r := \psi_r(f)_*[\Delta_r(f)] \in H_{n-rk}(N;\mathbb{Z}_2)$, respectively. If the manifolds $\widetilde{\Delta}_r(f)$, $\Delta_r(f)$ are orientable, these also represent classes in $\mathbb{Z}$-homology. Note that $m_1 = [M]_2$ or $[M]$, and $n_1 = f_*[M]_2$ or $f_*[M]$. In this chapter we address the following problem.

Problem 3.2. *How are the homology classes m_r, n_r related, for different values of $r \geq 1$?*

For instance, trivially we have $f_*m_1 = n_1$. The first non-trivial (partial) answer to this question was proposed by Whitney in his 1941 paper [41].

3.1 Whitney's Formula

In this Section, homology and cohomology will be taken with coefficients in $\mathbb{Z}$ if M and N are oriented and in $\mathbb{Z}_2$ otherwise. Since both M and N are closed manifolds, there are Poincaré Duality isomorphisms

$$\mathcal{D}_M\colon\ H^l(M) \xrightarrow{\cong} H_{n-k-l}(M),$$

$$\mathcal{D}_N\colon\ H^l(N) \xrightarrow{\cong} H_{n-l}(N),$$

for all l. In each case the isomorphism is given by cap product with the fundamental homology class.

Denote by $\lambda(f)$ the image of the fundamental class of M under the composition

$$H_{n-k}(M) \xrightarrow{f_*} H_{n-k}(N) \xrightarrow{\mathcal{D}_N^{-1}} H^k(N) \xrightarrow{f^*} H^k(M) \xrightarrow{\mathcal{D}_M} H_{n-2k}(M).$$

Let $e \in H^k(M)$ denote the Euler class of the normal bundle $\nu(f)$ of f.

Proposition (Whitney) 3.3.

$$m_2 = \lambda(f) - \mathcal{D}_M(e) \in H_{n-2k}(M).$$

As noted by R. J. Herbert in the introduction to [15], the content of this formula may be interpreted as "the homology class Poincaré dual to the Euler class of the normal bundle for an immersion $f\colon M \looparrowright N$ [is] represented by the intersection of M with a 'deformed position' of M minus the 'distant intersections' of f (Whitney's terminology for the double points of f)".

In 1959, Lashof and Smale [19] attempted to generalise Whitney's formula to the r-fold multiple points, where $r \geq 2$. They asserted that

$$m_r = \pm(\lambda(f) - \mathcal{D}_M(e))^{r-1} \in H_{n-rk}(M),$$

where the sign depends on n, k, and r, and the product in homology is Poincaré dual to the cup product in cohomology. For coefficients to be taken in $\mathbb{Z}$, we should additionally assume the codimension k to be even. However, Lashof and Smale's formula turned out to be false.

3.2 Herbert's Formula

A complete answer to Problem 3.2 did not arrive until 1981, with the thesis of Herbert [15]. Although Herbert's formula lives in homology, we give the (seemingly more natural) statement in cohomology. Let $\widetilde{m}_r = \mathcal{D}_M^{-1}(m_r)$ be the cohomology class dual to the homology class represented by $\mu_r(f)$, and let $\widetilde{n}_r = \mathcal{D}_N^{-1}(n_r)$ be dual to the homology class represented by $\psi_r(f)$. Again, let $e \in H^k(M)$ be the Euler class of $\nu(f)$.

Theorem (Herbert) 3.4.

$$\widetilde{m}_{r+1} = f^*\widetilde{n}_r - \widetilde{m}_r \cup e \in H^{rk}(M)$$

for $r \geq 1$, where coefficients are taken in $\mathbb{Z}$ if M and N are oriented and k is even, and in $\mathbb{Z}_2$ otherwise.

Note that Whitney's formula is indeed the special case $r = 1$ of Herbert's Theorem. Unfortunately, the proof of this result given in [15] is long and complicated. An alternative, shorter proof was given by F. Ronga in [27] using the idea of 'clean intersections' as introduced by D. Quillen in [25]. However, one might hope for a cleaner, more conceptual proof of this geometric result. We will give such a proof of the analogue of Herbert's Theorem in the setting of the bordism of immersions. In Chapter 4 we show how to derive Theorem 3.4, and its generalisations to other cohomology theories, from our Theorem 3.7.

3.3 Herbert's Formula in the Bordism of Immersions

Let $f\colon M^{n-k} \looparrowright N^n$ be a self-transverse immersion of closed manifolds, and suppose f has a ζ-structure given by $v\colon \nu(f) \to \zeta$, where ζ is a k-dimensional real Riemannian bundle. Such an f represents a class $[f] = [M, f, v] \in \mathcal{I}(N; \zeta)$. Since M is compact, f is proper, and so may also be regarded as a morphism in the categories $\mathcal{D}_0$ and $\mathcal{D}$. So f induces a morphism $f^*\colon \mathcal{I}(N;\zeta) \to \mathcal{I}(M;\zeta)$ in the category CMon.

The unit disc bundle of $\nu(f)$, denoted $D\nu(f)$, is a compact n-manifold with boundary the sphere bundle $\partial D\nu(f) = S\nu(f)$. The zero section $i\colon M \hookrightarrow D\nu(f)$ is an embedding with normal bundle $\nu(f)$. Hence i represents an element $[M, i, v] \in \mathcal{I}(D\nu(f), S\nu(f); \zeta)$. Since $i^{-1}(S\nu(f)) = \emptyset$, the zero section also represents a morphism in $\mathcal{D}$. Hence we may define an element

$$e_{\mathcal{I}} := i^*[i] \in \mathcal{I}(M; \zeta),$$

which we shall call the *Euler class* of $f\colon M^{n-k} \looparrowright N^n$.

The analogue of Whitney's formula may now be stated as follows.

Theorem 3.5.

$$f^*[f] = [\mu_2(f)] + e_{\mathcal{I}} \in \mathcal{I}(M; \zeta).$$

Proof. By Theorem A.7 in the Appendix, we may find an immersion $F\colon D\nu(f)^n \looparrowright N^n$ of the normal disc bundle which extends f and is injective on fibres. We may also choose F such that $F|_{S\nu(f)}\colon S\nu(f) \looparrowright N$ is transverse to f. This F is a morphism in the category $\mathcal{D}$, since $\partial N = \emptyset$. Hence we have a factorisation of f in the category $\mathcal{D}$ as $f = F \circ i$. Therefore, a good first step towards analysing $f^*[f] = i^*F^*[f]$ would be an analysis of the class $F^*[f] \in \mathcal{I}(D\nu(f), S\nu(f); \zeta)$.

Note that the immersion F is automatically transverse to f, by virtue of having zero codimension, and $F|_{S\nu(f)}$ is transverse to f by assumption. When we form the pullback square,

$$\begin{tikzcd} \Theta \arrow[r, "\rho"] \arrow[d, "\delta"'] & M^{n-k} \arrow[d, "f"] \\ D\nu(f)^n \arrow[r, "F"] & N^n \end{tikzcd}$$

the space

$$\Theta = \{(v, m) \in D\nu(f) \times M \mid F(v) = f(m)\}$$

is a compact $(n-k)$-manifold with boundary, and $\delta\colon \Theta \looparrowright D\nu(f)$ is an immersion. The triple $(\Theta, \delta, v \circ \overline{\rho})$ represents the bordism class $F^*[f]$. Now, using the fact that F

is injective on fibres, we can see that Θ splits as a disjoint union $\Theta = \Theta_0 \sqcup \Theta_1$, where

$$\Theta_0 = \{(0_m, m) \in \Theta\},$$

$$\Theta_1 = \{(v, m) \in \Theta \mid v \in \nu(f)_{m'},\, m \neq m'\}.$$

Hence in $\mathcal{I}(D\nu(f), S\nu(f); \zeta)$ we may write

$$F^*[f] = [\Theta_0, \delta|_{\Theta_0}, v \circ \overline{\rho}|_{\Theta_0}] + [\Theta_1, \delta|_{\Theta_1}, v \circ \overline{\rho}|_{\Theta_1}].$$

The restriction $\rho|_{\Theta_0}\colon\ \Theta_0 \to M$ given by $(0_m, m) \mapsto m$ is clearly a diffeomorphism satisfying $i \circ \rho|_{\Theta_0} = \delta|_{\Theta_0}$, and in fact the triples $(\Theta_0, \delta|_{\Theta_0}, v \circ \overline{\rho}|_{\Theta_0})$ and (M, i, v) are diffeomorphic. Thus

$$F^*[f] = [i] + [\Theta_1, \delta|_{\Theta_1}, v \circ \overline{\rho}|_{\Theta_1}],$$

and on applying the monoid homomorphism $i^*\colon\ \mathcal{I}(D\nu(f), S\nu(f); \zeta) \to \mathcal{I}(M; \zeta)$ we obtain

$$f^*[f] = i^* F^*[f] = e_{\mathcal{I}} + i^*[\Theta_1, \delta|_{\Theta_1}, v \circ \overline{\rho}|_{\Theta_1}].$$

All that remains is to identify the bordism classes $[\mu_2(f)]$ and $i^*[\Theta_1, \delta|_{\Theta_1}, v \circ \overline{\rho}|_{\Theta_1}]$ in $\mathcal{I}(M; \zeta)$.

We claim that self-transversality of f implies that i and $\delta|_{\Theta_1}$ are already transverse as maps to $D\nu(f)$. In fact, suppose that $i(m_1) = \delta(v, m)$, for some $m_1 \in M$ and $(v, m) \in \Theta_1$. This means that $v = 0_{m_1} \in \nu(f)_{m_1}$, which implies that $m_1 \neq m$ and $f(m_1) = f(m)$. Consider the following commutative diagram.

$$\begin{tikzcd}
 & \Theta_1 \arrow[r, "\rho|_{\Theta_1}"] \arrow[d, "\delta|_{\Theta_1}"'] & M \arrow[d, "f"] \\
M \arrow[r, "i"] & D\nu(f) \arrow[r, "F"] & N
\end{tikzcd}$$

Since both $\rho|_{\Theta_1}$ and F are local diffeomorphisms, we have

$$\begin{aligned}
& di(TM_{m_1}) + d\delta(T\Theta_{(v,m)}) = TD\nu(f)_{0_{m_1}} \\
\Leftrightarrow \quad & df(TM_{m_1}) + df(TM_m) = TN_{f(m)},
\end{aligned}$$

and the latter is easily seen to be a consequence of self-transversality of f. Hence i and $\delta|_{\Theta_1}$ are transverse.

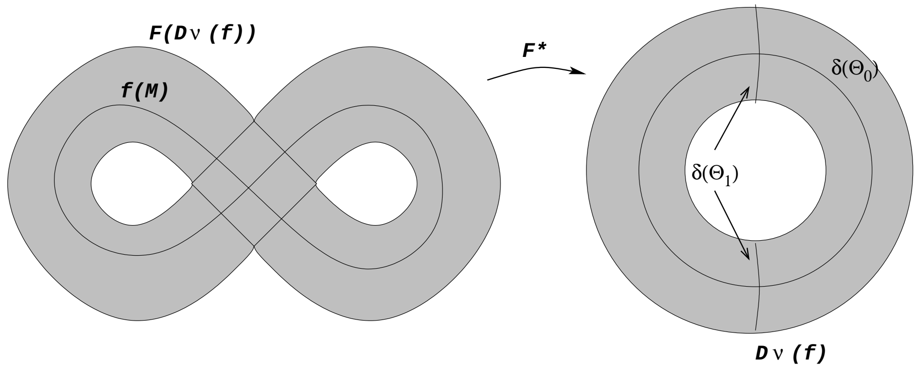

Figure 3.1: Illustrating Theorem 3.5 for the figure eight immersion.

Recall from section 2.4 that $[\mu_2(f)]$ is represented by the triple $(\widetilde{\Delta}_2(f), \mu_2(f), \widetilde{\mathcal{S}_2}(v))$, where

$$\widetilde{\Delta}_2(f) = \overline{\Delta_2}(f) = \{(m_1, m_2) \in \mathcal{F}(M; 2) \mid f(m_1) = f(m_2)\},$$

$$\mu_2(f)(m_1, m_2) = m_1,$$

and $\widetilde{\mathcal{S}_2}(v)\colon\ \nu(\mu_2(f)) \to \mathcal{S}_1\zeta = \zeta$ is the bundle map defined in Lemma 2.11. When we complete the above diagram by pulling back along i, it is easy to see that the resulting triple is diffeomorphic with this one. $\square$

We will illustrate the preceding proof with an example.

Example 3.6. Consider S^1 to be the unit circle in $\mathbb{C}$, and S^2 to be the one-point compactification of $\mathbb{R}^2$. Let $f\colon\ S^1 \looparrowright S^2$ be the self-transverse immersion given by $f(e^{i\theta}) = (\cos\theta, \sin 2\theta)$, whose image is a figure eight. This is shown on the left of Figure 3.1, together with the image of an immersion $F\colon\ D\nu(f) \looparrowright S^2$ of the (trivial) normal bundle. When we pull back f by F we obtain an immersion $\delta\colon\ \Theta \looparrowright D\nu(f)$ whose image is shown on the right. This example clearly shows $\delta(\Theta_0)$ as the image of the zero section $i\colon\ M \hookrightarrow D\nu(f)$, and the intersection of i with $\delta(\Theta_1)$ consists of the double points of f.

As an immediate corollary of Theorem 3.5 we obtain the following analogue of Herbert's Theorem in $\mathcal{I}(M; \mathcal{S}_r\zeta)$.

Theorem 3.7. *For $r \geq 1$,*

$$f^*[\psi_r(f)] = [\mu_{r+1}(f)] + [\mu_r(f)] \cup e_{\mathcal{I}} \in \mathcal{I}(M; \mathcal{S}_r\zeta).$$

Proof. This follows from the double point case above using Proposition 2.12, and the fact that $e_{\mathcal{I}} \in \mathcal{I}(M;\zeta)$ is represented by an embedding. The argument is as follows:

$$\begin{aligned}
f^*[\psi_r(f)] &= \psi_r f^*[f] \quad \text{by } \textbf{(iii)} \text{ of } 2.12 \\
&= \psi_r([\mu_2(f)] + e_{\mathcal{I}}) \quad \text{by Theorem } 3.5 \\
&= \sum_{i=0}^{r} \psi_{r-i}[\mu_2(f)] \cup \psi_i(e_{\mathcal{I}}) \quad \text{by } \textbf{(iv)} \\
&= \psi_r[\mu_2(f)] + \psi_{r-1}[\mu_2(f)] \cup e_{\mathcal{I}} \quad \text{by } \textbf{(ii)} \\
&= [\mu_{r+1}(f)] + [\mu_r(f)] \cup e_{\mathcal{I}} \quad \text{by } \textbf{(v)}.
\end{aligned}$$

□

Chapter 4

Geometric Cobordism and Steenrod Operations

In this chapter we give a brief survey of the various *cobordism theories*. These are generalised cohomology theories represented by Thom spectra, whose definitions and basic properties are outlined in Section 4.1. In Section 4.2 we give D. Quillen's geometric interpretation of cobordism as equivalence classes of suitably oriented proper maps of manifolds [25]. This will enable us to describe the relationship between such cobordism theories and the functor $\mathcal{I}(-;-)$ of Chapter 2. This is done in Section 4.3, where the analogues of Herbert's Theorem 3.4 in cobordism and other cohomology theories are deduced. In the final section we will define and discuss Steenrod operations in the setting of generalised cohomology theories, and give their construction (due to T. tom Dieck [37]) in the case of cobordism from both the homotopical and geometrical viewpoint.

4.1 Thom Spectra

Definition 4.1. *Suppose we are given a family $\boldsymbol{X}_\Gamma = \{X_k, f_k, g_k\}_{k\geq 0}$, where for each non-negative integer k we have a space X_k, a fibration $f_k\colon X_k \to BO(k)$ and a map*

$g_k\colon X_k \to X_{k+1}$ *making the following diagram commute.*

$$\begin{tikzcd} X_k \arrow[r, "f_k"] \arrow[d, "g_k"'] & BO(k) \arrow[d, "Bi_k"] \\ X_{k+1} \arrow[r, "f_{k+1}"] & BO(k+1) \end{tikzcd}$$

Here $O(k)$ denotes the k-dimensional orthogonal group, $BO(k)$ is its classifying space, and $i_k\colon O(k) \hookrightarrow O(k+1)$ is the standard inclusion homomorphism. If γ_k is the universal $O(k)$-bundle over $BO(k)$ with fibres $\mathbb{R}^k$, we have $(Bi_k)^*\gamma_{k+1} \cong \gamma_k \oplus \varepsilon^1$, where ε^n denotes the trivial bundle with fibres $\mathbb{R}^n$. From these data we get a sequence of vector bundles $\Gamma = \{\Gamma_k := f_k^*\gamma_k\}_{k\geq 0}$, with $dim(\Gamma_k) = k$, and bundle maps $\overline{g}_k\colon \Gamma_k \oplus \varepsilon^1 \to \Gamma_{k+1}$. Passing to Thom spaces gives a spectrum, denoted $\mathsf{M}\Gamma$, whose k-th space $\mathsf{M}\Gamma(k)$ is $T\Gamma_k$ and whose structure maps are the maps $T\overline{g}_k\colon T\Gamma_k \wedge S^1 \to T\Gamma_{k+1}$. Any spectrum arising in this way is known as a Thom spectrum.

We will always assume that our Thom spectrum $\mathsf{M}\Gamma$ is a ring spectrum. This means that there are maps of spectra $u\colon \mathsf{S} \to \mathsf{M}\Gamma$ (where S denotes the sphere spectrum with k-th space S^k) and $\mu\colon \mathsf{M}\Gamma \wedge \mathsf{M}\Gamma \to \mathsf{M}\Gamma$, making the following diagrams commute up to homotopy:

$$\begin{tikzcd} \mathsf{M}\Gamma \wedge \mathsf{M}\Gamma \wedge \mathsf{M}\Gamma \arrow[r, "\mu \wedge 1"] \arrow[d, "1 \wedge \mu"'] & \mathsf{M}\Gamma \wedge \mathsf{M}\Gamma \arrow[d, "\mu"] \\ \mathsf{M}\Gamma \wedge \mathsf{M}\Gamma \arrow[r, "\mu"] & \mathsf{M}\Gamma \end{tikzcd} \qquad \begin{tikzcd} \mathsf{S} \wedge \mathsf{M}\Gamma \arrow[r, "u \wedge 1"] \arrow[dr, "\cong"'] & \mathsf{M}\Gamma \wedge \mathsf{M}\Gamma \arrow[d, "\mu"] & \mathsf{M}\Gamma \wedge \mathsf{S} \arrow[l, "1 \wedge u"'] \arrow[dl, "\cong"] \\ & \mathsf{M}\Gamma & \end{tikzcd}$$

The maps u and μ are called the *unit* and *multiplication* of the spectrum. They arise by passage to Thom spaces from a compatible system of bundle maps

$$u_k\colon \mathbb{R}^k \to \Gamma_k, \quad \mu_{k,l}\colon \Gamma_k \times \Gamma_l \to \Gamma_{k+l},$$

where $k, l \in \mathbb{N}$ and each u_k is inclusion of a fibre. Examples will be given at the end of this section.

We will need some notion of orientability of a manifold N^n with respect to the spectrum $\mathsf{M}\Gamma$.

Definition 4.2. *A manifold* N^n *is called* Γ-orientable *if there exists a pair* (e, σ)*, where* $e\colon N \hookrightarrow \mathbb{R}^{n+l}$ *is an embedding with* $l > 0$*, and* $\sigma\colon \nu(e) \to \Gamma_l$ *is a bundle map isomorphic on fibres. Note that for any such pair* (e, σ) *and any integer* $j \geq l$ *we can define another pair* (e^j, σ^j) *by setting* $e^j = \iota_l^j \circ e\colon N \hookrightarrow \mathbb{R}^{n+j}$*, where* $\iota_l^j\colon \mathbb{R}^{n+l} \hookrightarrow \mathbb{R}^{n+j}$ *is the standard inclusion, and* $\sigma^j\colon \nu(e^j) \cong \nu(e) \oplus \varepsilon^{j-l} \to \Gamma_j$ *is the composition*

$$\sigma^j = \overline{g}_{j-1} \circ \ldots \circ (\overline{g}_{l+1} \oplus \varepsilon^{j-l-2}) \circ (\overline{g}_l \oplus \varepsilon^{j-l-1}) \circ (\sigma \oplus \varepsilon^{j-l}).$$

We may put an equivalence relation on such pairs as follows. Two pairs (e_0, σ_0) *and* (e_1, σ_1) *are equivalent if there is an integer* $j \geq \max\{l_0, l_1\}$*, and a third pair* (E, Σ) *giving a level isotopy between the pairs* (e_0^j, σ_0^j) *and* (e_1^j, σ_1^j)*. To be precise,* $E\colon N \times I \hookrightarrow \mathbb{R}^{n+j} \times I$ *is an embedding such that* $E(n, i) = (e_i^j(n), i)$ *for all* $n \in N$*,* $i \in \{0, 1\}$*, and* $\Sigma\colon \nu(E) \to \Gamma_j$ *is a bundle map which restricts to* σ_i^j *over* $N \times \{i\}$*. A* Γ-orientation of N *is then an equivalence class of such pairs. A* Γ-manifold *is a manifold* N *together with a* Γ*-orientation of* N*.*

Put succinctly, a Γ-manifold is a manifold with a Γ-structure on its stable normal bundle.

Our Thom spectra give rise to generalised (co)homology theories, known as the (co)bordism theories. The *k-th unreduced* Γ*-cobordism group* of the (un-pointed) space X is defined as

$$\mathsf{M}\Gamma^k(X) := \lim_{l \to \infty} [\Sigma^l X_+, \mathsf{M}\Gamma(k+l)],$$

where $[-,-]$ denotes pointed homotopy classes of maps, Σ^l denotes the l-th reduced suspension functor, $+$ denotes a disjoint base point, and the maps in the direct limit are given by suspension and the structure maps in the spectrum $\mathsf{M}\Gamma$. Similarly, the $(n-k)$*-th unreduced* Γ*-bordism group* of X is defined as

$$\mathsf{M}\Gamma_{n-k}(X) := \lim_{l \to \infty} [S^l, X_+ \wedge \mathsf{M}\Gamma(l-(n-k))].$$

Our assumption that $\mathsf{M}\Gamma$ was a ring spectrum ensures that there are external products in both bordism and cobordism, and an internal product in cobordism making the direct sum

$$\mathsf{M}\Gamma^*(X) := \bigoplus_{k \in \mathbb{Z}} \mathsf{M}\Gamma^k(X)$$

into a graded ring with unit, the Γ-*cobordism ring* of X.

The following are examples of Thom spectra which are also ring spectra.

Examples 4.3. **(0)** The sphere spectrum S is the most fundamental example of a ring spectrum. It can also be regarded as the Thom spectrum of the family $\mathbf{X}_{\{1\}}$ with $X_k = pt$, $f_k\colon pt \to BO(k)$ the inclusion of a point, and $g_k\colon pt \to pt$ the identity. Each Γ_k is the trivial bundle $\mathbb{R}^k$ over pt, and a $\{1\}$-orientation of a manifold N is a trivialisation of the stable normal bundle of N. The resulting cobordism theory $\mathsf{M}\{1\}^*$, known as *framed cobordism*, is isomorphic to stable cohomotopy.
(1) Let $\mathbf{X}_O$ be the family with $X_k = BO(k)$, $f_k = \mathbf{1}\colon BO(k) \to BO(k)$, and $g_k = Bi_k$. Then each Γ_k is the universal $O(k)$-bundle γ_k, and every manifold has a unique O-orientation. The resulting O-cobordism theory is known as *unoriented cobordism*. The ring structure comes from the classifying bundle maps $\gamma_k \times \gamma_l \to \gamma_{k+l}$.
(2) Let $\mathbf{X}_U$ be the family with $X_{2n} = BU(n) = X_{2n+1}$ (where $BU(n)$ is the classifying space of the unitary group $U(n)$), f_{2n} and f_{2n+1} are the classifying maps of the bundles γ_n^U and $\gamma_n^U \oplus \varepsilon^1$ respectively (where γ_n^U is the universal n-dimensional complex bundle), and the g_k are the obvious inclusions. Then a U-manifold is called a *weakly almost complex manifold*, and U-cobordism is called *complex cobordism*. Similarly, one may take a family $\mathbf{X}_{Sp}$ with $X_{4n} = X_{4n+1} = X_{4n+2} = X_{4n+3} = BSp(n)$, the classifying space of the symplectic group $Sp(n)$, and obtain *symplectic cobordism*.
(3) Let $\mathbf{X}_{SO}$ be the family with $X_k = BSO(k)$ (where $SO(k)$ is the special orthogonal group) and $f_k\colon BSO(k) \to BO(k)$ the classifying map of the universal oriented bundle γ_k^{SO} over $BSO(k)$. The product bundle $\gamma_k^{SO} \times \gamma_l^{SO}$ has a product orientation, classified by a bundle map to γ_{k+l}^{SO}, so MSO is a ring spectrum. An SO-orientation of N is an orientation (in the usual sense) of its stable normal bundle, which is equivalent to an orientation of N itself, so MSO is the spectrum of *oriented cobordism*. Similarly one can define the spectrum MSU.

The standard reference for the concepts found in this section is R. Stong's book [32]. For a very readable account of O- and SO-(co)bordism, see [9].

4.2 Geometric Cobordism

The bordism groups described above were, in the first instance, defined geometrically by M. F. Atiyah as the groups in a "...'Singular homology' theory based on differentiable manifolds" [2]. In fact, define a *singular* Γ-*manifold in* X to be a pair (M^m, f), where M^m is a closed Γ-manifold and $f\colon M \to X$ a continuous map. Two singular Γ-manifolds in X, (M_0, f_0) and (M_1, f_1), are said to be *bordant*, written $(M_0, f_0) \sim (M_1, f_1)$, if there is a pair (W^{m+1}, F) consisting of a Γ-manifold W with boundary $\partial W \approx M_0 \sqcup M_1$, whose Γ-orientation restricts to those of M_0, M_1 at the boundary, and a continuous map $F\colon W \to X$ such that $F|_{M_i} = f_i\colon M_i \to X$ for $i = 0, 1$. Then it is well known (see [9]) that

$$\mathsf{M}\Gamma_{n-k}(X) \cong \{(n-k)\text{-dimensional singular } \Gamma\text{-manifolds in } X\}/\sim .$$

In the same paper [2], Atiyah gave the above homotopy definition of the cobordism groups. The geometric interpretation of cobordism did not appear until ten years later, in a paper of Quillen. To quote A. Dold (from the introduction to [11]), "...homology of X is given by finite chains, cohomology by (infinite but) locally finite chains. In (co)bordism finite chains become maps $W \to X$ of compact manifolds W, whereas locally finite chains become *proper* maps $W \to X$ of arbitrary manifolds W".

We shall need to assume that our un-pointed space X is a manifold without boundary. This does not represent a serious restriction, since any finite complex has the homotopy type of such a manifold (simply take an open regular neighbourhood of an embedding into Euclidean space). The following definition of Γ-orientation of a map $f\colon M \to X$ generalises the notion of Γ-orientation of M, which is the case when X is a point.

Definition 4.4. *A map of manifolds without boundary* $f\colon M^{n-k} \to X^n$ *is* Γ-orientable *if there exists a factorisation of* f *as*

$$M^{n-k} \overset{e}{\hookrightarrow} E^{n+l} \xrightarrow{\pi} X^n,$$

where $\pi\colon E \to X$ *is a smooth* l*-dimensional vector bundle over* X*, and* e *is an embedding with* Γ*-structure* $\sigma\colon \nu(e) \to \Gamma_{k+l}$ *on its normal bundle. Such a factorisation* *of f through E is equivalent to a second one through E', if there exists a third vector bundle E'' over X containing E and E' as sub-bundles, such that the pairs (e, σ) and (e', σ') are level isotopic in E'' (compare Definition 4.2). A* Γ-orientation *of the map $f\colon M \to X$ is then an equivalence class of factorisations.*

We remark here that, assuming X^n is finite dimensional, each Γ-orientation of $f\colon M \to X$ contains a factorisation of the form

$$M \hookrightarrow X \times \mathbb{R}^m \xrightarrow{pr} X.$$

This follows from the fact that for any vector bundle E over a finite dimensional manifold X, there exists a bundle F over X such that $E \oplus F \cong \varepsilon^m$, for some integer m (see any introductory text on K-theory, such as [14]). Hence any vector bundle over X is a sub-bundle of a trivial bundle.

Proposition 4.5. *Let $f\colon M \to X$ be a Γ-oriented map, and let $g\colon Q \to X$ be a map of manifolds which is transverse to f. Then the map δ in the following pullback diagram*

$$\begin{tikzcd}
Q \times_X M \arrow[r, "\rho"] \arrow[d, "\delta"'] & M \arrow[d, "f"] \\
Q \arrow[r, "g"] & X
\end{tikzcd}$$

has a well-defined induced Γ-*orientation.*

Proof. Let the Γ-orientation of f be given by a factorisation

$$M \overset{e}{\hookrightarrow} E \xrightarrow{\pi} X,$$

and a bundle map $\sigma\colon \nu(e) \to \Gamma_l$. Then the above diagram factorises into two pullback squares, as below.

$$\begin{tikzcd}
Q \times_X M \arrow[r, "\rho"] \arrow[d, "e'"'] & M \arrow[d, "e"] \\
g^*E \arrow[r, "\overline{g}"] \arrow[d] & E \arrow[d, "\pi"] \\
Q \arrow[r, "g"] & X
\end{tikzcd}$$

Note that since f and π are both transverse to g, the embedding e is transverse to the bundle map $\overline{g}$. The embedding e' has a Γ-structure given by

$$\nu(e') \cong \rho^*\nu(e) \xrightarrow{\overline{\rho}} \nu(e) \xrightarrow{\sigma} \Gamma_l,$$

and hence the maps down the left hand side represent a Γ-orientation of δ. $\square$

We are now ready to put an equivalence relation on the class of proper Γ-oriented maps to X of a given codimension.

Definition 4.6. *Two proper Γ-oriented maps $f_0\colon M_0^{n-k} \to X^n$ and $f_1\colon M_1^{n-k} \to X^n$ of codimension k are* cobordant *if there exists a proper Γ-oriented map $F\colon W^{n-k+1} \to X^n \times \mathbb{R}$, again of codimension k, such that the embedding $e_i\colon X \hookrightarrow X \times \mathbb{R}$ given by $e_i(x) = (x, i)$ is transverse to F for $i \in \{0, 1\}$, and such that the pull-back of F by e_i with induced Γ-orientation is the Γ-oriented map f_i.*

Cobordism is an equivalence relation, and we have the following Proposition of Quillen [25].

Proposition 4.7. *The set of cobordism classes of proper Γ-oriented maps to X of codimension k is isomorphic to $\mathsf{M}\Gamma^k(X)$.*

The proof is a generalisation of R. Thom's original proof for the coefficient groups [36], and is omitted. One can also check that the structure of the cobordism ring $\mathsf{M}\Gamma^*(X)$ admits the following geometric interpretation in terms of proper maps.

Addition. Let x_1 and x_2 be classes in $\mathsf{M}\Gamma^k(X)$ represented by proper Γ-oriented maps $f_1\colon M_1^{n-k} \to X$ and $f_2\colon M_2^{n-k} \to X$. Then $x_1 + x_2 \in \mathsf{M}\Gamma^k(X)$ is represented by the map $f_1 \sqcup f_2\colon M_1 \sqcup M_2 \to X$, whose Γ-orientation is described as follows.

By the remark following Definition 4.4 and standard theorems of isotopy and embedding, we can find some large integer l such that for $i = 1, 2$ the Γ-orientation of f_i is given by a factorisation of the form

$$M_i \xhookrightarrow{e_i} X \times \mathbb{R}^l \to X.$$

By judicious use of embeddings

$$V_+, V_-\colon \mathbb{R}^l \hookrightarrow \mathbb{R}^l,$$

$$V_{\pm}(y_1, y_2, \ldots, y_l) = (\pm e^{y_1}, y_2, \ldots, y_l), \quad y_i \in \mathbb{R}$$

we can also arrange that e_1 (respectively e_2) embeds M_1 (M_2) in the upper (lower) half-space of $X \times \mathbb{R}^l$. Then the Γ-orientation of $f_1 \sqcup f_2$ is given by the factorisation

$$M_1 \sqcup M_2 \overset{e_1 \sqcup e_2}{\hookrightarrow} X \times \mathbb{R}^l \to X.$$

Given a cobordism class x represented by $f\colon M \to X$, whose Γ-orientation is given as

$$M \overset{e}{\hookrightarrow} X \times \mathbb{R}^l \to X, \quad \sigma\colon \nu(e) \to \Gamma_{k+l},$$

its negative $-x$ is represented by the same map with the negative Γ-orientation, described as follows. Composing e with the *reflection map*

$$r\colon X \times \mathbb{R}^l \to X \times \mathbb{R}^l, \quad r(x, y_1, y_2, \ldots y_l) = (x, -y_1, y_2, \ldots, y_l)$$

gives an embedding $r \circ e$ which is isotopic to e, hence has isomorphic normal bundle. The negative Γ-orientation is represented by

$$M \overset{r \circ e}{\hookrightarrow} X \times \mathbb{R}^l \to X, \quad \sigma \circ \overline{r}\colon \nu(r \circ e) \cong \nu(e) \to \Gamma_{k+l}.$$

One checks that the addition so defined is associative and commutative. The class of the empty manifold and map acts as a zero element, and for any class x we have $x - x = 0$, making $\mathsf{M}\Gamma^k(X)$ an Abelian group.

Functoriality. For each integer k, we have a contravariant functor $\mathsf{M}\Gamma^k(-)$ from the category of manifolds without boundary to the category of Abelian groups. This contravariance is described by a pull-back construction, as in Proposition 2.6. Let $x \in \mathsf{M}\Gamma^k(X)$ be represented by a proper Γ-oriented map $f\colon M^{n-k} \to X^n$, and let $g\colon Q^{n-l} \to X^n$ be a map of manifolds. By the transversality theorems we may find a representative $f'\colon M \to X$ of x which is transverse to g as maps to X. Then, as in Proposition 4.5, we pull back f' by g to obtain a proper Γ-oriented map $\delta\colon Q \times_X M \to X$, which represents the class $g^*x \in \mathsf{M}\Gamma^k(Q)$.

Products. The *external product*

$$\times\colon \mathsf{M}\Gamma^{k_1}(X_1) \otimes \mathsf{M}\Gamma^{k_2}(X_2) \to \mathsf{M}\Gamma^{k_1+k_2}(X_1 \times X_2)$$

is given by Cartesian product of representing maps. So if $x_i \in \mathsf{M}\Gamma^{k_i}(X_i)$ is represented by $f_i\colon M_i \to X_i$ for $i = 1, 2$, then the class $x_1 \times x_2 \in \mathsf{M}\Gamma^{k_1+k_2}(X_1 \times X_2)$ is represented by $f_1 \times f_2\colon M_1 \times M_2 \to X_1 \times X_2$, with the *product orientation* (this is described using the compatible system of bundle maps $\mu_{k,l}\colon \Gamma_k \times \Gamma_l \to \Gamma_{k+l}$ which define the ring spectrum structure on $\mathsf{M}\Gamma$.)

The *internal product*

$$\cup\colon \mathsf{M}\Gamma^k(X) \otimes \mathsf{M}\Gamma^l(X) \to \mathsf{M}\Gamma^{k+l}(X)$$

is given by $x_1 \cup x_2 = \triangle^*(x_1 \times x_2)$, where $\triangle\colon X \hookrightarrow X \times X$ is the diagonal embedding.

One may verify that with these definitions of sums and products, the graded group

$$\mathsf{M}\Gamma^*(X) = \bigoplus_{k \in \mathbb{Z}} \mathsf{M}\Gamma^k(X)$$

has the structure of a graded ring, with the class of the identity map $\mathbf{1}\colon X \to X$ (with trivial Γ-orientation) acting as the unit $1 \in \mathsf{M}\Gamma^0(X)$. The map $g^*\colon \mathsf{M}\Gamma^*(X) \to \mathsf{M}\Gamma^*(Q)$ induced by $g\colon Q \to X$ is a ring homomorphism.

Now suppose that X^n is a closed Γ-manifold, and let $f\colon M^{n-k} \to X^n$ be a proper Γ-oriented map. Then clearly $M = f^{-1}X$ is also closed, and it is not hard to verify that the Γ-orientations of X and f furnish M with a canonical Γ-orientation. Hence a proper Γ-oriented map to X is nothing but a singular Γ-manifold in X. This sheds considerable light on the following duality theorem of Atiyah [2].

Theorem (Poincaré-Atiyah Duality) 4.8. *Let X^n be a closed Γ-manifold. Then for each $k \in \mathbb{Z}$ there is an additive isomorphism*

$$\mathsf{M}\Gamma^k(X) \cong \mathsf{M}\Gamma_{n-k}(X),$$

given by the identity on representing maps.

4.3 From Bordism of Immersions to Cobordism

We now recall some notation from Chapter 2. Let $\mathcal{D}_0$ be the category of finite dimensional manifolds with empty boundary, and proper immersions. Let CMon be the category of commutative monoids. Note that by restriction of both the domain and the range categories, the k-th Γ-cobordism (for a given $k \geq 0$) may be regarded as a contravariant functor

$$\mathsf{M}\Gamma^k \colon \mathcal{D}_0 \to \mathsf{CMon}.$$

This is closely related to the functor given by bordism of immersions with Γ_k-structure,

$$\mathcal{I}(-;\Gamma_k) \colon \mathcal{D}_0 \to \mathsf{CMon},$$

where Γ_k is the k-dimensional real bundle which appeared in the Definition 4.1 of the Thom spectrum $\mathsf{M}\Gamma$. We now give the details of this relationship.

Proposition 4.9. *Let* $\mathsf{M}\Gamma$ *be a Thom spectrum. For each* $k \geq 0$*, there is a natural transformation of functors*

$$T_k \colon \mathcal{I}(-;\Gamma_k) \to \mathsf{M}\Gamma^k(-),$$

where $\mathsf{M}\Gamma^k$ *is regarded as a functor from* $\mathcal{D}_0$ *to* CMon.

Proof. Let N^n be a manifold, and suppose the class $[f] \in \mathcal{I}(N;\Gamma_k)$ is represented by a triple (M, f, v). Since M^{n-k} is compact, the immersion $f \colon M^{n-k} \looparrowright N^n$ is a proper map; we shall show that it has a canonical Γ-orientation.

First, choose an embedding $\tilde{f} \colon M \hookrightarrow \mathbb{R}^l$, where $l \geq 2(n-k)+2$. Such an embedding always exists and is unique up to isotopy, by Theorem A.1. Then we have the following factorisation of f through a trivial bundle,

$$M \overset{(f,\tilde{f})}{\hookrightarrow} N \times \mathbb{R}^l \overset{pr}{\to} N.$$

Note that $\nu(f,\tilde{f}) \cong \nu(f) \oplus \nu(\tilde{f}) \oplus TM \cong \nu(f) \oplus \varepsilon^l$, and so we obtain a Γ-orientation of f via the composition of bundle maps

$$\sigma \colon \nu(f) \oplus \varepsilon^l \xrightarrow{v \oplus \varepsilon^l} \Gamma_k \oplus \varepsilon^l \xrightarrow{\overline{g}_{k+l-1} \circ \ldots \circ (\overline{g}_{k+1} \oplus \varepsilon^{l-2}) \circ (\overline{g}_k \oplus \varepsilon^{l-1})} \Gamma_{k+l}$$

This is indeed a canonical Γ-orientation of f, since isotopic embeddings $\tilde{f}$ lead to the same orientation class.

If the two triples (M_0, f_0, v_0) and (M_1, f_1, v_1) are bordant, a simple check of the definitions shows that the proper Γ-oriented maps f_0 and f_1 are cobordant. Hence there is a well-defined function

$$T_k\colon\ \mathcal{I}(N;\Gamma_k) \to \mathsf{M}\Gamma^k(N),$$

for each N, which takes the bordism class of an immersion with Γ_k-structure to its cobordism class as a Γ-oriented map. It is also an easy exercise to check that each T_k is a monoid map, and is natural in N. □

We now wish to describe the behaviour of the maps T_k with respect to products. The products in $\mathsf{M}\Gamma^*$ arise from a compatible system of bundle maps

$$\mu_{k,l}\colon\ \Gamma_k \times \Gamma_l \to \Gamma_{k+l},$$

where there is one such μ for each pair $k, l \in \mathbb{N}$. Using these maps we obtain product maps

$$\mathcal{I}(N_1;\Gamma_k) \times \mathcal{I}(N_2;\Gamma_l) \xrightarrow{\times} \mathcal{I}(N_1 \times N_2;\Gamma_{k+l}),$$

$$\mathcal{I}(N;\Gamma_k) \times \mathcal{I}(N;\Gamma_l) \xrightarrow{\cup} \mathcal{I}(N;\Gamma_{k+l}),$$

by composing the products from Section 2.3 with the induced maps $(\mu_{k,l})_*$.

Proposition 4.10. *The natural transformations T_k preserve products, that is to say, the following diagrams commute for all $k, l \in \mathbb{N}$:*

$$\begin{tikzcd}
\mathcal{I}(N_1;\Gamma_k) \times \mathcal{I}(N_2;\Gamma_l) \arrow[r, "T_k \times T_l"] \arrow[d, "\times"] & \mathsf{M}\Gamma^k(N_1) \times \mathsf{M}\Gamma^l(N_2) \arrow[d, "\times"] \\
\mathcal{I}(N_1 \times N_2;\Gamma_{k+l}) \arrow[r, "T_{k+l}"] & \mathsf{M}\Gamma^{k+l}(N_1 \times N_2),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{I}(N;\Gamma_k) \times \mathcal{I}(N;\Gamma_l) \arrow[r, "T_k \times T_l"] \arrow[d, "\cup"] & \mathsf{M}\Gamma^k(N) \times \mathsf{M}\Gamma^l(N) \arrow[d, "\cup"] \\
\mathcal{I}(N;\Gamma_{k+l}) \arrow[r, "T_{k+l}"] & \mathsf{M}\Gamma^{k+l}(N).
\end{tikzcd}$$

Proof. Again, this is a straightforward check of the definitions. Note that the second diagram follows from the first, by naturality and the diagonal $\triangle\colon N \hookrightarrow N \times N$. $\square$

We now wish to deduce the analogue of Herbert's Theorem 3.4 in $\mathsf{M}\Gamma^*$, when $f\colon M^{n-k} \looparrowright N^n$ is an immersion with Γ_k-structure. By Theorem 3.7, we have

$$f^*[\psi_r(f)] = [\mu_{r+1}(f)] + e_{\mathcal{I}} \cup [\mu_r(f)] \in \mathcal{I}(M; \mathcal{S}_r\Gamma_k).$$

We shall need two preliminary definitions and an observation.

Definition 4.11. *Let* $\mathbf{X}_\Gamma$ *be a family as in Definition 4.1, resulting in a sequence of vector bundles* $\Gamma = \{\Gamma_k\}_{k\geq 0}$. *Fix* $k \geq 0$. *Suppose there is a bundle map*

$$\rho_r\colon\ \mathcal{S}_r\Gamma_k \to \Gamma_{rk}$$

for each $r \geq 0$, *where* $\mathcal{S}_r\Gamma_k$ *is the* r*-th extended power of the bundle* Γ_k, *and that the diagram*

$$\begin{tikzcd}
\mathcal{S}_p\Gamma_k \times \mathcal{S}_q\Gamma_k \arrow[r] \arrow[d, "\rho_p \times \rho_q"] & \mathcal{S}_{p+q}\Gamma_k \arrow[d, "\rho_{p+q}"] \\
\Gamma_{pk} \times \Gamma_{qk} \arrow[r, "\mu_{pk,qk}"] & \Gamma_{(p+q)k}
\end{tikzcd}$$

commutes up to bundle homotopy for all $p, q \geq 0$ *(the top map was defined in the proof of Proposition 2.12). Then we shall say that the resulting Thom spectrum* $\mathsf{M}\Gamma$ has self-intersections in codimension k.

Definition 4.12. *Let* ζ^k *be a vector bundle of dimension* k *over a manifold* M, *and suppose that* ζ *admits a bundle map* $v\colon\ \zeta \to \Gamma_k$. *Then the zero section* $i\colon\ M \hookrightarrow E(\zeta)$ *is a proper* Γ*-oriented map, and the element* $e_\Gamma(\zeta) = i^*[i] \in \mathsf{M}\Gamma^k(M)$ *is called the* Γ-cobordism Euler class *of the bundle* ζ.

Observe that if $f\colon\ M^{n-k} \looparrowright N^n$ is an immersion with Γ_k-structure, and $e_{\mathcal{I}} \in \mathcal{I}(M; \Gamma_k)$ is its Euler class defined as at the beginning of Section 3.3, then $T_k(e_{\mathcal{I}}) = e_\Gamma(\nu(f))$. From this point forward we shall abuse notation and write $[f] \in \mathsf{M}\Gamma^k(N)$ for the image of $[f] \in \mathcal{I}(N; \Gamma_k)$ under the natural map T_k.

Proposition 4.13. *Let* $\mathsf{M}\Gamma$ *be a Thom spectrum with self-intersections in codimension* k*. Let* $f\colon M^{n-k} \looparrowright N^n$ *be an immersion with* Γ_k*-structure, where* M *is closed. Then*

$$f^*[\psi_r(f)] = [\mu_{r+1}(f)] + e_\Gamma(\nu(f)) \cup [\mu_r(f)] \in \mathsf{M}\Gamma^{rk}(M).$$

Proof. This follows directly from Theorem 3.7 and the results of this Section. ☐

Examples 4.14. **(1)** When $\Gamma = O$, there are bundle maps $\rho_r\colon \mathcal{S}_r\gamma_k \to \gamma_{rk}$ which classify the r-th extended power of each universal $O(k)$-bundle γ_k. Hence MO has self-intersections in every codimension, and we obtain Herbert's Theorem in MO^*.
(2) When $\Gamma = U$, there are bundle maps $\rho_r\colon \mathcal{S}_r\gamma_k^U \to \gamma_{rk}^U$ for each $k \in \mathbb{N}$, since the extended power of the universal $U(k)$-bundle γ_k^U has a canonical complex structure. Since γ_k^U has real dimension $2k$, we can say that MU has self-intersections in all even codimensions. Similarly, MSp has self-intersections in all codimensions divisible by 4. Hence we deduce Herbert's Theorem in MU^* (MSp^*) when f is of even codimension (codimension divisible by 4) and has a complex (symplectic) structure on its normal bundle.
(3) When $\Gamma = SO$, the situation is slightly different. The extended power $\mathcal{S}_r\widetilde{\gamma}_k$ of the universal $SO(k)$-bundle $\widetilde{\gamma}_k$ is orientable if k is even, since then a permutation of the factors of $\widetilde{\gamma}_k^{(r)}$ preserves the product orientation. Hence MSO has self-intersections in even codimensions, so if f has even codimension and $\nu(f)$ is oriented, Herbert's Theorem holds in MSO^*. Similarly, MSU has self-intersections in codimensions divisible by 4.

We can also deduce analogues of Herbert's Theorem in other multiplicative cohomology theories, using the following definition.

Definition 4.15. *Let* E *be a ring spectrum. A* Thom class *is a ring map* $t\colon \mathsf{M}\Gamma \to \mathsf{E}$ *of ring spectra, where* $\mathsf{M}\Gamma$ *is a Thom spectrum.*

A Thom class $t\colon \mathsf{M}\Gamma \to \mathsf{E}$ induces a multiplicative natural transformation of cohomology theories $\overline{t}\colon \mathsf{M}\Gamma^* \to \mathsf{E}^*$. If $e_\Gamma(\zeta) \in \mathsf{M}\Gamma^k(M)$ is the Γ-cobordism Euler class of ζ, then $\overline{t}(e_\Gamma(\zeta)) = e_{\mathsf{E}}(\zeta)$ is an Euler class for ζ in E^*. Since a cobordism class

$\alpha \in \mathsf{M}\Gamma^k(N)$ is represented by a proper Γ-oriented map $f\colon M^{n-k} \to N^n$, we may also regard the class $\overline{t}(\alpha) \in \mathsf{E}^k(N)$ as being represented by the map f, and for this reason we will often write $[f] \in \mathsf{E}^k(N)$ instead of $\overline{t}(\alpha)$.

Proposition 4.16. *Let* $\mathsf{M}\Gamma$ *and* f *be as in Corollary 4.13, and let* $t\colon \mathsf{M}\Gamma \to \mathsf{E}$ *be a Thom class. Then the analogue of Herbert's Theorem holds in* E^**, that is,*

$$f^*[\psi_r(f)] = [\mu_{r+1}(f)] + e_{\mathsf{E}}(\nu(f)) \cup [\mu_r(f)] \in \mathsf{E}^{rk}(M).$$

Examples 4.17. (1) The universal Thom class $t\colon \mathsf{MO} \to \mathsf{H}\mathbb{Z}_2$.

(2) The Conner-Floyd Thom classes $t\colon \mathsf{MU} \to \mathsf{K}$, $t\colon \mathsf{MSp} \to \mathsf{KO}$, where K, KO are the spectra of complex and real K-theory respectively [10].

(3) The oriented Thom class $t\colon \mathsf{MSO} \to \mathsf{H}\mathbb{Z}$ to integral cohomology, and the Conner-Floyd map $t\colon \mathsf{MSU} \to \mathsf{KO}$ to real K-theory [10].

Finally in this section, we would like to deduce Herbert's Theorem in the dual homology theories. To do so we assume that N^n is a Γ-manifold. We observe that an immersion $f\colon M^{n-k} \looparrowright N^n$ with Γ_k-structure is also a singular Γ-manifold in N, since M receives a canonical Γ-orientation from those of N and the map f. The bordism class of this singular manifold depends only on the class $[f] \in \mathcal{I}(N;\Gamma_k)$. Thus for any Γ-manifold N^n there is well-defined map $\mathcal{I}(N;\Gamma_k) \to \mathsf{M}\Gamma_{n-k}(N)$.

Recall (Theorem 4.8) that since M^{n-k} is a closed Γ-manifold, there is a Poincaré-Atiyah duality isomorphism

$$\mathsf{M}\Gamma^l(M) \cong \mathsf{M}\Gamma_{n-k-l}(M)$$

for all l. This is given by cap product with a fundamental class $[M]_{\mathsf{M}\Gamma} \in \mathsf{M}\Gamma_{n-k}(M)$, which is represented by the singular Γ-manifold $\mathbf{1}\colon M \to M$. Now a Thom class $t\colon \mathsf{M}\Gamma \to \mathsf{E}$ also induces a natural transformation of homology theories $\underline{t}\colon \mathsf{M}\Gamma_* \to \mathsf{E}_*$, which maps $[M]_{\mathsf{M}\Gamma}$ to a fundamental class $[M]_{\mathsf{E}} \in \mathsf{E}_{n-k}(M)$. Hence the manifold M is oriented with respect to the spectrum E, and so there is a Poincaré duality isomorphism

$$\mathsf{E}^l(M) \cong \mathsf{E}_{n-k-l}(M)$$

for all l. Note that if $g\colon Q^q \to M$ is a singular Γ-manifold in M representing $[g] \in \mathsf{M}\Gamma_q(M)$, then $\underline{t}[g] = g_*[Q]_{\mathsf{E}} \in \mathsf{E}_q(M)$. The natural transformations $\overline{t}, \underline{t}$ satisfy the identity

$$\overline{t}(x) \cap \underline{t}(y) = \underline{t}(x \cap y) \in \mathsf{E}_*(M),$$

for all $x \in \mathsf{M}\Gamma^*(M), y \in \mathsf{M}\Gamma_*(M)$.

The above information is summarised in the following commutative diagram.

$$\begin{tikzcd}
\mathcal{I}(M;\Gamma_{rk}) \arrow[r, "\mathcal{T}_{rk}"] \arrow[dr] & \mathsf{M}\Gamma^{rk}(M) \arrow[r, "\overline{t}"] \arrow[d, "\cong"] & \mathsf{E}^{rk}(M) \arrow[d, "\cong"] \\
& \mathsf{M}\Gamma_{n-(r+1)k}(M) \arrow[r, "\underline{t}"'] & \mathsf{E}_{n-(r+1)k}(M)
\end{tikzcd}$$

Proposition 4.18. *Let $\mathsf{M}\Gamma$, f and t be as in Corollary 4.16, and suppose that N is a Γ-manifold. Then the analogues of Herbert's Theorem hold in $\mathsf{M}\Gamma_*$ and E_*, that is*

$$\mathcal{D}_M f^*[\psi_r(f)] = \mathcal{D}_M[\mu_{r+1}(f)] + \mathcal{D}_M(e(\nu(f)) \cup [\mu_r(f)]),$$

where $\mathcal{D}_M$ denotes Poincaré duality in M.

4.4 Steenrod Operations

Steenrod operations are cohomology operations arising from higher homotopy commutativity properties of the product in a multiplicative generalised cohomology theory. They were discovered in the case of $\mathbb{Z}_p$-cohomology by N.E. Steenrod [30]; analogous operations were found to exist in the cobordism cohomology theories by T. tom Dieck [37], and in K-theory by Atiyah [3]. We are mainly interested in the cobordism theories, where Quillen's interpretation of cobordism classes as classes of proper maps allows for a particularly nice geometric construction of the so-called Steenrod-tom Dieck operations.

We begin with some generalities. Let r be a positive integer and let G be a subgroup of the symmetric group S_r. Let EG be a contractible space with a free right G-action, and let Y be a *pointed* space with base point $*$. The group G acts on the left of the r-fold smash product $Y^{\wedge r}$ of Y, by permutation of the factors.

Definition 4.19. *The quotient space*

$$EG \ltimes_G Y^{\wedge r} := \frac{EG \times_G Y^{\wedge r}}{EG \times_G \{*\}}$$

is denoted by $D_G Y$. The r-th extended power *of Y is the space $D_{S_r} Y = D_r Y$.*

Compare the Definition 2.10 of the r-th extended power of a bundle. In fact we have the following well-known result (see [37], Lemma 3.5) relating the two.

Proposition 4.20. *For any bundle ζ there is a homeomorphism*

$$T\mathcal{S}_G\zeta \approx D_G T\zeta,$$

where $T\zeta$ denotes the Thom space of ζ.

Corollary 4.21. *For any space X (pointed or un-pointed) there is a homeomorphism*

$$(EG \times_G X^{(r)})_+ \approx D_G(X_+),$$

where $+$ denotes a disjoint base point.

Proof. Apply Proposition 4.20 to the 0-dimensional bundle $\mathbf{1}\colon X \to X$. □

Note that given any point $e \in EG$ we may define a map $i_e\colon Y^{\wedge r} \to D_G Y$ by setting $i_e([y_1, \ldots, y_r]) = [e, y_1, \ldots, y_r]$. We shall denote any map obtained in this way by i, since it is independent of e up to homotopy.

Let E be a commutative ring spectrum, and let $\widetilde{\mathsf{E}}^n(Y)$ denote the n-th *reduced* cohomology group of the pointed space Y in the cohomology theory represented by E. Recall that G is a subgroup of the symmetric group S_r.

Definition 4.22. *For any positive integer d, an* external Steenrod operation of type (G, d) in $\widetilde{\mathsf{E}}^*$ *is a family $P = (P^{kd} \mid k \in \mathbb{Z})$ of natural transformations*

$$P^{kd}\colon \widetilde{\mathsf{E}}^{kd}(Y) \to \widetilde{\mathsf{E}}^{rkd}(D_G Y)$$

with the additional property that the composition

$$i^* \circ P^{kd}\colon \widetilde{\mathsf{E}}^{kd}(Y) \to \widetilde{\mathsf{E}}^{rkd}(Y^{\wedge r})$$

is the r-fold exterior product $y \mapsto y^{\wedge r}$.

Thus there is a sense in which these operations 'extend the operation of raising x to the r-th power'. One may also define internal operations, using the *extended diagonal map*

$$\triangle_G\colon BG \wedge Y = EG \ltimes_G Y \to D_G Y$$

which maps $[e, y]$ to $[e, y, \ldots, y]$ (here $BG := EG/G$).

Definition 4.23. *An external Steenrod operation P of type (G, d) in $\widetilde{\mathsf{E}}^*$ gives an* internal Steenrod operation $(\mathcal{P}^{kd} \mid k \in \mathbb{Z})$ *by setting*

$$\mathcal{P}^{kd} = \triangle_G^* \circ P^{kd}\colon \widetilde{\mathsf{E}}^{kd}(Y) \to \widetilde{\mathsf{E}}^{rkd}(BG \wedge Y).$$

The above definitions are due to T. tom Dieck, based on the work of Steenrod. One may give additional axioms for the operations, from which a myriad of properties may be derived (see for example [37], [31]). An example of such is the following.

Proposition 4.24. *Let $P = (P^{kd} \mid k \in \mathbb{Z})$ be a Steenrod operation of type (G, d) in $\widetilde{\mathsf{E}}^*$, and let $t \in \widetilde{\mathsf{E}}^{kd}(T\zeta)$ be a Thom class for the kd-dimensional bundle ζ. Then $P^{kd}(t) \in \widetilde{\mathsf{E}}^{rkd}(D_G T\zeta)$ is a Thom class for $\mathcal{S}_G\zeta$.*

Proof. Let $i\colon T\zeta^{\wedge r} \to D_G T\zeta$ be the map induced by the inclusion of a point in EG. By Definition 4.22,

$$i^* \circ P^{kd}(t) = t^{\wedge r} \in \mathsf{E}^{rkd}(T\zeta^{\wedge r}),$$

which is a Thom class for $\zeta^{(r)}$. The inclusion of a compactified fibre of $\mathcal{S}_G\zeta$ factors as

$$S^{rkd} = (S^{kd})^{\wedge r} \to T\zeta^{\wedge r} \xrightarrow{i} D_G T\zeta.$$

The Proposition follows by the definition of a Thom class. □

To illustrate how such operations are constructed, we consider the Eilenberg-Mac Lane spectrum $\mathsf{H}\mathbb{Z}_p$, the spectrum of ordinary $\mathbb{Z}_p$-cohomology. This is an example of an Ω-*spectrum*, which is a spectrum E whose structure maps $\sigma_k\colon \Sigma\mathsf{E}_k \to \mathsf{E}_{k+1}$ are adjoint to homeomorphisms $\widetilde{\sigma}_k\colon \mathsf{E}_k \to \Omega\mathsf{E}_{k+1}$.

Example 4.25. Take $\mathsf{E} = \mathsf{H}\mathbb{Z}_p$ with p prime, and $G = \mathbb{Z}_p \leq S_p$. Since $\mathsf{H}\mathbb{Z}_p$ is an Ω-spectrum, a cohomology class $\alpha \in \widetilde{H}^k(Y; \mathbb{Z}_p)$ is represented by a pointed map $h\colon Y \to K(\mathbb{Z}_p, k)$. The construction of $D_G Y$ given above extends to a functor from the category of pointed spaces and maps to itself. Hence we have a map $D_{\mathbb{Z}_p} h\colon D_{\mathbb{Z}_p} Y \to D_{\mathbb{Z}_p} K(\mathbb{Z}_p, k)$. For each k we can construct a map $\xi_k\colon D_{\mathbb{Z}_p} K(\mathbb{Z}_p, k) \to K(\mathbb{Z}_p, pk)$ using commutativity properties of the cohomology $\times$-product. Composing we get a map

$$\xi_k \circ D_{\mathbb{Z}_p} h\colon D_{\mathbb{Z}_p} Y \to K(\mathbb{Z}_p, pk)$$

which represents the class $P(\alpha) \in \widetilde{H}^{pk}(D_{\mathbb{Z}_p} Y; \mathbb{Z}_p)$. This gives a Steenrod operation of type $(\mathbb{Z}_p, 1)$ in $\mathsf{H}\mathbb{Z}_p^*$.

In the case $p = 2$, the familiar Steenrod squares

$$Sq^i\colon \widetilde{H}^k(Y; \mathbb{Z}_2) \to \widetilde{H}^{k+i}(Y; \mathbb{Z}_2)$$

are obtained as follows. The corresponding internal operation applied to an element $\alpha \in \widetilde{H}^k(Y; \mathbb{Z}_2)$ yields an element $\mathcal{P}^k(\alpha) \in \widetilde{H}^{2k}(\mathbb{R}P^\infty \wedge Y; \mathbb{Z}_2)$. By the Künneth Theorem,

$$\widetilde{H}^*(\mathbb{R}P^\infty \wedge Y; \mathbb{Z}_2) \cong \mathbb{Z}_2[w] \otimes \widetilde{H}^*(Y; \mathbb{Z}_2),$$

where $w \in \widetilde{H}^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$ is the Euler class of the canonical line bundle. The action of the Sq^i on α are determined by the formula

$$\mathcal{P}^k(\alpha) = \sum_{i \geq 0} w^{k-i} \otimes Sq^i(\alpha).$$

We now sketch tom Dieck's proof [37] of the existence of Steenrod operations in $\widetilde{\mathsf{M}\Gamma}^*$, where $\mathsf{M}\Gamma$ is one of the Thom spectra of Examples 4.3.

Proposition (tom Dieck) 4.26. *Suppose* $\mathsf{M}\Gamma$ *has self-intersections in all codimensions divisible by* d. *Then for every* $r \geq 1$ *there exists an external Steenrod operation* $P = (P^{kd} \mid k \in \mathbb{Z})$ *of type* (S_r, d) *in* $\widetilde{\mathsf{M}\Gamma}^*$.

Proof. Let α be an element of

$$\widetilde{\mathsf{M}\Gamma}^{kd}(Y) \cong \lim_{l \to \infty} [\Sigma^l Y, \mathsf{M}\Gamma(l + kd)];$$

we must construct an element $P(\alpha) \in \widetilde{\mathsf{M}\Gamma}^{rkd}(D_r Y)$. Assuming that Y is a finite complex, α is represented by a map

$$h\colon \Sigma^{jd} Y \to \mathsf{M}\Gamma((j + k)d) = T\Gamma_{(j+k)d}.$$

Applying the functor D_r, we obtain a map

$$D_r h\colon\ D_r\Sigma^{jd}Y \to D_r T\Gamma_{(j+k)d}.$$

Now we observe that there is a bundle map $\rho_r\colon\ \mathcal{S}_r\Gamma_{(j+k)d} \to \Gamma_{r(j+k)d}$ for every natural number $(j+k)$, by our assumption on $\mathsf{M}\Gamma$. Passing to Thom spaces gives a map

$$T\rho_r\colon\ D_r T\Gamma_{(j+k)d} \to T\Gamma_{r(j+k)d}$$

which represents the canonical Thom class t_{j+k} of $\mathcal{S}_r\Gamma_{(j+k)d}$. Composing we get a map

$$T\rho_r \circ D_r h\colon\ D_r\Sigma^{j+d}Y \to T\Gamma_{r(j+k)d}$$

which represents the class $(D_r h)^* t_{j+k} \in \widetilde{\mathsf{M}\Gamma}^{r(j+k)d}(D_r\Sigma^{jd}Y)$.

For any vector bundle ζ over a pair (Y, A), with Thom class $v \in \widetilde{\mathsf{M}\Gamma}^{k}(T\zeta)$, there is a *relative Thom isomorphism*

$$\Phi_v\colon\ \widetilde{\mathsf{M}\Gamma}^{n}(Y/A) \cong \widetilde{\mathsf{M}\Gamma}^{n+k}(T\zeta/T(\zeta|_A)).$$

Let ε^{jd} be the trivial jd-dimensional bundle over Y (Note that $T\varepsilon^{jd}/T(\varepsilon^{jd}|_*) \approx \Sigma^{jd}Y$). We shall apply the Thom Isomorphism in the case of the bundle $\mathcal{S}_r\varepsilon^{jd}$ over the pair $(ES_r \times_{S_r} Y^{(r)}, A)$, where A is the subspace of $ES_r \times_{S_r} Y^{(r)}$ consisting of points with at least one of their Y coordinates equal to the base point. This bundle is classified by the composition of bundle maps

$$\mathcal{S}_r\varepsilon^{jd} \xrightarrow{\mathcal{S}_r u_{jd}} \mathcal{S}_r\Gamma_{jd} \xrightarrow{\rho_r} \Gamma_{rjd},$$

where $u_{jd}\colon\ \varepsilon^{jd} \to \Gamma^{jd}$ is determined by the maps giving the unit of the spectrum $\mathsf{M}\Gamma$ (see Section 4.1). On passage to Thom spaces this represents a Thom class $v \in \mathsf{M}\Gamma^{rjd}(D_r\Sigma^{jd}Y_+)$. Clearly there is a homeomorphism

$$\frac{ES_r \times_{S_r} Y^{(r)}}{A} \approx D_r Y,$$

and an adaptation of Proposition 4.20 gives

$$\frac{T(\mathcal{S}_r\varepsilon^{jd})}{T(\mathcal{S}_r\varepsilon^{jd}|_A)} \approx D_r\Sigma^{jd}Y.$$

Hence there is a Thom Isomorphism

$$\Phi_v\colon \widetilde{\mathsf{M}\Gamma}^{rkd}(D_rY) \cong \widetilde{\mathsf{M}\Gamma}^{r(j+k)d}(D_r\Sigma^{jd}Y),$$

and we may set $P(\alpha) = \Phi_v^{-1}((D_rh)^*t_{j+k})$. Using properties of Thom isomorphisms and the fact that D_r is a functor, it is not difficult to check that $P^{kd}\colon \widetilde{\mathsf{M}\Gamma}^{kd}(-) \to \widetilde{\mathsf{M}\Gamma}^{rkd}(D_r(-))$ is a well-defined natural transformation for each $k \in \mathbb{Z}$. The property $i^* \circ P(\alpha) = \alpha^{\wedge r}$ can be deduced from naturality of P and the fact that $i^*t_k = t^{\wedge r}$, where $t \in \widetilde{\mathsf{M}\Gamma}^{kd}(T\Gamma_{kd})$ is the canonical Thom class represented by the identity map and $t_k \in \widetilde{\mathsf{M}\Gamma}^{rkd}(D_rT_{kd})$ is the Thom class of $\mathcal{S}_r\Gamma_{kd}$. □

Remarks 4.27. (i) Note that each operation in $\widetilde{\mathsf{M}\Gamma}^*$ is uniquely defined, thanks to the structure we imposed on the spectrum $\mathsf{M}\Gamma$. A different choice of unit map $u\colon \mathsf{S} \to \mathsf{M}\Gamma$ may lead to a different operation.

(ii) Let G be a subgroup of the *alternating group* A_r (the subgroup of S_r consisting of even permutations). Then there are bundle maps $\mathcal{S}_G\widetilde{\gamma}_k \to \widetilde{\gamma}_{rk}$ for each k, since an even permutation of the factors of $\widetilde{\gamma}_k^{(r)}$ preserves the product orientation. Hence there is a Steenrod operation of type $(G, 1)$ in $\widetilde{\mathsf{MSO}}^*$ (and similarly of type $(G, 2)$ in $\widetilde{\mathsf{MSU}}^*$).

(iii) We have deliberately chosen our language to suggest a relationship between the self-intersection operations ψ_r and Steenrod operations (see also the 'Cartan formula' of Proposition 2.12). This will be the subject of Chapter 6. In fact, the statement that "$\mathsf{M}\Gamma$ has self-intersections in codimensions divisible by d" can be interpreted as "$\mathsf{M}\Gamma$ has (part of) the structure of an H_∞^d ring spectrum". See [7] for a survey of H_∞ ring spectra and their connection with Steenrod operations.

Finally in this chapter we give the geometric interpretation of the Steenrod operations. We restrict attention to the case $r = 2$ and $S_2 = \mathbb{Z}_2$, giving an external Steenrod operation $P = (P^{kd} \mid k \in \mathbb{Z})$ of type $(\mathbb{Z}_2, d)$ in $\widetilde{\mathsf{M}\Gamma}^*$. As $E\mathbb{Z}_2$ we take the infinite sphere $S^\infty = \bigcup_l S^l$ with the direct limit topology, and $\mathbb{Z}_2$ action given by the antipodal map.

Given a manifold X^n we wish to describe the maps

$$P^{kd}\colon \widetilde{\mathsf{M}\Gamma}^{kd}(X_+) \cong \mathsf{M}\Gamma^{kd}(X) \to \mathsf{M}\Gamma^{2kd}(S^\infty \times_{\mathbb{Z}_2} X^{(2)}) \cong \widetilde{\mathsf{M}\Gamma}^{2kd}(D_2X_+)$$

in terms of proper maps of manifolds.

Note that for each $l \in \mathbb{N}$, the usual inclusion $\jmath_l\colon S^l \hookrightarrow S^\infty$ extends to a $\mathbb{Z}_2$-equivariant map $\jmath_l \times \mathbf{1}\colon S^l \times X^{(2)} \hookrightarrow S^\infty \times X^{(2)}$, which factors to give an inclusion

$$\imath_l := \jmath_l \times_{\mathbb{Z}_2} \mathbf{1}\colon S^l \times_{\mathbb{Z}_2} X^{(2)} \hookrightarrow S^\infty \times_{\mathbb{Z}_2} X^{(2)}.$$

Let $\alpha \in \mathsf{M}\Gamma^{kd}(X)$ be represented by a proper map $f\colon M^{n-kd} \to X^n$ with Γ-orientation given by

$$M \overset{e}{\hookrightarrow} E \overset{\pi}{\to} X, \quad \sigma\colon \nu(e) \to \Gamma_{(j+k)d}.$$

The following fact is well known to workers in the field (see for example [37], Satz 14.1).

Proposition 4.28. *For each $l \in \mathbb{N}$, the class $\imath_l^* P^{kd}(\alpha) \in \mathsf{M}\Gamma^{2kd}(S^l \times_{\mathbb{Z}_2} X^{(2)})$ is represented by the proper map*

$$\lambda_l(f) := 1 \times_{\mathbb{Z}_2} f^{(2)}\colon S^l \times_{\mathbb{Z}_2} M^{(2)} \to S^l \times_{\mathbb{Z}_2} X^{(2)},$$

with the following Γ-orientation. A factorisation of $\lambda_l(f)$ is given by

$$S^l \times_{\mathbb{Z}_2} M^{(2)} \xrightarrow{1 \times_{\mathbb{Z}_2} e^{(2)}} S^l \times_{\mathbb{Z}_2} E^{(2)} \xrightarrow{1 \times_{\mathbb{Z}_2} \pi^{(2)}} S^l \times_{\mathbb{Z}_2} X^{(2)}.$$

Note that $1 \times_{\mathbb{Z}_2} e^{(2)}$ has normal bundle $S^l \times_{\mathbb{Z}_2} \nu(e)^{(2)}$, so an orientation is given by the composition of bundle maps

$$S^l \times_{\mathbb{Z}_2} \nu(e)^{(2)} \xrightarrow{\imath_l} \mathcal{S}_2\nu(e) \xrightarrow{\mathcal{S}_2\sigma} \mathcal{S}_2\Gamma_{(j+k)d} \xrightarrow{\rho_{(j+k)d}} \Gamma_{2(j+k)d}.$$

The corresponding internal operation

$$\mathcal{P}^{kd}\colon \widetilde{\mathsf{M}\Gamma}^{kd}(X_+) \cong \mathsf{M}\Gamma^{kd}(X) \to \mathsf{M}\Gamma^{2kd}(\mathbb{R}P^\infty \times X) \cong \widetilde{\mathsf{M}\Gamma}^{2kd}(\mathbb{R}P^\infty \wedge X_+)$$

is defined as $\triangle_2^* \circ P^{kd}$, where $\triangle_2\colon \mathbb{R}P^\infty \times X \hookrightarrow S^\infty \times_{\mathbb{Z}_2} X^{(2)}$ is the extended diagonal map. This map can be viewed as the direct limit of maps $\triangle_2^l\colon \mathbb{R}P^l \times X \hookrightarrow S^l \times_{\mathbb{Z}_2} X^{(2)}$. Note that there is an inclusion

$$\ell_l\colon \mathbb{R}P^l \times X \hookrightarrow \mathbb{R}P^\infty \times X$$

for each $l \in \mathbb{N}$, and we have

$$\imath_l \circ \triangle_2^l = \triangle_2 \circ \ell_l\colon \mathbb{R}P^l \times X \to S^\infty \times_{\mathbb{Z}_2} X^{(2)}.$$

Corollary 4.29. *For each $l \in \mathbb{N}$, the class $\ell_l^* \mathcal{P}^{kd}(\alpha) \in \mathsf{M}\Gamma^{2kd}(\mathbb{R}P^l \times X)$ is represented by the map $\xi_l(f)$ in the following pull back diagram.*

$$\begin{tikzcd} \Sigma(f) \arrow[r] \arrow[d, "\xi_l(f)"'] & S^l \times_{\mathbb{Z}_2} M^{(2)} \arrow[d, "\lambda_l(f)'"] \\ \mathbb{R}P^l \times X \arrow[r, "\triangle_2^l"] & S^l \times_{\mathbb{Z}_2} X^{(2)}, \end{tikzcd}$$

Here $\lambda_l(f)'$ is a representative of $[\lambda_l(f)]$ which is transverse to $\triangle_2^l$, and $\xi_l(f)$ is given the Γ-orientation induced from $\lambda_l(f)'$.

Chapter 5

Bordism of Spreadings

In this Chapter we introduce the notion of a *spreading of type* Y *in* X, where X and Y are pointed topological spaces. Spreadings were first considered by P. Vogel in his 1974 paper [38] (with the French name 'étalements') as a tool for studying the bordism of immersions, and the results here are from that paper. The bordism of spreadings gives a bifunctor $\mathcal{J}(-;-)$, closely related to the bifunctor $\mathcal{I}(-;-)$ of Chapter 2. In fact, one of Vogel's key insights was that the bordism class in $\mathcal{I}(N;\zeta)$ of a triple (M, f, v) depends only on the bordism class of a particular spreading of type $T\zeta$ in the one-point compactification N_c of N, whose data consists of a 'spreading' of f (an extension of f to an immersion of its normal disc bundle) and the homotopy data of the structure map v near the zero section $i(M) \subseteq \nu(f)$.

5.1 Definitions

We first give the definition of a spreading. Let (X, A) be a pair of topological spaces, and let Y be a pointed space with base point y.

Definition 5.1. *A* spreading of type Y in (X, A) *consists of a triple* (K, α, β)*, where* K *is a topological space,* $\alpha\colon K \to X$ *is a proper, closed continuous map, and* $\beta\colon (K, \alpha^{-1}(A)) \to (Y, y)$ *is a continuous map such that the restriction of* α *to* $K - \beta^{-1}(y)$ *is a local homeomorphism.*

By a local homeomorphism $h\colon W \to Z$ we mean a map h such that every $w \in W$ has an open neighbourhood U such that $h|_U\colon U \to h(U)$ is a homeomorphism.

We may put a bordism relation on the class of all such triples as follows.

Definition 5.2. *Two spreadings (K_0, α_0, β_0) and (K_1, α_1, β_1) of type Y in (X, A) are* bordant *if there is a spreading (L, Ψ, Φ) of type Y in $(X \times I, A \times I)$ such that the squares in the following diagram are pullback squares,*

$$\begin{tikzcd} K_0 \arrow[r] \arrow[d, "\alpha_0"'] & L \arrow[d, "\Psi"] & K_1 \arrow[l] \arrow[d, "\alpha_1"] \\ X \times \{0\} \arrow[r] & X \times I & X \times \{1\} \arrow[l] \end{tikzcd}$$

and $\Phi|_{K_i} = \beta_i$ for $i = 0, 1$.

Note that bordism is an equivalence relation on the class of all spreadings of type Y in (X, A). We denote the bordism class of a triple (K, α, β) by $[K, \alpha, \beta]$. If (X, x) and (Y, y) are both pointed topological spaces, we denote the set of bordism classes of spreadings of type Y in X by $\mathcal{J}(X; Y)$, omitting reference to the base points. This set has the structure of a commutative monoid, with the addition defined by disjoint union of representing spreadings, so

$$[K, \alpha, \beta] + [K', \alpha', \beta'] = [K \sqcup K', \alpha \sqcup \alpha', \beta \sqcup \beta'] \in \mathcal{J}(X; Y).$$

The empty spreading acts as the zero element.

To illustrate these concepts, and for future reference, we record the following Proposition which says that the bordism class of a spreading (K, α, β) of type Y in X is not affected by 'throwing away' almost all of $\beta^{-1}(y)$.

Proposition 5.3. *Let (K, α, β) be a spreading of type Y in X. Let $C \subseteq K$ be the closure of $K - \beta^{-1}(y)$. Then*

$$[C, \alpha|_C, \beta|_C] = [K, \alpha, \beta] \in \mathcal{J}(X; Y)$$

Proof. Let $i\colon C \hookrightarrow K$ be the inclusion. The mapping cylinder

$$M_i = \frac{C \times I \sqcup K}{(c, 1) \sim i(c)},$$

along with the obvious maps $M_i \to X \times I$ and $M_i \to Y$, gives the required bordism of spreadings. $\square$

5.2 Functoriality

The bordism monoid $\mathcal{J}(X;Y)$ is functorial in both variables, X and Y. Let $\mathscr{T}_\bullet$ denote the category of pointed topological spaces and pointed continuous maps. Let $\underline{\mathscr{T}}_\bullet$ be the category with the same objects, but morphisms the *proper* pointed maps. Obviously, homotopies in these categories are required to preserve base points.

Proposition 5.4. *Bordism of spreadings is a homotopy bifunctor*

$$\mathcal{J}(-;-)\colon\ \underline{\mathscr{T}}_\bullet^{\mathsf{op}} \times \mathscr{T}_\bullet \to \mathsf{CMon}.$$

Proof. We begin with covariance. Let $t\colon\ Y_1 \to Y_2$ be a pointed map, and let (K, α, β) be a spreading of type Y_1 in X. Then the triple $(K, \alpha, t \circ \beta)$ is a spreading of type Y_2 in X, whose bordism class depends only on the bordism class of (K, α, β). Hence we get a well-defined induced map

$$t_*\colon\ \mathcal{J}(X;Y_1) \to \mathcal{J}(X;Y_2),$$

which is clearly a monoid map and makes $\mathcal{J}(X;-)$ a functor. If $T\colon\ Y_1 \times I \to Y_2$ is a pointed homotopy from t to another pointed map $t_1\colon\ Y_1 \to Y_2$, then the spreading $(K \times I, \alpha \times \mathbf{1}, T \circ (\beta \times \mathbf{1}))$ of type Y_2 in $(X \times I, x \times I)$ is a bordism from $(K, \alpha, t \circ \beta)$ to $(K, \alpha, t_1 \circ \beta)$. Hence $\mathcal{J}(X;-)$ is a homotopy functor.

Now suppose we have a proper pointed map $\phi\colon\ X_1 \to X_2$. We define the induced map

$$\phi^*\colon\ \mathcal{J}(X_2;Y) \to \mathcal{J}(X_1;Y),$$

as follows. Given a spreading (K, α, β) of type Y in X_2, form the pullback of α by ϕ. This gives a diagram

$$\begin{tikzcd}
X_1 \times_{X_2} K \arrow[r, "\rho"] \arrow[d, "\delta"'] & K \arrow[r, "\beta"] \arrow[d, "\alpha"] & Y \\
X_1 \arrow[r, "\phi"] & X_2 &
\end{tikzcd}$$

and the triple $(X_1 \times_{X_2} K, \delta, \beta \circ \rho)$ is a spreading of type Y in X_1. Setting

$$\phi^*[K, \alpha, \beta] = [X_1 \times_{X_2} K, \delta, \beta \circ \rho]$$

gives a well-defined monoid map depending only on the homotopy class of ϕ, and the Proposition is proved. $\square$

Note that *one-point compactification* describes a covariant functor

$$(-)_c \colon \mathcal{D}_0 \to \underline{\mathscr{T}_\bullet},$$

and passage to Thom spaces describes a covariant *Thomification* functor

$$T \colon \mathsf{Vect} \to \mathscr{T}_\bullet.$$

With these functors we describe the relationship between $\mathcal{J}(-;-)$ and $\mathcal{I}(-;-)$.

Proposition 5.5. *The following diagram of functors commutes up to natural isomorphism.*

$$\begin{tikzcd}
\mathcal{D}_0^{\mathsf{op}} \times \mathsf{Vect} \arrow[rr, "(-)_c \times T"] \arrow[dr, "\mathcal{I}(-;-)"'] & & \underline{\mathscr{T}_\bullet}^{\mathsf{op}} \times \mathscr{T}_\bullet \arrow[dl, "\mathcal{J}(-;-)"] \\
& \mathsf{CMon} &
\end{tikzcd}$$

Proof. We must find a natural transformation of functors

$$\Theta \colon \mathcal{I}(-;-) \to \mathcal{J}((-)_c; T(-)),$$

each component of which is invertible. Hence we must describe, for any pair of objects (N, ζ) in $\mathcal{D}_0 \times \mathsf{Vect}$, a monoid isomorphism

$$\Theta \colon \mathcal{I}(N;\zeta) \to \mathcal{J}(N_c; T\zeta)$$

such that if $g \colon Q \to N$ is a proper immersion and $\eta \colon \zeta \to \xi$ is a bundle map, then the following diagrams commute.

$$\begin{tikzcd}
\mathcal{I}(N;\zeta) \arrow[r, "\Theta"] \arrow[d, "g^*"'] & \mathcal{J}(N_c;T\zeta) \arrow[d, "(g_c)^*"] \\
\mathcal{I}(Q;\zeta) \arrow[r, "\Theta"] & \mathcal{J}(Q_c;T\zeta)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}(N;\zeta) \arrow[r, "\Theta"] \arrow[d, "\eta_*"'] & \mathcal{J}(N_c;T\zeta) \arrow[d, "(T\eta)_*"] \\
\mathcal{I}(N;\xi) \arrow[r, "\Theta"] & \mathcal{J}(N_c;T\xi)
\end{tikzcd}$$

Given $[M, f, v] \in \mathcal{I}(N; \zeta)$, the bundle map $v\colon\ \nu(f) \to \zeta$ is isometric on fibres, so that it induces a map of unit disc bundles. Composing with the collapse map $D\zeta \to T\zeta$ gives a continuous map

$$\widetilde{v}\colon\ D\nu(f) \to T\zeta,$$

and if $* \in T\zeta$ is the base point we have $\widetilde{v}^{-1}(*) = S\nu(f)$.

As in the proof of Theorem 3.5, let $F\colon\ D\nu(f)^n \looparrowright N^n$ be an immersion of the normal disc bundle which extends f and is injective on each fibre. We may regard F as a map to the one-point compactification N_c. Then the triple $(D\nu(f), F, \widetilde{v})$ is easily seen to be a spreading of type $T\zeta$ in N_c. We set

$$\Theta([M, f, v]) = [D\nu(f), F, \widetilde{v}].$$

To check that Θ is well-defined, we must show that different choices of F lead to bordant spreadings. But any two such F are regularly homotopic, by Proposition A.7 in the Appendix, and a regular homotopy from F to F' gives a bordism of triples, $(D\nu(f), F, \widetilde{v}) \sim (D\nu(f), F', \widetilde{v})$. We must also check that bordant triples (M_i, f_i, v_i), $i = 0, 1$, lead to bordant spreadings. Again, this is clear.

The map Θ is evidently a monoid homomorphism. To exhibit an inverse

$$\Theta^{-1}\colon\ \mathcal{J}(N_c; T\zeta) \to \mathcal{I}(N; \zeta),$$

we use a modification of the Pontrjagin-Thom construction [36].

Consider a spreading (K, α, β) of type $(T\zeta, *)$ in N_c. By the definition of a spreading,

$$\alpha|_{K-\beta^{-1}(*)}\colon\ K - \beta^{-1}(*) \to N$$

is a local homeomorphism, hence $K - \beta^{-1}(*)$ has the structure of an n-manifold without boundary. The construction of Θ^{-1} will proceed by finding a closed submanifold, corresponding to the inverse image of the base space X in $T\zeta$ under β. Note that we may assume that ζ is a smooth k-vector bundle over a manifold X, whose total space $E(\zeta)$ is therefore a manifold. This is because $\mathcal{I}(N; -)$ and $\mathcal{J}(N_c; -)$ are homotopy functors and X has the homotopy type of a manifold. For the same reason we may also assume (as observed by Thom in [36], Chapter IV) that the map β has the following properties:

(i) $\beta|_{K-\beta^{-1}(*)}\colon K-\beta^{-1}(*) \to E(\zeta)$ is smooth and transverse to the zero section $X \hookrightarrow E(\zeta)$;

(ii) $M := \beta^{-1}(X)$ is a codimension k closed submanifold of $K-\beta^{-1}(*)$, whose normal bundle ν admits a bundle map $v\colon \nu \to \zeta$.

(iii) β is in *standard form*, meaning that β agrees with v on an open tubular neighbourhood $U \approx \nu$ of M, and maps $K-U$ to the base point $* \in T\zeta$.

The restriction of α to the submanifold $M \subseteq K-\beta^{-1}(*)$ is an immersion $f\colon M^{n-k} \looparrowright N^n$ which again has normal bundle ν. Hence we may set

$$\Theta^{-1}([K,\alpha,\beta]) = [M,f,v],$$

which is easily seen to be well-defined.

The composition $\Theta^{-1}\circ\Theta$ is the identity on $\mathcal{I}(N;\zeta)$ by construction. To see that

$$\Theta\circ\Theta^{-1}[K,\alpha,\beta] = [K,\alpha,\beta] \in \mathcal{J}(N_c;T\zeta),$$

we apply Proposition 5.3 to the spreading (K,α,β). With the aid of this inverse to Θ, the naturality statements can be verified; we omit the details. □

5.3 Products

Let (X,x), (Y,y), (X',x') and (Y',y') be objects in $\mathscr{T}_\bullet$. The smash products $(X\wedge X', x\wedge x')$ and $(Y\wedge Y', y\wedge y')$ are also objects in $\mathscr{T}_\bullet$. There are obvious quotient maps $p\colon X\times X' \to X\wedge X'$ and $q\colon Y\times Y'\to Y\wedge Y'$ (so that $x\wedge x' = p(X\vee X')$ and $y\wedge y' = q(Y\vee Y')$).

Definition 5.6. *Given spreadings (K,α,β) of type Y in X and (K',α',β') of type Y' in X', their* external *or* smash product *is the triple*

$$(K\times K', p\circ(\alpha\times\alpha'), q\circ(\beta\times\beta')),$$

which is a spreading of type $Y\wedge Y'$ in $X\wedge X'$.

One may check that this gives a well-defined smash product *pairing on bordism classes*

$$\mathcal{J}(X;Y) \times \mathcal{J}(X';Y') \xrightarrow{\wedge} \mathcal{J}(X \wedge X'; Y \wedge Y').$$

The following Proposition says that the natural isomorphism

$$\Theta\colon \mathcal{I}(-;-) \to \mathcal{J}((-)_c; T(-))$$

maps Cartesian products to smash products. It is a well known fact that for vector bundles ζ and ζ' over base spaces X and X', there is a homeomorphism

$$T\zeta \wedge T\zeta' \approx T(\zeta \times \zeta').$$

Also note that for manifolds N, N' there is a homeomorphism $N_c \wedge N'_c \approx (N \times N')_c$.

Proposition 5.7. *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{I}(N;\zeta) \times \mathcal{I}(N';\zeta') \arrow[r, "\Theta \times \Theta"] \arrow[d, "\times"'] & \mathcal{J}(N_c;T\zeta) \times \mathcal{J}(N'_c;T\zeta') \arrow[d, "\wedge"] \\
\mathcal{I}(N \times N';\zeta \times \zeta') \arrow[r, "\Theta"] & \mathcal{J}((N \times N')_c; T\zeta \wedge T\zeta')
\end{tikzcd}$$

Proof. If we trace an arbitrary element

$$([M,f,v],[M',f',v']) \in \mathcal{I}(N;\zeta) \times \mathcal{I}(N';\zeta')$$

around the diagram, first right, then down, we obtain the class

$$[D\nu(f) \times D\nu(f'), p \circ (F \times F'), q \circ (\widetilde{v} \times \widetilde{v'})] \in \mathcal{J}((N \times N')_c; T(\zeta \times \zeta')),$$

where

$$p\colon\ N_c \times N'_c \to N_c \wedge N'_c \approx (N \times N')_c,$$

$$q\colon\ T\zeta \times T\zeta' \to T\zeta \wedge T\zeta' \approx T(\zeta \times \zeta').$$

Since the map $\widetilde{v} \times \widetilde{v'}$ is induced by the product of bundle maps $v \times v'\colon\ \nu(f) \times \nu(f') \to \zeta \times \zeta'$, it is transverse to the zero section $X \times X' \hookrightarrow T(\zeta \times \zeta')$, with inverse image $M \times M'$. It is therefore easy to see that Θ^{-1} applied to the above class yields the class

$$[M \times M', f \times f', v \times v'] \in \mathcal{I}(N \times N'; \zeta \times \zeta'),$$

as required. ∎

5.4 The Self-intersection Operations Ψ_r

Let ζ be an object in Vect. In Proposition 2.12 we defined, after Koschorke and Sanderson [18], operations

$$\psi_r\colon\ \mathcal{I}(-;\zeta) \to \mathcal{I}(-;\mathcal{S}_r\zeta)$$

for $r \geq 0$ which were natural transformations of set-valued co-functors. As an immediate corollary of Proposition 5.5, we may define similar operations in the bordism of spreadings. Recall from Proposition 4.20 the homeomorphism $T\mathcal{S}_r\zeta \approx D_rT\zeta$. We define $D_0T\zeta = S^0$.

Proposition 5.8. *For each $r \geq 0$ there is a natural transformations of set-valued co-functors*

$$\Psi_r\colon\ \mathcal{J}((-)_c;T\zeta) \to \mathcal{J}((-)_c;D_rT\zeta)$$

defined by commutativity of the following diagram.

$$\begin{tikzcd}
\mathcal{I}(-;\zeta) \arrow[r, "\Theta"] \arrow[d, "\psi_r"'] & \mathcal{J}((-)_c;T\zeta) \arrow[d, "\Psi_r"] \\
\mathcal{I}(-;\mathcal{S}_r\zeta) \arrow[r, "\Theta"] & \mathcal{J}((-)_c;D_rT\zeta)
\end{tikzcd}$$

We now describe the action of Ψ_r on an arbitrary element $[K,\alpha,\beta] \in \mathcal{J}(N_c;T\zeta)$. Since Θ is an isomorphism, we may find $[M,f,v] \in \mathcal{I}(N;\zeta)$ such that $[K,\alpha,\beta] = \Theta[M,f,v] = [D\nu(f),F,\widetilde{v}]$. Here $F\colon\ D\nu(f) \looparrowright N$ is an immersion which extends f. In what follows, the reader is invited to keep in mind the spreading of the figure-eight immersion, depicted in Figure 3.1.

As in Section 2.4, let $\mathcal{F}(D\nu(f);r)$ denote the configuration space of ordered r-tuples of distinct points in $D\nu(f)$. We can form the pull-back square

$$\begin{tikzcd}
\overline{\Delta}_r(F) \arrow[r] \arrow[d, "\overline{\Psi}_r(F)"'] & \mathcal{F}(D\nu(f);r) \arrow[d, "F^{(r)}|"] \\
N \arrow[r, "\triangle"] & N^{(r)}
\end{tikzcd}$$

Note that the space

$$\overline{\Delta}_r(F) = \{(k_1, \ldots, k_r) \in \mathcal{F}(D\nu(f); r) \mid F(k_1) = \ldots = F(k_r)\}$$

is compact since F is an immersion. The group S_r acts on $\overline{\Delta}_r(F)$ freely with quotient space $\Delta_r(F)$, and acts trivially on N. The closed, proper map $\overline{\Psi}_r(F)$ is equivariant with respect to these actions, so induces a closed, proper map $\Psi_r(F)\colon \Delta_r(F) \to N$ (in the case of Figure 3.1, the image of this map for $r = 2$ is the central square).

Now recall from Lemma 2.11 that to define the $\mathcal{S}_r\zeta$-structure on $\nu(\psi_r(f))$ we fixed an embedding $\lambda\colon \nu(f) \hookrightarrow \mathbb{R}^\infty$. This restricts to give an embedding of the unit normal disc bundle, which we also denote by λ. We may define a map $\mathcal{S}_r(\widetilde{v})\colon \Delta_r(F) \to D_rT\zeta$ by setting

$$\mathcal{S}_r(\widetilde{v})([k_1, \ldots, k_r]) = [\lambda(k_1), \ldots, \lambda(k_r), \widetilde{v}(k_1), \ldots, \widetilde{v}(k_r)].$$

Proposition 5.9. *The triple*

$$(\Delta_r(F), \Psi_r(F), \mathcal{S}_r(\widetilde{v}))$$

is a spreading of type $D_rT\zeta$ in N_c, which represents $\Psi_r[K, \alpha, \beta] \in \mathcal{J}(N_c, D_rT\zeta)$.

Proof. To verify that the above triple is a spreading, we only need check that $\Psi_r(F)$ is a local homeomorphism away from $\mathcal{S}_r(\widetilde{v})^{-1}(*)$. This is true since F is a local homeomorphism away from the boundary $S\nu(f)$. To see that $[\Delta_r(F), \Psi_r(F), \mathcal{S}_r(\widetilde{v})] = \Psi_r[K, \alpha, \beta]$, we must apply Θ^{-1} to this class and obtain the class $\psi_r[M, f, v] = [\Delta_r(f), \psi_r(f), \mathcal{S}_r(v)]$.

The map $\mathcal{S}_r(\widetilde{v})\colon \Delta_r(F) \to D_rT\zeta$ is already transverse to the zero section

$$\mathcal{F}(\mathbb{R}^\infty; r) \times_{S_r} X^{(r)} \hookrightarrow \mathcal{F}(\mathbb{R}^\infty; r) \times_{S_r} E(\zeta)^{(r)}$$

by virtue of being constructed from a product of bundle maps $v\colon \nu(f) \to E(\zeta)$. We then see that

$$\mathcal{S}_r(\widetilde{v})^{-1}(\mathcal{F}(\mathbb{R}^\infty; r) \times_{S_r} X^{(r)}) = \Delta_r(f) \hookrightarrow \Delta_r(F).$$

Since the immersion $\psi_r(f)$ factorises as

$$\Delta_r(f) \hookrightarrow \Delta_r(F) \xrightarrow{\Psi_r(F)} N,$$

its normal bundle is isomorphic to the normal bundle of $\Delta_r(f)$ in $\Delta_r(F)$. Thus

$$\Theta^{-1}[\Delta_r(F), \Psi_r(F), \mathcal{S}_r(\widetilde{v})] = [\Delta_r(f), \psi_r(f), \mathcal{S}_r(v)]$$

as claimed. $\square$

Chapter 6

Relations Between Operations

For any $l \in \mathbb{N}$, the *standard double cover* of $\mathbb{R}P^l$ is the map $c_l\colon\ S^l \to \mathbb{R}P^l$ which sends a unit vector in $\mathbb{R}^{l+1}$ to the line it spans. This covering map is a local diffeomorphism, and hence is a codimension zero immersion of closed manifolds. The normal bundle of c_l is the zero dimensional bundle $\mathbf{1}\colon\ S^l \to S^l$. Thus c_l represents a bordism class $[c_l] \in \mathcal{I}(\mathbb{R}P^l;\star)$, where $\star$ denotes the point bundle over a point. Let $f\colon\ M^{n-k} \looparrowright N^n$ be a self-transverse immersion of closed manifolds. Assume that f has a ζ-structure, so represents a class $[f] \in \mathcal{I}(N;\zeta)$. The Cartesian product immersion $c_l \times f\colon\ S^l \times M \to \mathbb{R}P^l \times N$ therefore represents a class

$$[c_l \times f] \in \mathcal{I}(\mathbb{R}P^l \times N; \star \times \zeta) \cong \mathcal{I}(\mathbb{R}P^l \times N;\zeta).$$

The main result of this Chapter is an analysis of the class

$$\psi_2[c_l \times f] \in \mathcal{I}(\mathbb{R}P^l \times N; \mathcal{S}_2\zeta).$$

Since $c_l \times f$ is *not self-transverse*, we cannot immediately describe the double-point immersion $\psi_2(c_l \times f)\colon\ \Delta_2(c_l \times f) \looparrowright \mathbb{R}P^l \times N$. However, Proposition 5.8 tells us that we can study its bordism class by applying the double-point operation Ψ_2 to a spreading of $c_l \times f$, and Proposition 5.9 tells us how to do this. We find that the resulting class in $\mathcal{J}((\mathbb{R}P^l \times N)_c; D_2T\zeta)$ splits as a disjoint union of spreadings, which under the isomorphism of Proposition 5.5 correspond to a disjoint union of familiar immersions built from f.

6.1 The Main Theorem

We first recall some notation from Section 4.4. Let $f\colon M \to N$ be a map of manifolds. For each $l \in \mathbb{N}$ we defined in Proposition 4.28 a map

$$\lambda_l(f) := \mathbf{1} \times_{\mathbb{Z}_2} f^{(2)}\colon S^l \times_{\mathbb{Z}_2} M^{(2)} \to S^l \times_{\mathbb{Z}_2} N^{(2)},$$

which may be thought of as an *extended power* of f. Here $\mathbb{Z}_2$ acts on S^l antipodally and on the product $f \times f$ by switching the factors. We also defined in Corollary 4.29 a map $\xi_l(f)$ of the same codimension, by pulling back along the extended diagonal $\triangle_2^l\colon \mathbb{R}P^l \times N \to S^l \times_{\mathbb{Z}_2} N^{(2)}$ a transverse representative $\lambda_l(f)'$ of the homotopy class of $\lambda_l(f)$:

$$\begin{tikzcd}
\Sigma(f) \arrow[r] \arrow[d, "\xi_l(f)"'] & S^l \times_{\mathbb{Z}_2} M^{(2)} \arrow[d, "\lambda_l(f)'"] \\
\mathbb{R}P^l \times N \arrow[r, "\triangle_2^l"] & S^l \times_{\mathbb{Z}_2} N^{(2)}.
\end{tikzcd}$$

If f is a Γ-oriented map of codimension k, the codimension $2k$ maps $\lambda_l(f)$ and $\xi_l(f)$ inherit canonical Γ-orientations from that of f, at least when the Thom spectrum $\mathsf{M}\Gamma$ has self-intersections in codimension k. If $f\colon M \looparrowright N$ is an immersion of closed manifolds, the maps $\lambda_l(f)$ and $\xi_l(f)$ are also immersions of closed manifolds.

Lemma 6.1. *Let $f\colon M \looparrowright N$ be an immersion with ζ-structure $v\colon \nu(f) \to \zeta$. Then the immersions $\lambda_l(f)$ and $\xi_l(f)$ have $\mathcal{S}_2\zeta$-structures.*

Proof. Since $\xi_l(f)$ is defined as a pullback of $\lambda_l(f)$, it suffices by Proposition 2.6 to show that $\lambda_l(f)$ has an $\mathcal{S}_2\zeta$-structure. If $p\colon \nu(f) \to M$ is the bundle projection of $\nu(f)$, then the normal bundle $\nu(\lambda_l(f))$ has projection

$$\mathbf{1} \times_{\mathbb{Z}_2} p^{(2)}\colon S^l \times_{\mathbb{Z}_2} \nu(f)^{(2)} \to S^l \times_{\mathbb{Z}_2} M^{(2)}.$$

We fix, for the remainder of this Chapter, an embedding $\rho'\colon D^l \times \nu(f) \hookrightarrow \mathbb{R}^\infty$ which restricts to an embedding $\rho\colon S^l \times \nu(f) \hookrightarrow \mathbb{R}^\infty$. Then define a bundle map $\rho(v)\colon \nu(\lambda_l(f)) \to \mathcal{S}_2\zeta$ by setting

$$\rho(v)[w, x_1, x_1] = [\rho(w, x_1), \rho(-w, x_2), v(x_1), v(x_2)],$$

where $w \in S^l$ and $x_1, x_2 \in \nu(f)$. $\square$

We write $[\xi_l(f)]$ for the class

$$(\triangle_2^l)^*[S^l \times_{\mathbb{Z}_2} M^{(2)}, \lambda_l(f), \rho(v)] \in \mathcal{I}(\mathbb{R}P^l \times N; \mathcal{S}_2\zeta).$$

Theorem 6.2.

$$\psi_2[c_l \times f] = [\xi_l(f)] + [c_l] \times \psi_2[f] \in \mathcal{I}(\mathbb{R}P^l \times N; \mathcal{S}_2\zeta).$$

Proof. We apply the natural isomorphism

$$\Theta\colon \mathcal{I}(\mathbb{R}P^l \times N; \mathcal{S}_2\zeta) \xrightarrow{\simeq} \mathcal{J}((\mathbb{R}P^l \times N)_c; D_2T\zeta)$$

and Propositions 5.7 and 5.8 to reduce the statement of the Theorem to the equivalent statement

$$\Psi_2\Theta[c_l \times f] = (\triangle_2^l)_c^*\Theta[\lambda_l(f)] + \Theta[c_l] \wedge \Psi_2\Theta[f] \in \mathcal{J}((\mathbb{R}P^l \times N)_c; D_2T\zeta).$$

We first describe the class $\Theta[c_l \times f]$. The normal bundle of $c_l \times f$ is $S^l \times \nu(f)$ with ζ-structure $v_0\colon\ S^l \times \nu(f) \to \zeta$ given by $v_0(w, x) = v(x)$. Hence an extension of $c_l \times f$ to the unit normal disc bundle is given by

$$c_l \times F\colon\ S^l \times D\nu(f) \looparrowright \mathbb{R}P^l \times N,$$

where $F\colon\ D\nu(f) \looparrowright N$ is an immersion extending f. Thus we have

$$\Theta[c_l \times f] = [S^l \times D\nu(f), c_l \times F, \widetilde{v_0}] \in \mathcal{J}((\mathbb{R}P^l \times N)_c; T\zeta).$$

We now apply the operation $\Psi_2\colon\ \mathcal{J}((\mathbb{R}P^l \times N)_c; T\zeta) \to \mathcal{J}((\mathbb{R}P^l \times N)_c; D_2T\zeta)$. By Proposition 5.9 we have

$$\Psi_2\Theta[c_l \times f] = [\Delta_2(c_l \times F), \Psi_2(c_l \times F), \mathcal{S}_2(\widetilde{v_0})],$$

where the map $\mathcal{S}_2(\widetilde{v_0})$ is given by

$$\mathcal{S}_2(\widetilde{v_0})[(w, x_1), (w, x_2)] = [\rho(w, x_1), \rho(w, x_2), \widetilde{v}(x_1), \widetilde{v}(x_2)],$$

with ρ as in Lemma 6.1. Now $\Delta_2(c_l \times F)$ is the space

$$\{[(w_1, x_1), (w_2, x_2)] \in \mathcal{F}(S^l \times D\nu(f); 2)/\mathbb{Z}_2 \mid (c_l \times F)(w_1, x_1) = (c_l \times F)(w_2, x_2)\}.$$

This space splits as a disjoint union $\Delta_2(c_l \times F) = \Sigma_1 \sqcup \Sigma_2$, where

$$\Sigma_1 = \{[(w, x_1), (-w, x_2)] \in \Delta_2(c_l \times F)\},$$

$$\Sigma_2 = \{[(w, x_1), (w, x_2)] \in \Delta_2(c_l \times F) \mid x_1 \neq x_2\}.$$

Hence $\Psi_2\Theta[c_l \times f]$ splits as a sum of bordism classes

$$\Psi_2\Theta[c_l \times f] = [\Sigma_1, \Psi_2(c_l \times F)|_{\Sigma_1}, \mathcal{S}_2(\widetilde{v_0})|_{\Sigma_1}] + [\Sigma_2, \Psi_2(c_l \times F)|_{\Sigma_2}, \mathcal{S}_2(\widetilde{v_0})|_{\Sigma_2}].$$

We claim that the triple $(\Sigma_1, \Psi_2(c_l \times F)|_{\Sigma_1}, \mathcal{S}_2(\widetilde{v_0})|_{\Sigma_1})$ represents $(\triangle_2^l)_c^*\Theta[\lambda_l(f)] \in \mathcal{J}(\mathbb{R}P^l \times N; D_2T\zeta)$. From Lemma 6.1 we may write

$$\Theta[\lambda_l(f)] = [S^l \times_{\mathbb{Z}_2} D\nu(f)^{(2)}, \mathbf{1} \times_{\mathbb{Z}_2} F^{(2)}, \widetilde{\rho(v)}] \in \mathcal{J}((S^l \times_{\mathbb{Z}_2} N^{(2)})_c; D_2T\zeta),$$

since $\mathbf{1} \times_{\mathbb{Z}_2} F^{(2)}\colon\ S^l \times_{\mathbb{Z}_2} D\nu(f)^{(2)} \looparrowright S^l \times_{\mathbb{Z}_2} N^{(2)}$ is a spreading of $\lambda_l(f)$. To obtain $(\triangle_2^l)_c^*\Theta[\lambda_l(f)]$ we must form the pullback

$$\begin{tikzcd} \Sigma(F) \arrow[r, "i"] \arrow[d, "\xi_l(F)"'] & S^l \times_{\mathbb{Z}_2} D\nu(f)^{(2)} \arrow[d, "1 \times_{\mathbb{Z}_2} F^{(2)}"] \\ \mathbb{R}P^l \times N \arrow[r, "\triangle_2^l"] & S^l \times_{\mathbb{Z}_2} N^{(2)}. \end{tikzcd}$$

Here i is the inclusion

$$\Sigma(F) = \{[w, x_1, x_2] \in S^l \times_{\mathbb{Z}_2} D\nu(f)^{(2)} \mid F(x_1) = F(x_2)\} \xhookrightarrow{i} S^l \times_{\mathbb{Z}_2} D\nu(f)^{(2)}$$

and $\xi_l(F)[w, x_1, x_2] = ([w], F(x_1))$. Hence

$$(\triangle_2^l)_c^*\Theta[\lambda_l(f)] = [\Sigma(F), \xi_l(F), \widetilde{\rho(v)} \circ i] \in \mathcal{J}((\mathbb{R}P^l \times N)_c; D_2T\zeta).$$

Note that there is a homeomorphism $h_1\colon\ \Sigma_1 \to \Sigma(F)$ given by $h_1[(w, x_1), (-w, x_2)] = [w, x_1, x_2]$, and that $\xi_l(F) \circ h_1 = \Psi_2(c_l \times F)|_{\Sigma_1}$ and $\widetilde{\rho(v)} \circ i \circ h_1 = \mathcal{S}_2(\widetilde{v_0})|_{\Sigma_1}$. This proves the claim.

It remains only to show that the triple $(\Sigma_2, \Psi_2(c_l \times F)|_{\Sigma_2}, \mathcal{S}_2(\widetilde{v_0})|_{\Sigma_2})$ represents the product spreading $\Theta[c_l] \wedge \Psi_2\Theta[f]$. We know that

$$\Theta[c_l] = [S^l, c_l, \pi] \in \mathcal{J}((\mathbb{R}P^l)_c; S^0) \text{ and}$$

$$\Psi_2\Theta[f] = [\Delta_2(F), \Psi_2(F), \mathcal{S}_2(\widetilde{v})] \in \mathcal{J}(N_c; D_2T\zeta),$$

where $\pi\colon S^l \to S^0$ is projection onto the non-base point. By Definition 5.6 of the smash product,

$$\begin{aligned}\Theta[c_l] \wedge \Psi_2\Theta[f] &= [S^l \times \Delta_2(F), p \circ (c_l \times \Psi_2(F)), q \circ (\pi \times \mathcal{S}_2(\widetilde{v}))] \\ &= [S^l \times \Delta_2(F), p \circ (c_l \times \Psi_2(F)), \eta]\end{aligned}$$

where $p\colon \mathbb{R}P^l_c \times N_c \to (\mathbb{R}P^l \times N)_c$ and $q\colon S^0 \times D_2T\zeta \to S^0 \wedge D_2T\zeta = D_2T\zeta$ are the obvious maps. Here $\eta\colon S^l \times \Delta_2(F) \to D_2T\zeta$ is the map given by $\eta(w, [x_1, x_2]) = [\rho(\underline{1}, x_1), \rho(\underline{1}, x_2), \widetilde{v}(x_1), \widetilde{v}(x_2)]$, where $\underline{1} = (1, 0, \ldots, 0) \in S^l$. This equality comes from choosing the embedding $\rho(\underline{1}, -)\colon D\nu(f) \hookrightarrow \mathbb{R}^\infty$ in the definition of $\mathcal{S}_2(\widetilde{v})$.

Finally note that there is a homeomorphism $h_2\colon \Sigma_2 \to S^l \times \Delta_2(F)$ given by $h_2[(w, x_1), (w, x_2)] = (w, [x_1, x_2])$, and that $p \circ (c_l \times \Psi_2(F)) \circ h_2 = \Psi_2(c_l \times F)|_{\Sigma_2}$. Also the composition $\eta \circ h_2$ is given by

$$\eta \circ h_2[(w, x_1), (w, x_2)] = [\rho(\underline{1}, x_1), \rho(\underline{1}, x_2), \widetilde{v}(x_1), \widetilde{v}(x_2)],$$

which is homotopic to $\mathcal{S}_2(\widetilde{v_0})|_{\Sigma_2}$ since ρ extends to the embedding $\rho'\colon D^l \times D\nu(f) \hookrightarrow \mathbb{R}^\infty$. This proves the claim, and the Theorem. $\square$

6.2 Corollaries of Theorem 6.2

Here we derive some corollaries of Theorem 6.2 in the unoriented case, which relate the self-intersection operations to the internal Steenrod operations of Section 4.4. The word 'unoriented' here means that the only structure we impose on our immersion $f\colon M^{n-k} \looparrowright N^n$ is the classifying bundle map $v\colon \nu(f) \to \gamma_k$ to the universal $O(k)$-bundle. On applying the monoid homomorphism induced by the classifying map $\mathcal{S}_2\gamma_k \to \gamma_{2k}$ of Examples 4.14 **(1)**, we find that

$$\psi_2[c_l \times f] = [\xi_l(f)] + [c_l] \times \psi_2[f] \in \mathcal{I}(\mathbb{R}P^l \times N; \gamma_{2k}).$$

We then apply the natural transformation $T_{2k}\colon \mathcal{I}(-;\gamma_{2k}) \to \mathsf{MO}^{2k}(-)$ of Proposition 4.9, which regards an immersion of a closed manifold as a proper map. We will abuse notation slightly and continue to write $[f]$ for the cobordism class in $\mathsf{MO}^k(N)$ represented by an immersion $f\colon M^{n-k} \looparrowright N^n$. Since every proper map of manifolds has a unique O-orientation, we find that $[\xi_l(f)] = \ell_l^* \mathcal{P}^k[f]$ by Corollary 4.29, where $\mathcal{P}^k\colon \mathsf{MO}^k(N) \to \mathsf{MO}^{2k}(\mathbb{R}P^\infty \times N)$ is the internal Steenrod operation of type $(\mathbb{Z}_2, 1)$ in MO^*, and $\ell_l\colon \mathbb{R}P^l \times N \hookrightarrow \mathbb{R}P^\infty \times N$ is the standard inclusion.

Corollary 6.3. *For any $l \in \mathbb{N}$,*

$$\psi_2[c_l \times f] = \ell_l^* \mathcal{P}^k[f] + [c_l] \times \psi_2[f] \in \mathsf{MO}^{2k}(\mathbb{R}P^l \times N).$$

Rather than have one result for each $l \in \mathbb{N}$, we should now collect them together by passing to the limit. $\mathbb{R}P^\infty \times N$ is an infinite dimensional manifold which is filtered by finite dimensional sub-manifolds $\mathbb{R}P^l \times N$, forming a direct system of embeddings

$$\ldots \hookrightarrow \mathbb{R}P^l \times N \overset{j_l \times \mathbf{1}}{\hookrightarrow} \mathbb{R}P^{l+1} \times N \hookrightarrow \ldots \hookrightarrow \mathbb{R}P^\infty \times N,$$

where $j_l\colon \mathbb{R}P^l \hookrightarrow \mathbb{R}P^{l+1}$ is the usual inclusion. Hence we get an inverse system of Abelian groups and homomorphisms

$$\ldots \leftarrow \mathsf{MO}^{2k}(\mathbb{R}P^l \times N) \overset{(j_l \times \mathbf{1})^*}{\longleftarrow} \mathsf{MO}^{2k}(\mathbb{R}P^{l+1} \times N) \leftarrow \ldots$$

The following two results imply that $\mathsf{MO}^{2k}(\mathbb{R}P^\infty \times N) \cong \lim_l \mathsf{MO}^{2k}(\mathbb{R}P^l \times N)$.

Proposition 6.4. *Let Y be a CW-complex filtered by finite sub-complexes Y_n, $n \in \mathbb{N}$. If E is a spectrum such that $\pi_i(\mathsf{E})$ is a finite Abelian group for all i, then the map*

$$\mathsf{E}^k(Y) \to \varprojlim \mathsf{E}^k(Y_n)$$

is an isomorphism for all k.

Proof. See Y.B. Rudyak [28], Corollary III.4.17. $\square$

Theorem 6.5. *The coefficient groups of unoriented bordism form a polynomial ring over $\mathbb{Z}_2$,*

$$\pi_*(\mathsf{MO}) \cong \mathbb{Z}_2[x_2, x_4, x_5, x_6, x_8, \ldots]$$

on generators x_i of dimension i, one for each $i \geq 1$ not of the form $2^s - 1$.

Proof. See Thom [36], Theorem IV.12. $\square$

Since $j_l^*[c_{l+1}] = [c_l] \in \mathsf{MO}^0(\mathbb{R}P^l)$, we may easily check that

$$(j_l \times \mathbf{1})^* \psi_2[c_{l+1} \times f] = \psi_2[c_l \times f] \in \mathsf{MO}^{2k}(\mathbb{R}P^l \times N),$$

$$(j_l \times \mathbf{1})^*([c_{l+1}] \times \psi_2[f]) = [c_l] \times \psi_2[f] \in \mathsf{MO}^{2k}(\mathbb{R}P^l \times N),$$

$$\text{and } (j_l \times \mathbf{1})^* \ell_{l+1}^* \mathcal{P}^k[f] = \ell_l^* \mathcal{P}^k[f] \in \mathsf{MO}^{2k}(\mathbb{R}P^l \times N).$$

Therefore we may define elements

$$\psi_2[c \times f] = \lim_{\leftarrow} \psi_2[c_l \times f] \in \mathsf{MO}^{2k}(\mathbb{R}P^\infty \times N),$$

$$[c] \times \psi_2[f] = \lim_{\leftarrow} [c_l] \times \psi_2[f] \in \mathsf{MO}^{2k}(\mathbb{R}P^\infty \times N),$$

$$\text{and } \mathcal{P}^k[f] = \lim_{\leftarrow} \ell_l^* \mathcal{P}^k[f] \in \mathsf{MO}^{2k}(\mathbb{R}P^\infty \times N).$$

Corollary 6.6.

$$\psi_2[c \times f] = \mathcal{P}^k[f] + [c] \times \psi_2[f] \in \mathsf{MO}^{2k}(\mathbb{R}P^\infty \times N).$$

We may obtain a similar result in $\mathbb{Z}_2$-cohomology, where the internal Steenrod operation of type $(\mathbb{Z}_2, 1)$ will also be denoted

$$\mathcal{P}^k \colon H^k(N; \mathbb{Z}_2) \to H^{2k}(\mathbb{R}P^\infty \times N; \mathbb{Z}_2).$$

The next result says that the universal Thom class $t \colon \mathsf{MO} \to \mathsf{H}\mathbb{Z}_2$ induces a natural transformation of cohomology theories which preserves the corresponding external Steenrod operations.

Proposition 6.7. *The following diagram commutes for all pointed spaces X and all $k \in \mathbb{Z}$.*

$$\begin{tikzcd}
\widetilde{\mathsf{MO}}^k(X) \arrow[r, "\bar{t}"] \arrow[d, "P^k"'] & \widetilde{H}^k(X; \mathbb{Z}_2) \arrow[d, "P^k"] \\
\widetilde{\mathsf{MO}}^{2k}(D_2X) \arrow[r, "\bar{t}"] & \widetilde{H}^{2k}(D_2X; \mathbb{Z}_2)
\end{tikzcd}$$

Proof. By naturality it suffices to prove that

$$\bar{t} \circ P^k(u) = P^k \circ \bar{t}(u) \in \widetilde{H}^{2k}(D_2 MO(k); \mathbb{Z}_2),$$

where $u \in \widetilde{\mathsf{MO}}^k(MO(k))$ is the cobordism Thom class of γ_k, represented by the identity map $\mathbf{1}\colon\ MO(k) \to MO(k)$.

If ζ^n is an n-dimensional vector bundle with cobordism Thom class $v \in \widetilde{\mathsf{MO}}^n(T\zeta)$, then $\bar{t}(v) \in \widetilde{H}^n(T\zeta; \mathbb{Z}_2)$ is a cohomology Thom class. Hence by Proposition 4.24, both sides of the above equality are the unique Thom class in $\widetilde{H}^{2k}(D_2 MO(k); \mathbb{Z}_2)$ of the extended power bundle $\mathcal{S}_2\gamma_k$. □

Since each internal operation is defined by setting $\mathcal{P}^k = \triangle_2^* \circ P^k$, we immediately deduce that the following diagram commutes.

$$\begin{tikzcd}
\mathsf{MO}^k(N) \arrow[r, "\bar{t}"] \arrow[d, "\mathcal{P}^k"] & H^k(N; \mathbb{Z}_2) \arrow[d, "\mathcal{P}^k"] \\
\mathsf{MO}^{2k}(\mathbb{R}P^\infty \times N) \arrow[r, "\bar{t}"] & H^{2k}(\mathbb{R}P^\infty \times N; \mathbb{Z}_2).
\end{tikzcd}$$

We continue to write $[f]$ for the cohomology class $\bar{t}[f] \in H^k(N; \mathbb{Z}_2)$ represented by an immersion $f\colon\ M^{n-k} \looparrowright N^n$.

Corollary 6.8.

$$\psi_2[c \times f] = \mathcal{P}^k[f] + [c] \times \psi_2[f] \in H^{2k}(\mathbb{R}P^\infty \times N; \mathbb{Z}_2).$$

Note that for $l \in \mathbb{N}$ the map of closed manifolds $c_l\colon\ S^l \to \mathbb{R}P^l$ has degree ± 2 when l is odd and 0 when l is even. Thus $(c_l)_*[S^l] = 0 \in H_l(\mathbb{R}P^l; \mathbb{Z}_2)$ for all l, so the class $[c] \in H^0(\mathbb{R}P^\infty; \mathbb{Z}_2)$, which is defined as the limit over l of the Poincaré duals of these elements, must be zero.

Corollary 6.9. *Let $\alpha \in H^k(N; \mathbb{Z}_2)$ be represented by an immersion $f\colon\ M^{n-k} \looparrowright N^n$ (meaning $\mathcal{D}_N(\alpha) = f_*[M] \in H_{n-k}(N; \mathbb{Z}_2)$). Then $\mathcal{P}^k(\alpha)$ is represented by the double point immersion $\psi_2(c \times f)$, where $c\colon\ S^\infty \to \mathbb{R}P^\infty$ is the universal principal $\mathbb{Z}_2$-bundle.*

6.3 Possible Extensions

In this section we discuss several possible avenues of future research based on Theorem 6.2.

The Complex and Symplectic Cases. The Thom spectra MU and MSp of complex and symplectic cobordism have self-intersections in codimensions divisible by 2 and 4 respectively. Hence each of the theories MU^* and MSp^* carries an internal Steenrod operation of type $\mathbb{Z}_2$, and the reader may wonder why we did not obtain the analogue of Corollary 6.6 in these theories.

To illustrate the difficulty in the complex case, let $f\colon M^{n-2k} \looparrowright N^n$ be an immersion with complex structure $v\colon \nu(f) \to \gamma_k^U$. As in Proposition 4.9 we may regard f as a proper complex oriented map, by factorising it as

$$M \overset{(f,\tilde{f})}{\hookrightarrow} N \times \mathbb{R}^l \overset{pr}{\to} N$$

where $\tilde{f}\colon M \hookrightarrow \mathbb{R}^l$ is some embedding. This complex orientation of f induces a complex orientation of the map $\xi_l(f)$ in a canonical way (see Proposition 4.28 and Corollary 4.29) and with this orientation $\xi_l(f)$ represents the class $\ell_l^* \mathcal{P}^{2k}[f] \in \mathsf{MU}^{4k}(\mathbb{R}P^l \times N)$.

The map $\xi_l(f)$ may also be regarded as an immersion with $\mathcal{S}_2\gamma_k^U$-structure, as in Lemma 6.1. With this structure it is of course true that

$$\psi_2[c_l \times f] = [\xi_l(f)] + [c_l] \times \psi_2[f] \in \mathcal{I}(\mathbb{R}P^l \times N; \gamma_{2k}^U).$$

However when we apply the natural transformation $T_{4k}\colon \mathcal{I}(-;\gamma_{2k}^U) \to \mathsf{MU}^{4k}(-)$ it is not immediately obvious that $T_{4k}[\xi_l(f)] = \ell_l^* \mathcal{P}^{2k}[f]$. Although both cobordism classes are represented by the same map $\xi_l(f)$, we have given this map two different complex orientations. It may be the case that these two orientations are cobordant; this is work in progress.

$\mathbb{Z}_p$ **Operations.** Having obtained a relationship between the double-point operation ψ_2 and the Steenrod operation in $\mathsf{H}\mathbb{Z}_2$, one may ask if a similar approach will yield relations between ψ_p and the $\mathbb{Z}_p$ operations in $\mathsf{H}\mathbb{Z}_p$, where p is an odd prime.

We may regard $\mathbb{Z}_p$ as the set $\{1, \omega, \ldots, \omega^{p-1}\} \subseteq \mathbb{C}$ under multiplication, where ω is a p-th root of unity. The universal $\mathbb{Z}_p$-bundle is a map $d\colon S^\infty \to L_p$, where L_p is an infinite lens space. This may be viewed as a limit of immersions of closed manifolds $d_l\colon S^{2l-1} \to L_p^{2l-1}$, which map a point $w \in S^{2l-1} \subseteq \mathbb{C}^l$ to its orbit under the $\mathbb{Z}_p$ action on S^{2l-1} given by left multiplication.

Given an immersion $f\colon M \looparrowright N$ with ζ-structure there are immersions $\lambda_l^p(f)$ and $\xi_l^p(f)$, defined entirely analogously to the $\mathbb{Z}_2$ case, and these can be shown to have $\mathcal{S}_p\zeta$-structures. When we carry out the analysis of the element $\psi_p[d_l \times f] \in \mathcal{I}(L_p^{2l-1} \times N; \mathcal{S}_p\zeta)$ in the same way as was done in Section 6.1 for the $\mathbb{Z}_2$ case, we find that

$$\psi_p[d_l \times f] = [\xi_l^p(f)] + [d_l] \times \psi_p[f] + \text{other more complicated terms},$$

with the number of terms on the right hand side being equal to the number of partitions of p. It is probably unlikely that this will yield any sensible result, because of the discrepancy between $\mathbb{Z}_p$ and S_p when $p > 2$.

Self-intersections of Singular Maps. Corollary 6.9 gives an alternative geometric construction of $\mathcal{P}^k(\alpha) \in H^{2k}(\mathbb{R}P^\infty \times N; \mathbb{Z}_2)$ when $\alpha \in H^k(N; \mathbb{Z}_2)$ is represented by an immersion. However, it is not known that every $\mathbb{Z}_2$-cohomology class can be represented in this way. René Thom, in answer to a problem posed by Steenrod, proved that the map

$$\underline{t}\colon \mathsf{MO}_{n-k}(N) \to H_{n-k}(N; \mathbb{Z}_2)$$

is surjective, thus showing that every class $\alpha \in H^k(N; \mathbb{Z}_2)$ can be represented by a map of manifolds $f\colon M^{n-k} \to N^n$ with M closed [36]. The question of which bordism classes in $\mathsf{MO}_{n-k}(N)$ contain immersions remains only partially resolved (see [21] for a recent example), and so the map f representing α may have singularities in general.

Thus the following question arises: can we define the r-fold self-intersection map $\psi_r(f)\colon \Delta_r(f) \to N$ of a map $f\colon M \to N$ with singularities? We may give the same definition of self-transversality as we did in the immersion case in Definition 2.9, and self-transverse maps are generic in the sense that they form a dense subspace of the space of all smooth maps ([13], Proposition III.3.2). If we try to define $\psi_r(f)$ in the same way as we did for f an immersion, though, we quickly run into trouble as the manifold $\Delta_r(f)$ is *not closed.*

We are led to consider the following problem. Given a self-transverse singular map $f\colon M^{n-k} \to N^n$, can we define a closed manifold $\Delta_r(f)$ and a singular map $\psi_r(f)\colon \Delta_r(f) \to N$ with image $\{n \in N \mid |f^{-1}(n)| \geq r\}$, such that we recover the classical definition when f is an immersion?

R. Rimányi and A. Szűcs have recently constructed a classifying space for bordism of maps with prescribed singularities [26]. Self-intersection operations in this bordism theory (if they exist) should be induced by combinatorially defined maps between such classifying spaces (compare Section 7.3).

Chapter 7

Homotopy Classification

The beauty and utility of any bordism theory stems from the fact that it may be translated into homotopy theory by means of a Pontrjagin-Thom construction [36], and hence studied (and often computed) using methods of Algebraic Topology. The bordism of immersions is no exception. In 1966 R. Wells proved that for $k > 0$,

$$\begin{aligned} \mathcal{I}(\mathbb{R}^{n+k}; \gamma_k) &\cong \lim_{l\to\infty} \pi_{n+k+l}(\Sigma^l MO(k)) \\ &=: \pi^S_{n+k}(MO(k)), \end{aligned}$$

thus exhibiting bordism of immersions as the stable homotopy of Thom spaces [39]. His result was generalised in the 1970's, both by Koschorke and Sanderson [18], and independently by Vogel [38]. They showed that for every $N^n \in \mathcal{D}_0$ and $\zeta \in \mathsf{Vect}$,

$$\mathcal{I}(N; \zeta) \cong [N_c, C(\mathbb{R}^\infty, T\zeta)],$$

where $C(\mathbb{R}^\infty, T\zeta)$ is a combinatorially defined model of the weak homotopy type of $QT\zeta = \lim_{l\to\infty} \Omega^l \Sigma^l T\zeta$, built from configuration spaces, and N_c is the one-point compactification of N. We will give Vogel's proof, which essentially elucidates Koschorke and Sanderson's proof using the language of spreadings, in Section 7.2. In Section 7.3 we remark that, under this isomorphism, the self-intersection operations are induced by certain James-Hopf maps

$$h^r\colon\ C(\mathbb{R}^\infty, T\zeta) \to C(\mathbb{R}^\infty, D_r T\zeta).$$

7.1 Configuration space models

In this section we give a brief introduction to configuration space models. These are functors on pointed spaces, originally devised to give usable combinatorial models of iterated loop-suspension functors.

We begin by defining, following F.R. Cohen, J.P. May and L.R. Taylor [8], a functor

$$C(Z,-)\colon\ \mathscr{T}_\bullet \to \mathscr{T}_\bullet$$

for any topological space Z. Given an integer $r \geq 1$ let $\mathcal{F}(Z;r)$ denote the configuration space

$$\{(z_1,\ldots,z_r) \in Z^{(r)} \mid i \neq j \Rightarrow z_i \neq z_j\},$$

with the subspace topology from $Z^{(r)}$. For any $(X,*) \in \mathscr{T}_\bullet$, the symmetric group S_r acts on the right of $\mathcal{F}(Z;r)$ and on the left of $X^{(r)}$, in each case by permutation of factors. Hence for every $r \geq 1$ we have a space

$$\mathcal{F}(Z;r) \times_{S_r} X^{(r)},$$

whose points may be thought of as sets of r points in Z, each with a label from X. We will write points of this space in the form

$$[(z_1,x_1),(z_2,x_2),\ldots,(z_r,x_r)],$$

where each $z_i \in Z$ and each $x_i \in X$.

Definition 7.1. *Let Z be any topological space. There is a functor*

$$C(Z,-)\colon\ \mathscr{T}_\bullet \to \mathscr{T}_\bullet,$$

whose value on an object $(X,*) \in \mathscr{T}_\bullet$ is the pointed space

$$C(Z,X) = \left(\coprod_{r\geq 1} \mathcal{F}(Z;r) \times_{S_r} X^{(r)}\right) / \sim,$$

where $\sim$ is the equivalence relation generated by

$$[(z_1,x_1),\ldots,(z_r,x_r)] \sim [(z_1,x_1),\ldots,(z_r,x_r),(z_{r+1},*)]$$

$$\text{and } [z, *] \sim [z', *],$$

and the base point is the class $[z, *]$. *This space is filtered by finite subspaces*

$$C_n(Z, X) = \left(\coprod_{1 \le r \le n} \mathcal{F}(Z; r) \times_{S_r} X^{(r)} \right) / \sim,$$

for $n \ge 1$, *and is given the* weak *or* direct limit *topology with respect to this direct system.*

The value of the functor on a pointed map $f\colon X \to X'$ *is the map*

$$C(Z, f)\colon C(Z, X) \to C(Z, X')$$

given by $C(Z, f)([(z_1, x_1), \ldots, (z_r, x_r)]) = [(z_1, f(x_1)), \ldots, (z_r, f(x_r))]$.

To form $C(Z, X)$ we take all configurations of points in Z with labels from X, and identify those configurations which differ only by points labelled by the base point $* \in X$. The class of any configuration all of whose labels are $*$ then acts as the base point.

Note also that an *injective* map $g\colon Z \to Z'$ induces a natural transformation

$$C(g, -)\colon C(Z, -) \to C(Z', -)$$

in an obvious manner, since g sends a configuration of points in Z to a configuration in Z'. Hence C may be regarded as a functor from the category $\mathscr{U}$ of topological spaces and injective continuous maps, to the functor category $\mathsf{Fun}(\mathscr{T}_\bullet)$.

Proposition 7.2. *Let* Z *be a topological space with a fixed injective map* $g\colon Z \sqcup Z \to Z$. *Then* $C(Z, X)$ *has the structure of a commutative topological monoid.*

Proof. Given any pair of spaces W and W', there is a symmetric, associative pairing

$$\mu\colon C(W, X) \times C(W', X) \to C(W \sqcup W', X),$$

which simply takes a disjoint union of configurations of labelled points. The pairing on $C(Z, X)$ is then given by the composition

$$C(Z, X) \times C(Z, X) \xrightarrow{\mu} C(Z \sqcup Z, X) \xrightarrow{C(g, X)} C(Z, X),$$

with the base point $[z, *] \in C(Z, X)$ acting as a two-sided identity. $\square$

For example, the spaces $C(\mathbb{R}^l, X)$ and $C(\mathbb{R}^\infty, X)$, where $\mathbb{R}^\infty$ is the infinite union $\bigcup_l \mathbb{R}^l$ with the weak topology, are commutative topological monoids. An embedding $V_+ \sqcup V_-\colon \mathbb{R}^l \sqcup \mathbb{R}^l \hookrightarrow \mathbb{R}^l$ for $1 \leq l \leq \infty$ is provided by the maps V_+ and V_- of Section 4.2 (see the discussion following Proposition 4.7), which embed $\mathbb{R}^l$ into the upper and lower half space of $\mathbb{R}^l$.

Let Ω and Σ denote the *loop space* and *reduced suspension* functors on pointed spaces. These are *adjoint functors*, meaning that for pointed spaces X and Y there is a bijection $Maps(\Sigma X, Y) \cong Maps(X, \Omega Y)$. By composition and iteration of these functors, we obtain a functor

$$\Omega^l \Sigma^l\colon \mathscr{T}_\bullet \to \mathscr{T}_\bullet$$

for each $l \in \mathbb{N}$. In fact the space $\Omega^l \Sigma^l X$ is a topological monoid, since we may add maps $f, g\colon S^l \to \Sigma^l X$ using the pinch map $S^l \to S^l \vee S^l$ (see for example R. Switzer [33], Chapter 2). A point of $\Omega^l \Sigma^l X$ is a map f from S^l to $\Sigma^l X$, the suspension of which is therefore a map Σf from S^{l+1} to $\Sigma^{l+1} X$, which is a point in $\Omega^{l+1}\Sigma^{l+1} X$. In this way suspension describes an inclusion of functors $\Omega^l \Sigma^l \hookrightarrow \Omega^{l+1}\Sigma^{l+1}$ for each $l \in \mathbb{N}$. The direct limit of these inclusions

$$Q = \Omega^\infty \Sigma^\infty := \lim_{l \to \infty} \Omega^l \Sigma^l$$

is a functor of prime importance in stable homotopy theory. It allows one to think of stable maps as unstable ones, since by adjointness there is an isomorphism

$$[X, Y]_S := \lim_{l \to \infty} [\Sigma^l X, \Sigma^l Y] \cong [X, QY]$$

for any pointed spaces X and Y, where the maps $[\Sigma^l X, \Sigma^l Y] \to [\Sigma^{l+1} X, \Sigma^{l+1} Y]$ in the direct limit are given by suspension.

The usefulness of configuration space models stems from the following result, which was proved in [22].

Theorem 7.3. *There is a natural map of topological monoids*

$$\alpha_l\colon C(\mathbb{R}^l, X) \to \Omega^l \Sigma^l X$$

for every $1 \leq l \leq \infty$, which is a weak homotopy equivalence when X is connected.

7.2 Homotopy Classification of $\mathcal{I}(-;-)$

Recall from Chapter 5 the functors $(-)_c\colon \mathcal{D}_0 \to \mathscr{T}_\bullet$ and $T(-)\colon \mathsf{Vect} \to \mathscr{T}_\bullet$, given by one-point compactification and Thomification. In this section we give a homotopy interpretation of $\mathcal{J}((-)_c; T(-))$ (and hence of $\mathcal{I}(-;-)$) by exhibiting, for each pair of objects $(N,\zeta) \in \mathcal{D}_0 \times \mathsf{Vect}$, an isomorphism

$$\Upsilon\colon\ \mathcal{J}(N_c; T\zeta) \to [N_c, C(\mathbb{R}^\infty, T\zeta)]\,.$$

The inverse Υ^{-1} takes the homotopy class of a map $f\colon\ N_c \to C(\mathbb{R}^\infty, T\zeta)$ to the class $f^*[u]$, where $[u] \in \mathcal{J}(C(\mathbb{R}^\infty, T\zeta); T\zeta)$ is the class of some *universal spreading.*

We begin by defining a space $\widetilde{C(Z,X)}$ which is closely related to $C(Z,X)$. Note that for any given $r \geq 1$ the symmetric group S_{r-1} also acts on $\mathcal{F}(Z;r)$ and $X^{(r)}$, by permuting the last $(r-1)$ factors and leaving the first fixed. A point in the space $\mathcal{F}(Z;r) \times_{S_{r-1}} X^{(r)}$ will be written in the form

$$(z_1, x_1), [(z_2, x_2), \ldots, (z_r, x_r)],$$

and may be thought of as a *pointed* configuration of r points in Z with labels from X, meaning that one of the points of the configuration is distinguished from the rest.

Now define an un-pointed space

$$\widetilde{C(Z,X)} = \left(\coprod_{r\geq 1} \mathcal{F}(Z;r) \times_{S_{r-1}} X^{(r)} \right) / \approx,$$

where $\approx$ is the equivalence relation generated by

$$(z_1, x_1), [(z_2, x_2), \ldots, (z_r, x_r)] \approx (z_1, x_1), [(z_2, x_2), \ldots, (z_r, x_r), (z_{r+1}, *)].$$

This space is topologised as the direct limit of finite subspaces

$$\widetilde{C_n(Z,X)} = \left(\coprod_{1\leq r\leq n} \mathcal{F}(Z;r) \times_{S_{r-1}} X^{(r)} \right) / \approx,$$

and comes equipped with maps $\gamma\colon\ \widetilde{C(Z,X)} \to C(Z,X)$ and $\beta\colon\ \widetilde{C(Z,X)} \to X$ defined by

$$\gamma((z_1, x_1), [(z_2, x_2), \ldots]) = [(z_1, x_1), (z_2, x_2), \ldots],$$

$$\beta((z_1, x_1), [(z_2, x_2), \ldots]) = x_1.$$

These maps may be thought of respectively as 'forgetting' and 'taking the label of' the base point of the configuration.

We are going to show that the triple $(\widetilde{C(Z,X)}, \gamma, \beta)$ is a spreading when Z and X are Hausdorff, for which we shall need the following well known Lemma about the direct limit topology (see [14]).

Lemma 7.4. *Let X be topologised as the direct limit of an increasing sequence of subspaces*

$$\ldots \subset X_i \subset X_{i+1} \subset \ldots \subset X,$$

and suppose each X_i is Hausdorff. Then a compact subset $C \subseteq X$ is contained in X_n for some $n \in \mathbb{N}$.

Proposition 7.5. *Let Z and X be Hausdorff spaces. Then the triple $(\widetilde{C(Z,X)}, \gamma, \beta)$ defines a spreading of type X in $C(Z,X)$.*

Proof. Recall the Definition 5.1 of a spreading. We must first check that the map $\gamma\colon \widetilde{C(Z,X)} \to C(Z,X)$ is closed and proper. For each $n \in \mathbb{N}$ the restriction $\gamma_n\colon \widetilde{C_n(Z,X)} \to C_n(Z,X)$ is a finite union of finite covering maps, and as such is open, closed and proper. These maps form a map of direct systems, and we may regard γ as the direct limit of these maps. We now use the general fact that a direct limit of closed maps is closed to deduce that γ is closed. To see that γ is proper, we apply Lemma 7.4. Since Z and X are Hausdorff, it follows that each of the $C_n(Z,X)$ are Hausdorff, and so a compact subset $K \subseteq C(Z,X)$ is contained in some $C_N(Z,X)$. Hence

$$\gamma^{-1}(K) = g_N(\gamma_N^{-1}(K))$$

is compact, where $g_N\colon \widetilde{C_N(Z,X)} \to \widetilde{C(Z,X)}$ is the map to the direct limit.

The next thing to check is that β maps $\gamma^{-1}[z,*]$ to the base point $* \in X$. This is immediate. So it only remains to prove that γ restricted to $\widetilde{C(Z,X)} - \beta^{-1}(*)$ is a local homeomorphism.

Let $y \in \widetilde{C(Z,X)} - \beta^{-1}(*)$ be an arbitrary point. By Lemma 7.4, y is contained in some finite subspace $\widetilde{C_n(Z,X)}$, hence is of the form

$$y = (z_1, x_1), [(z_2, x_2), \ldots, (z_n, x_n)],$$

where $x_1 \neq *$. Since Z is Hausdorff, we may find disjoint open subsets U and $\widetilde{U}$ of Z with $z_1 \in U$ and $z_2, \ldots, z_n \in \widetilde{U}$. Similarly we may find disjoint open subsets V and $\widetilde{V}$ of X with $x_1 \in V$ and $* \in \widetilde{V}$. Now define an open neighbourhood W_y of y in $\widetilde{C(Z,X)} - \beta^{-1}(*)$ by

$$(z_1', x_1'), [(z_2', x_2'), \ldots] \in W_y \subseteq \widetilde{C(Z,X)} - \beta^{-1}(*) \quad \Leftrightarrow \quad z_1' \in U, \;\; z_2', \ldots z_n' \in \widetilde{U}, \\ x_1' \in V, \;\; x_i' \in \widetilde{V} \text{ for } i > n.$$

This W_y is clearly open, and since γ as a direct limit of open maps is open, so is the image set $\gamma(W_y)$. A simple check shows that γ is a homeomorphism on W_y, and the Proposition is proved. $\square$

If ζ is a vector bundle over a Hausdorff space X then the Thom space $T\zeta$ is also Hausdorff, so we obtain a spreading $u = (\widetilde{C(\mathbb{R}^\infty, T\zeta)}, \gamma, \beta)$ of type $T\zeta$ in $C(\mathbb{R}^\infty, T\zeta)$. By Proposition 7.2 and the remarks following, the map $V_+ \sqcup V_- \colon \mathbb{R}^\infty \sqcup \mathbb{R}^\infty \hookrightarrow \mathbb{R}^\infty$ furnishes $C(\mathbb{R}^\infty, T\zeta)$ with the structure of a commutative topological monoid. We are now ready to prove the classification result.

Theorem 7.6. *The following diagram of functors commutes up to natural isomorphism.*

$$\begin{tikzcd}
\mathcal{D}_0^{\mathsf{op}} \times \mathsf{Vect} \arrow[r, "(-)_c \times T"] \arrow[d, "(-)_c \times T"'] & \underline{\mathscr{T}_\bullet}^{\mathsf{op}} \times \mathscr{T}_\bullet \arrow[d, "\mathcal{J}(-;-)"] \\
\underline{\mathscr{T}_\bullet}^{\mathsf{op}} \times \mathscr{T}_\bullet \arrow[r, "{[-, C(\mathbb{R}^\infty, -)]}"] & \mathsf{CMon}
\end{tikzcd}$$

Proof. Given $N^n \in \mathcal{D}_0$ and $\zeta \in \mathsf{Vect}$, we must define a monoid isomorphism

$$\Upsilon \colon \mathcal{J}(N_c; T\zeta) \to [N_c, C(\mathbb{R}^\infty, T\zeta)]$$

which is natural in N and ζ. If $[K, \alpha, \beta] \in \mathcal{J}(N_c; T\zeta)$, then by Proposition 5.5 we can find a unique class $[M, f, v] \in \mathcal{I}(N; \zeta)$ such that $[K, \alpha, \beta] = [D\nu(f), F, \widetilde{v}]$, where

$F\colon D\nu(f) \looparrowright N$ is an immersion of the unit normal disc bundle extending f. Since $D\nu(f)^n$ is a compact manifold with boundary, by Theorem A.1 in the Appendix we can find an embedding $\lambda\colon D\nu(f) \hookrightarrow \mathbb{R}^\infty$, and any two such λ are isotopic. Now define a map $g\colon N_c \to C(\mathbb{R}^\infty, T\zeta)$ as follows. For each $n \in N_c$, the set $F^{-1}(n) = \{k_1, \ldots, k_r\}$ is finite, since it is compact. We set

$$g(n) = \begin{cases} [(\lambda(k_1), \widetilde{v}(k_1)), \ldots, (\lambda(k_r), \widetilde{v}(k_r))] & \text{if } F^{-1}(n) = \{k_1, \ldots, k_r\} \\ [0, *] & \text{if } F^{-1}(n) = \emptyset \end{cases}$$

One verifies that $g\colon N_c \to C(\mathbb{R}^\infty, T\zeta)$ is a pointed, continuous map. We then set $\Upsilon([K, \alpha, \beta]) = [g]$.

To check that Υ is well defined, we first note that it does not depend on the choice of λ, since if we use another such map λ' to give a map $g'\colon N_c \to C(\mathbb{R}^\infty, T\zeta)$, there is an isotopy $\lambda \simeq \lambda'$ which induces a homotopy $g \simeq g'$. Now suppose $[D\nu(f_0), F_0, \widetilde{v_0}] = [D\nu(f_1), F_1, \widetilde{v_1}] \in \mathcal{J}(N_c; T\zeta)$. Then $[M_0, f_0, v_0] = [M_1, f_1, v_1]$ and we can find a bordism (W, K, V) from f_0 to f_1. From this triple we may build a spreading $(D\nu(K), \Psi, \Phi)$ of type $T\zeta$ in $N_c \times I$ such that

$$\begin{tikzcd} D\nu(f_0) \arrow[r] \arrow[d, "F_0"'] & D\nu(K) \arrow[d, "\Psi"] & D\nu(f_1) \arrow[l] \arrow[d, "F_1"] \\ N_c \times \{0\} \arrow[r] & N_c \times I & N_c \times \{1\} \arrow[l] \end{tikzcd}$$

is a pull-back diagram, and $\Phi|_{D\nu(f_i)} = \widetilde{v_1}$ for $i = 0, 1$. Since $D\nu(K)$ is a manifold with corners it admits an embedding $\Lambda\colon D\nu(K) \to \mathbb{R}^\infty$ such that $\Lambda|_{D\nu(f_i)} = \lambda_i$ (see [20]). It is then clear that we can construct a map $G\colon N_c \times I \to C(\mathbb{R}^\infty, T\zeta)$ using Ψ, Φ and Λ, which is a homotopy from g_0 to g_1. Hence Υ is well defined.

To see that Υ is a monoid map, note that a sum of classes in $\mathcal{J}(N_c; T\zeta)$ is represented by a spreading $(D\nu(f_0) \sqcup D\nu(f_1), F_0 \sqcup F_1, \widetilde{v_0} \sqcup \widetilde{v_1})$. Suppose that, for $i = 0, 1$, we have $\Upsilon([D\nu(f_i), F_i, \widetilde{v_i}]) = [g_i]$, where g_i is defined using an embedding $\lambda_i\colon D\nu(f_i) \hookrightarrow \mathbb{R}^\infty$. Define $\lambda\colon D\nu(f_0) \sqcup D\nu(f_1) \hookrightarrow \mathbb{R}^\infty$ by composing $\lambda_0 \sqcup \lambda_1\colon D\nu(f_0) \sqcup D\nu(f_1) \to \mathbb{R}^\infty \sqcup \mathbb{R}^\infty$ with $V_+ \sqcup V_-\colon \mathbb{R}^\infty \sqcup \mathbb{R}^\infty \to \mathbb{R}^\infty$. Then check that the class $\Upsilon([D\nu(f_0) \sqcup D\nu(f_1), F_0 \sqcup F_1, \widetilde{v_0} \sqcup \widetilde{v_1}])$ obtained using λ is represented by the composition

$$N_c \xrightarrow{\triangle} N_c \times N_c \xrightarrow{g_0 \times g_1} C(\mathbb{R}^\infty, T\zeta) \times C(\mathbb{R}^\infty, T\zeta) \longrightarrow C(\mathbb{R}^\infty, T\zeta).$$

To show that Υ is an isomorphism, we have a map $\Upsilon^{-1}\colon [N_c, C(\mathbb{R}^\infty, T\zeta)] \to \mathcal{J}(N_c; T\zeta)$ which sends $[f]$ to $f^*[u]$, where $u = (\widetilde{C(\mathbb{R}^\infty, T\zeta)}, \gamma, \beta)$ is the spreading of type $T\zeta$ in $C(\mathbb{R}^\infty, T\zeta)$ defined by Proposition 7.5. Suppose we start with a spreading $(D\nu(f), F, \widetilde{v})$, from which we define a map $g\colon N_c \to C(\mathbb{R}^\infty, T\zeta)$ as above. Note that for every $k \in D\nu(f)$, the non-empty set $F^{-1}(F(k)) = \{k, k_2, \ldots, k_r\}$ has k as a distinguished element. Thus we may define a map $\widetilde{g}\colon D\nu(f) \to \widetilde{C(\mathbb{R}^\infty, T\zeta)}$ by setting

$$\widetilde{g}(k) = (\lambda(k), \widetilde{v}(k)), [(\lambda(k_2), \widetilde{v}(k_2)), \ldots, (\lambda(k_r), \widetilde{v}(k_r))].$$

Then the following diagram commutes, and the square is a pull-back.

$$\begin{tikzcd}
 & T\zeta & \\
D\nu(f) \arrow[ur, "\widetilde{v}"] \arrow[rr, "\widetilde{g}"] \arrow[d, "F"'] & & \widetilde{C(\mathbb{R}^\infty, T\zeta)} \arrow[ul, "\beta"'] \arrow[d, "\gamma"] \\
N_c \arrow[rr, "g"] & & C(\mathbb{R}^\infty, T\zeta)
\end{tikzcd}$$

This shows that Υ and Υ^{-1} are mutual inverses. The statement about naturality is easily verified. $\square$

Corollary 7.7. *There are natural isomorphisms*

$$\mathcal{I}(-;-) \cong \mathcal{J}((-)_c; T(-)) \cong [(-)_c; C(\mathbb{R}^\infty, T(-))].$$

Hence for any $N^n \in \mathcal{D}_0$ *and* $\zeta \in \mathsf{Vect}$ *with* $\dim \zeta > 0$,

$$\mathcal{I}(N; \zeta) \cong [N_c, QT\zeta] \cong [N_c, T\zeta]_S.$$

Proof. The first statement is a combination of Proposition 5.5 and Theorem 7.6. The second follows immediately on applying Theorem 7.3, since $T\zeta$ is connected when $\dim \zeta > 0$. $\square$

7.3 Homotopy Interpretation of the Self-intersection Operations

An immediate consequence of the above Corollary is that for any $\zeta \in \mathsf{Vect}$ and $r \geq 1$ there is a natural isomorphism

$$\mathcal{I}(-;\mathcal{S}_r\zeta) \cong [(-)_c, C(R^\infty, D_rT\zeta)]$$

of functors from $\mathcal{D}_0$ to CMon. One may ask, therefore, if the self-intersection operations $\psi_r\colon \mathcal{I}(-;\zeta) \to \mathcal{I}(-;\mathcal{S}_r\zeta)$ are induced by maps $h^r\colon C(\mathbb{R}^\infty, T\zeta) \to C(\mathbb{R}^\infty, D_rT\zeta)$. This is indeed true if we invert weak equivalences in $\mathscr{T}_\bullet$, which is possible thanks to the model category structure on $\mathscr{T}_\bullet$ [24].

Definition 7.8. *Let $H\mathscr{T}_\bullet$ be the homotopy category associated to a suitable model structure on $\mathscr{T}_\bullet$, in which the weak equivalences are formally inverted. Then for each $r \geq 1$ and connected space $X \in \mathscr{T}_\bullet$, the* James-Hopf map

$$h^r\colon C(\mathbb{R}^\infty, X) \to C(\mathbb{R}^\infty, D_rX)$$

is a morphism in $H\mathscr{T}_\bullet$.

M. G. Barratt and P. J. Eccles constructed in [4] a topological monoid Γ^+X which is naturally homotopy equivalent to QX when X is connected. They also construct maps $\hbar^r\colon \Gamma^+X \to \Gamma^+D_rX$ for all $r \geq 1$ [5]. Let $k\colon \Gamma^+X \to QX$ be a homotopy eqivalence. Recalling the weak equivalence $\alpha_\infty\colon C(\mathbb{R}^\infty, X) \to QX$ of Theorem 7.3, the h^r may be defined by commutativity of the following diagram.

$$\begin{tikzcd}
C(\mathbb{R}^\infty, X) \arrow[r, "\alpha_\infty"] \arrow[d, "h^r"'] & QX \arrow[r, "k^{-1}"] & \Gamma^+X \arrow[d, "\hbar^r"] \\
C(\mathbb{R}^\infty, D_rX) & QD_rX \arrow[l, "\alpha_\infty^{-1}"'] & \Gamma^+D_rX \arrow[l, "k"']
\end{tikzcd}$$

Such maps appear in various guises in the literature, where they are used to prove stable splittings of the spaces $\Omega^n\Sigma^nX$ and QX; see for example [5] and [29]. The following Theorem, which is essentially folklore, says that the James-Hopf maps induce the self-intersection operations. Statements and partial proofs may be found in the papers [34], [35], [18] and [38].

Theorem 7.9. *For any vector bundle* $\zeta \in \mathsf{Vect}$ *with* $\dim \zeta \geq 1$, *the following diagram of set-valued functors commutes.*

$$\begin{tikzcd}
\mathcal{I}(-;\zeta) \arrow[r, "\simeq", "\Theta"'] \arrow[d, "\psi_r"'] & \mathcal{J}((-)_c;T\zeta) \arrow[r, "\simeq", "\Upsilon"'] \arrow[d, "\Psi_r"'] & {[(-)_c, C(\mathbb{R}^\infty, T\zeta)]} \arrow[d, "(h^r)_*"'] \\
\mathcal{I}(-;\mathcal{S}_r\zeta) \arrow[r, "\simeq", "\Theta"'] & \mathcal{J}((-)_c;D_rT\zeta) \arrow[r, "\simeq", "\Upsilon"'] & {[(-)_c, C(\mathbb{R}^\infty, D_rT\zeta)]}
\end{tikzcd}$$

Remark 7.10. This result, together with Remark 4.27**(iii)** and Theorem 6.2 and its Corollaries, provides formal evidence that the James-Hopf map h^2 and the H_∞ ring structures on the MO and $\mathsf{H}\mathbb{Z}_2$ spectra are somehow related. Therefore there should be concrete relations to be discovered, linking these two important constructs of stable homotopy theory.

Appendix A

Results from Differential Topology

In this appendix we collect some results from differential topology which are referred to in the rest of the work. Many of these results, although well-known to workers in the field, are hard to find in the literature. We do not attempt to give full proofs of such results, rather try to indicate how they might be proved.

The first Theorem, proved by Whitney in [40], concerns embeddings of manifolds in Euclidean spaces. Recall that an *embedding* is an immersion which is homeomorphic onto its image, and an *isotopy* is a homotopy through embeddings.

Theorem (Whitney) A.1. *Let N^n be a manifold of dimension n. If $l \geq 2n+1$ then N embeds in $\mathbb{R}^l$. Furthermore, if $l \geq 2n+2$ then any two embeddings of N into $\mathbb{R}^l$ are isotopic.*

We will also need several results concerning when a given immersion is regularly homotopic to an immersion with some given property P.

Let M and N be smooth manifolds without boundary, with M compact. Let $C^\infty(M,N)$ denote the set of all smooth maps from M to N, given the *weak topology* (see Hirsch [16]; note that this coincides with the *strong topology* since M is compact). The following fundamental and deep fact about this mapping space does not seem to have a clear proof in the literature.

Theorem A.2. *The space $C^\infty(M,N)$ is locally path connected.*

Proof. This says that for every smooth map $f\colon M \to N$ and every neighbourhood $U \subseteq C^\infty(M,N)$ of f, there is a path connected neighbourhood V with $f \in V \subseteq U$. But by Theorem 10.4 of [23], $C^\infty(M,N)$ is covered by open sets, each of which is homeomorphic to an open subset of some complete, locally convex vector space. □

Let P be some property of smooth maps from M to N. We define the following subspaces of $C^\infty(M,N)$:

$$C_P^\infty(M,N) = \{f \in C^\infty(M,N) \mid f \text{ has property } P\},$$
$$Im(M,N) = \{f \in C^\infty(M,N) \mid f \text{ is an immersion}\},$$
$$Im_P(M,N) = Im(M,N) \cap C_P^\infty(M,N).$$

Lemma A.3. *$Im(M,N) \subseteq C^\infty(M,N)$ is open.*

Proof. See Hirsch [16], Theorem 2.1.1. □

Now fix a proper map $g\colon Q \to N$, and let P be one of the following properties:

“f is transverse to g”

“f is self-transverse”.

Lemma A.4. *$C_P^\infty(M,N) \subseteq C^\infty(M,N)$ is dense.*

Proof. This is Proposition III.3.2 of [13] and exercise 14 (b) on page 84 of [16]. □

We now come to our first approximation result.

Proposition A.5. *Let $f\colon M \looparrowright N$ be an immersion. Then f is regularly homotopic to an immersion f' with property P.*

Proof. Since $C^\infty(M,N)$ is locally path connected and $Im(M,N) \subseteq C^\infty(M,N)$ is open, it is a standard result of general topology that each path component of $Im(M,N)$ is open in $C^\infty(M,N)$. In particular the path component of f, which is the set

$$U_f = \{f' \in Im(M,N) \mid f' \text{ is regularly homotopic to } f\}$$

is open in $C^\infty(M,N)$. Since $C_P^\infty(M,N)$ is dense in $C^\infty(M,N)$, there is some $f' \in U_f$ with property P. □

This result, along with Proposition 2.5, says that within every bordism class $[M, f, v] \in \mathcal{I}(N, \zeta)$ we can find a representative (M, f', v') where f' is regularly homotopic to f and has property P.

Proposition A.6. *Let $F\colon W \looparrowright N \times I$ be a bordism between two immersions $f_0\colon M_0 \looparrowright N$ and $f_1\colon M_1 \looparrowright N$.*

(a) *If f_0 and f_1 are transverse to the proper map $g\colon Q \to N$, there is a bordism $F'\colon W \looparrowright N \times I$ from f_0 to f_1 which is transverse to $g \times \mathbf{1}\colon Q \times I \to N \times I$.*

(b) *If f_0 and f_1 are self-transverse, there is a bordism $F'\colon W \looparrowright N \times I$ from f_0 to f_1 which is self-transverse.*

Proof. To prove this, one would need to prove relative versions of the results A2 through to A5. □

The next Theorem concerns the existence and uniqueness of immersed tubular neighbourhoods of immersions of closed manifolds, and was used in the proofs of Theorem 3.5 and Proposition 5.5.

Theorem A.7. *Let $f\colon M^{n-k} \looparrowright N^n$ be an immersion with M closed. There is an immersion $F\colon D\nu(f) \looparrowright N$ of the unit normal disc bundle of f which extends f (so $F \circ i = f$ where $i\colon M \hookrightarrow D\nu(f)$ is the zero section), and is injective on the fibres. Furthermore, two such immersions F_0 and F_1 are regularly homotopic by a homotopy which is stationary on $i(M)$.*

Proof. Furnish N with a Riemannian metric, and let $E\nu(f)$ be the total space of the normal bundle of f. We will define $F\colon D\nu(f) \looparrowright N$ with the aid of the exponential map

$$\mathrm{Exp}\colon E\nu(f) \to N,$$

which is defined as follows. Write a point of $E\nu(f)$ as (x, v), where $x \in M$ and $v \in \nu(f)_x$. If we view $\nu(f)$ as the orthogonal complement of the vector sub-bundle TM in f^*TN, then v may be viewed as a vector in $TN_{f(x)}$. Let $\gamma\colon I \to N$ be the parameterised geodesic arc in N of length $|v|$, with initial point $\gamma(0) = f(x)$ and initial velocity vector $d\gamma/dt|_{t=0} = v$. We then set $\mathrm{Exp}(x, v) = \gamma(1)$.

The usual considerations show that Exp is defined and a local diffeomorphism on an open neighbourhood of the zero section $i(M)$ in $E\nu(f)$. Hence we can choose a small disc bundle

$$D_\epsilon\nu(f) = \{(x, v) \in E\nu(f) \mid |v| \leq \epsilon\}$$

such that Exp restricted to $D_\epsilon\nu(f)$ is an immersion. Let $h\colon\ D\nu(f) \to D_\epsilon\nu(f)$ be fibrewise multiplication by ϵ. Then $F = \mathrm{Exp}|_{D_\epsilon\nu(f)} \circ h\colon\ D\nu(f) \looparrowright N$ is an immersion with the required properties.

For the statement that two such immersions F_0 and F_1 are regularly homotopic, the reader is asked to look at any proof of the fact that any two closed tubular neighbourhoods $G_0, G_1\colon\ D\nu(g) \hookrightarrow N$ of an embedding $g\colon\ M \hookrightarrow N$ are isotopic rel M (for example Hirsch [16], Theorem 4.5.3). He or she will see that the isotopy $H\colon\ D\nu(g)\times I \to N$ from G_0 to G_1 is constructed locally. Hence the same construction slightly modified gives a regular homotopy rel M from F_0 to F_1. We omit the details.

$\square$

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