The Poincaré–Hopf theorem for line fields revisited

(joint with D. Crowley)

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Line fields

Let M^m be a smooth manifold of dimension $m \geq 2$.

Definition

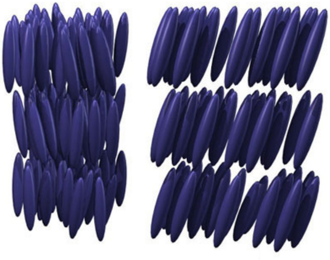
A line field on M is a smooth section $\xi : M \to PTM$ of the projectivized tangent bundle.

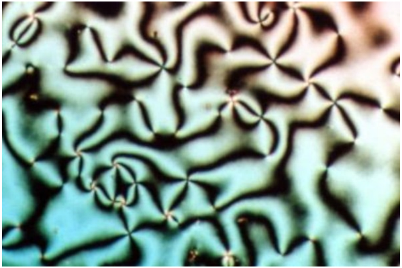
In other words, a line field is a smooth assignment

$$x \mapsto \xi(x) \subset TM_x$$

of a one-dimensional subspace of the tangent space at each point.

Line fields, or nematic fields, are of interest in soft-matter physics, where they are used to model nematic liquid crystals.





(Images: `https://en.wikipedia.org/wiki/Liquid_crystal`)

A nowhere zero vector field $v : M \to TM$ gives rise to a line field by setting

$$\xi(x) = \langle v(x) \rangle \subset TM_x$$

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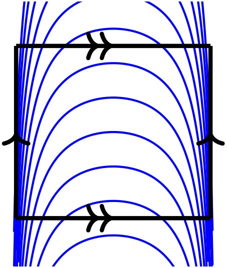
However, not every line field can be lifted to a nowhere zero vector field.



Proposition

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Proof: A line field ξ on M may be viewed as a line sub-bundle $\xi \subset TM$.

Fix a metric on M, then the sphere bundle

$$p_\xi : \widetilde{M} := S(\xi) \to M$$

is the associated double cover.

Note that $\widetilde{M}$ has a canonical nowhere zero vector field which lifts $p_\xi^*\xi$.

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By the multiplicativity of the Euler characteristic for covers,

$$0 = \chi(\widetilde{M}) = 2\,\chi(M),$$

hence $\chi(M) = 0$ and M admits a nowhere zero vector field. □

Theorem (Poincaré–Hopf)

Let $v : M \to TM$ be a vector field with isolated zeroes at $x_1, \ldots, x_n \in M$. Then

$$\sum_{i=1}^{n} \mathrm{ind}_v(x_i) = \chi(M).$$

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$$\sum_{i=1}^{n} \mathrm{ind}_v(x_i) = \chi(M).$$

The index $\mathrm{ind}_v(x_i) \in \mathbb{Z}$ is the degree of the composition

$$f : S \xrightarrow{v|_S} STM|_S \xrightarrow{\Phi} S \times S^{m-1} \xrightarrow{\pi_2} S^{m-1},$$

where:

- $v|_S$ is the restriction of (the normalization of) v to a small sphere S centred at x_i;
- Φ is a trivialisation, and
- π_2 is projection onto the second factor.

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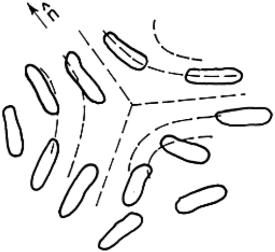
A vector field with zeroes determines a line field with singularities, but a line field with singularities need not lift to a vector field.

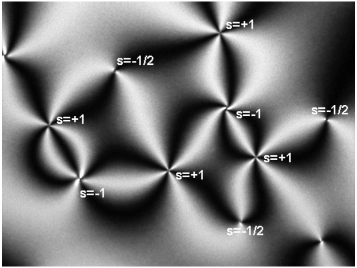
Question

What is the analogue of Poincaré–Hopf for line fields with singularities?

The singularities are known as topological defects in the Physics literature.

Of particular interest are point defects in 2 and 3 dimensions, and line defects or disclinations in 3 dimensions (which may be knotted).





(Images: `http://www.lassp.cornell.edu/sethna/OrderParameters/TopologicalDefects.html`, `http://www.personal.kent.edu/~bisenyuk/liquidcrystals/textures1.html`)

Hopf's result

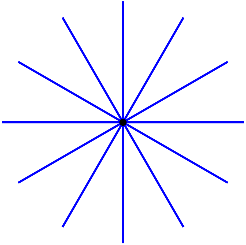
Theorem (Hopf)

A line field ξ with singularities $x_1, \ldots, x_n$ on a closed orientable surface Σ has

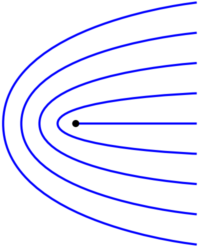
$$\sum_{i=1}^{n} \operatorname{h\,ind}_\xi(x_i) = \chi(\Sigma).$$

The Hopf index $\operatorname{h\,ind}_\xi(x_i) \in \frac{1}{2}\mathbb{Z}$ is the number of total rotations made by ξ as a simple closed curve around x_i is traversed.

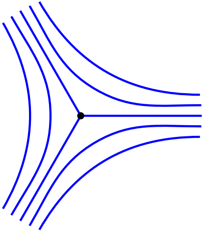
Reference: H. Hopf, *Differential Geometry in the Large*, LNM 1000, (1983) (Based on lectures given at Stanford University in 1956).



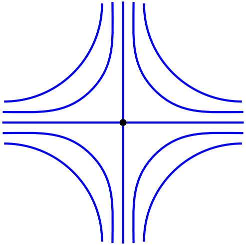
(a) $\operatorname{h\,ind}_\xi(x) = 1$



(b) $\operatorname{h\,ind}_\xi(x) = \frac{1}{2}$



(c) $\operatorname{h\,ind}_\xi(x) = -\frac{1}{2}$



(d) $\operatorname{h\,ind}_\xi(x) = -1$

Line field singularities and their Hopf indices.

Markus' result

Definition

A singularity x_i of a line field ξ on M^m is called (non)-orientable if the restriction of ξ to a small sphere S centred at x_i lifts (does not lift) to a vector field.

Equivalently, x_i is (non)-orientable if the restriction to S of the associated double cover $p_\xi|_S : \widetilde{S} \to S$ is (non)-trivial.

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If $m = 2$, then x_i is orientable if and only if $\mathrm{h\,ind}_\xi(x_i) \in \mathbb{Z}$.

If $m > 2$, then all singularities are orientable.

The Markus index $\mathrm{m\,ind}_\xi(x_i) \in \mathbb{Z}$ is defined as follows:

For m even, it is the degree of the composition

$$f : S \xrightarrow{\xi|_S} PTM|_S \xrightarrow{\Phi} S \times \mathbb{R}P^{m-1} \xrightarrow{\pi_2} \mathbb{R}P^{m-1}.$$

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For $m \geq 3$ odd, orienting ξ near x_i gives a lift $\tilde{f} : S \to S^{m-1}$ of $f : S \to \mathbb{R}P^{m-1}$. Choose base points and suspend, and take the degree of the composition

$$S^m \xrightarrow{\Sigma\tilde{f}} S^m \longrightarrow \mathbb{R}P^m.$$

Theorem (Markus)

A line field ξ with singularities $x_1, \ldots, x_n$ on a closed manifold M^m has

$$\sum_{i=1}^{n} \mathrm{m\,ind}_{\xi}(x_i) = 2\chi(M) - k,$$

where k is the number of non-orientable singularities.

Reference: L. Markus, *Line element fields and Lorentz structures on differentiable manifolds*, Ann. Math. 62, (1955)

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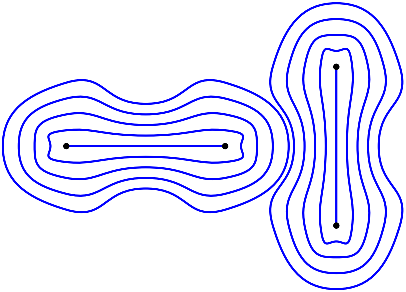
Unfortunately, there are counter-examples to Markus' Theorem for $m = 2$ and $m \geq 3$ odd.

Example: The baseball

There is a line field on S^2, known colloquially as "the baseball", with four non-orientable singularities of Hopf index $\frac{1}{2}$ and Markus index 1.



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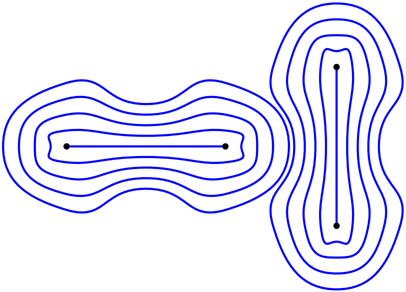
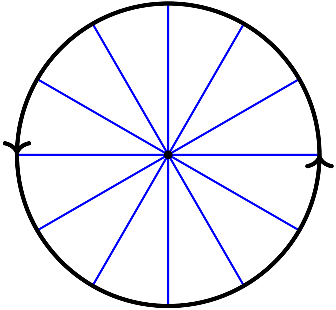


This contradicts Markus' Theorem, since

$$\sum_{i=1}^{n} \mathrm{m\,ind}_\xi(x_i) = 4 \neq 0 = 2\chi(S^2) - 4.$$

Example: The hedgehog

This is a line field on $\mathbb{R}P^m$ with a single orientable singularity of Hopf index 1 and Markus index 2.



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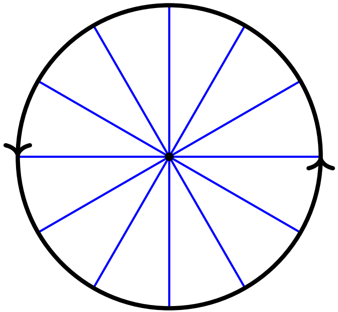

For $m \geq 3$ odd this contradicts Markus' Theorem, since

$$\sum_{i=1}^{n} \mathrm{m\,ind}_\xi(x_i) = 2 \neq 0 = 2\chi(\mathbb{R}P^m).$$

Our result

We define the projective index by

$$\operatorname{p\,ind}_\xi(x_i) = \begin{cases} \deg(f) & \in \mathbb{Z} \quad \text{if } m \text{ even}, \\ \deg_2(f) & \in \mathbb{Z}/2 \quad \text{if } m \text{ odd}, \end{cases}$$

where $f : S^{m-1} \to \mathbb{R}P^{m-1}$ is the composition

$$f : S \xrightarrow{\xi|_S} PTM|_S \xrightarrow{\Phi} S \times \mathbb{R}P^{m-1} \xrightarrow{\pi_2} \mathbb{R}P^{m-1}.$$

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A line field ξ with singularities $x_1, \ldots, x_n$ on a closed manifold M^m has

$$\sum_{i=1}^{n} \operatorname{p\,ind}_{\xi}(x_i) = 2\chi(M).$$

The equality is congruence mod 2 when m is odd.

Remarks

This corrects Markus' Theorem, and extends Hopf's Theorem to dimensions $m > 2$.

Our proof is similar to that of Markus, but we introduce normal indices to clarify some issues when $m = 2$.

Similar statements were given by Koschorke for $m > 2$ (1974) and Jänich (1984). Our contribution is a careful proof valid in all dimensions.

Normal indices

Let x be an isolated zero of the vector field $v : M \to TM$. Recall that $\mathrm{ind}_v(x)$ is the degree of the composition

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If $a \in S^{m-1}$ is a regular value of f, then $v|_S$ is transverse to the embedding $\sigma = \sigma_a : S \hookrightarrow STM|_S$ given by

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Then $\mathrm{ind}_v(x)$ equals the oriented intersection number

$$\sigma(S) \pitchfork v(S) \in \mathbb{Z}.$$

Suppose M endowed with a Riemannian metric. Then the outward unit normal to S defines an embedding $\eta : S \hookrightarrow STM|_S$.

Definition

The normal index $\mathrm{ind}_v^{\perp}(x) \in \mathbb{Z}$ is defined to be the oriented intersection number

$$\eta(S) \pitchfork v(S) \in \mathbb{Z}.$$

The normal index counts the number of times v points outwards on S (with signs).

Lemma

We have
$$\mathrm{ind}_v^{\perp}(x) = \mathrm{ind}_v(x) + (-1)^{m-1}.$$

Proof: Calculate intersection numbers in
$$H_*(S \times S^{m-1}) \cong H_*(S) \otimes H_*(S^{m-1}).$$

□

Now let x be an isolated singularity of the line field $\xi : M \to PTM$. Recall that $\operatorname{p\,ind}_\xi(x)$ is the degree of the composition

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$$\sigma(y) = \Phi^{-1}(y, a).$$

Then $\mathrm{p\,ind}_\xi(x)$ equals the intersection number

$$\mathrm{p\,ind}_\xi(x) = \begin{cases} \sigma(S) \pitchfork \xi(S) & \in \mathbb{Z} \quad \text{if } m \text{ even,} \\ \sigma(S) \pitchfork_2 \xi(S) & \in \mathbb{Z}/2 \quad \text{if } m \text{ odd.} \end{cases}$$

The normal line to S defines an embedding $\eta : S \hookrightarrow PTM|_S$.

Definition

The normal projective index is defined by

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The normal projective index counts the number of times ξ is normal to S (with signs if m is even).

Lemma

When m is even, we have

$$\operatorname{p\,ind}^{\perp}_{\xi}(x) = \operatorname{p\,ind}_{\xi}(x) - 2.$$

Proof: Calculate intersection numbers in

$$H_*(S \times \mathbb{R}P^{m-1}) \cong H_*(S) \otimes H_*(\mathbb{R}P^{m-1}).$$

□

Lemma

When $m \geq 3$ is odd, we have

$$\mathrm{p\,ind}_\xi(x) \equiv \mathrm{p\,ind}_\xi^\perp(x) \equiv 0 \in \mathbb{Z}/2.$$

Proof: The map $f : S \to \mathbb{R}P^{m-1}$ lifts through the standard double cover $S^{m-1} \to \mathbb{R}P^{m-1}$, and therefore $\mathrm{p\,ind}_\xi(x) = \deg_2(f) \equiv_2 0$. Since σ and η represent the same mod 2 homology class, the result follows. □

The proof

Theorem (Crowley–G.)

A line field ξ with singularities $x_1, \ldots, x_n$ on a closed manifold M^m has

$$\sum_{i=1}^{n} \mathrm{p\,ind}_{\xi}(x_i) = 2\chi(M).$$

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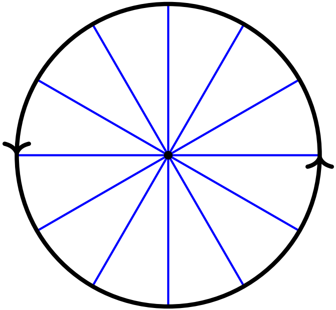
Proof: When $m \geq 3$ is odd, trivial consequence of $\mathrm{p\,ind}_\xi(x_i) \equiv_2 0$.

Remark: The Markus index $\mathrm{m\,ind}_\xi(x_i) \in \mathbb{Z}$ is not well-defined for m odd, since the two lifts $\widetilde{f} : S \to S^{m-1}$ differ by a map of degree $(-1)^m = -1$.

One may define an index in $\mathbb{N}_0$, but the hedgehog example suggests the above result is the best we can hope for.



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Let D_i be a coordinate disk containing x_i and no other singularities, and let $S_i = \partial D_i$. Then $N := M \setminus \bigcup \mathrm{int}(D_i)$ is a compact with boundary

$$\partial N \approx \bigsqcup_{i=1}^{n} S_i \approx \bigsqcup_{i=1}^{n} S^{m-1}.$$

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The restriction $\xi|_N$ is a line field with associated double cover $p : \widetilde{N} \to N$.

Each restriction $p|_{S_i} : \widetilde{S}_i \to S_i$ is a double cover of S^{m-1}, which is trivial if and only if x_i is orientable.

By gluing in m-disks along the boundary components of $\widetilde{N}$, we obtain a closed manifold $\widetilde{M}$ and a double cover

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The line field $\xi|_N$ lifts canonically to a vector field $\widetilde{\xi}$ on $\widetilde{N}$, which extends to a vector field v on $\widetilde{M}$.

Each pre-image $\pi^{-1}(x_i)$ consists of one or two isolated zeroes of v.

Lemma

For each singularity x_i of ξ, we have

$$\operatorname{p}\operatorname{ind}_{\xi}^{\perp}(x_i) = \sum_{y \in \pi^{-1}(x_i)} \operatorname{ind}_{v}^{\perp}(y).$$

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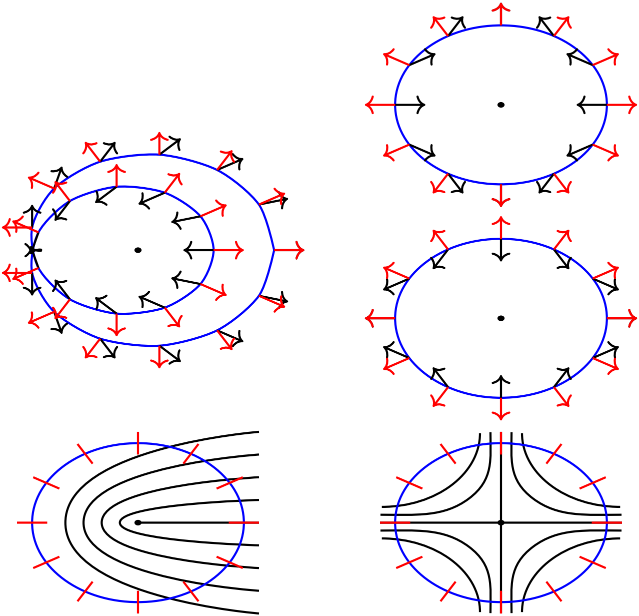
Intuitively: the number of times ξ is normal to S equals the number of times v agrees with the outward normal on $\widetilde{S}$.



Proof of Lemma: The double cover $\pi : \widetilde{M} \to M$ induces a 4-fold cover $\overline{\pi} : ST\widetilde{M}|_{\widetilde{S}} \to PTM|_S$, and there is pullback square

$$\begin{array}{ccc} \widetilde{S} \sqcup \widetilde{S} & \xrightarrow{\widetilde{\eta} \sqcup -\widetilde{\eta}} & ST\widetilde{M}|_{\widetilde{S}} \\ \downarrow{\scriptstyle p \sqcup p} & & \downarrow{\scriptstyle \overline{\pi}} \\ S & \xrightarrow{\eta} & PTM|_S \end{array}$$

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where $\widetilde{\eta} : \widetilde{S} \to ST\widetilde{M}|_{\widetilde{S}}$ denotes the outward unit normal to $\widetilde{S}$.

It follows that $\overline{\pi}^* \eta_!(1) = 2\, \widetilde{\eta}_!(1)$.

By a similar argument, $\overline{\pi}^*\xi_!(1) = 2\,v_!(1)$. Therefore,

$$4\,\mathrm{p}\,\mathrm{ind}^{\perp}_{\xi}(x) = 4\,\langle \eta_!(1) \cup \xi_!(1), [PTM|_S] \rangle$$

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&= \langle \eta_!(1) \cup \xi_!(1), 4[PTM|_S]\rangle \\
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&= \langle \overline{\pi}^*\eta_!(1) \cup \overline{\pi}^*\xi_!(1), [ST\widetilde{M}|_{\widetilde{S}}]\rangle \\
&= \langle 2\widetilde{\eta}_!(1) \cup 2v_!(1), [ST\widetilde{M}|_{\widetilde{S}}]\rangle
\end{aligned}$$

By a similar argument, $\overline{\pi}^*\xi_!(1) = 2\,v_!(1)$. Therefore,

$$\begin{aligned}
4\,\mathrm{p}\,\mathrm{ind}_\xi^\perp(x) &= 4\,\langle \eta_!(1) \cup \xi_!(1), [PTM|_S]\rangle \\
&= \langle \eta_!(1) \cup \xi_!(1), 4[PTM|_S]\rangle \\
&= \langle \eta_!(1) \cup \xi_!(1), \overline{\pi}_*[ST\widetilde{M}|_{\widetilde{S}}]\rangle \\
&= \langle \overline{\pi}^*\eta_!(1) \cup \overline{\pi}^*\xi_!(1), [ST\widetilde{M}|_{\widetilde{S}}]\rangle \\
&= \langle 2\widetilde{\eta}_!(1) \cup 2v_!(1), [ST\widetilde{M}|_{\widetilde{S}}]\rangle \\
&= 4 \sum_{y \in \pi^{-1}(x)} \mathrm{ind}_v^\perp(y),
\end{aligned}$$

and the conclusion follows. □

We now apply Riemann–Hurwitz and the classical Poincaré–Hopf formula.

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$$2\chi(M) = k + \chi(\widetilde{M}) = k + \sum_{i=1}^{n} \sum_{y \in \pi^{-1}(x_i)} \mathrm{ind}_v(y)$$

We now apply Riemann–Hurwitz and the classical Poincaré–Hopf formula.

$$\begin{aligned} 2\chi(M) = k + \chi(\widetilde{M}) &= k + \sum_{i=1}^{n} \sum_{y \in \pi^{-1}(x_i)} \mathrm{ind}_v(y) \\ &= k + \sum_{i=1}^{n} \sum_{y \in \pi^{-1}(x_i)} \Big(\mathrm{ind}_v^{\perp}(y) + 1 \Big) \end{aligned}$$

We now apply Riemann–Hurwitz and the classical Poincaré–Hopf formula.

$$\begin{aligned} 2\chi(M) = k + \chi(\widetilde{M}) &= k + \sum_{i=1}^{n} \sum_{y \in \pi^{-1}(x_i)} \operatorname{ind}_v(y) \\ &= k + \sum_{i=1}^{n} \sum_{y \in \pi^{-1}(x_i)} \left(\operatorname{ind}_v^{\perp}(y) + 1 \right) \\ &= k + (2n - k) + \sum_{i=1}^{n} \sum_{y \in \pi^{-1}(x_i)} \operatorname{ind}_v^{\perp}(y) \end{aligned}$$

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$$
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&= 2n + \sum_{i=1}^{n} \operatorname{p\,ind}_{\xi}^{\perp}(x_i)
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2\chi(M) = k + \chi(\widetilde{M}) &= k + \sum_{i=1}^{n} \sum_{y\in\pi^{-1}(x_i)} \mathrm{ind}_v(y) \\
&= k + \sum_{i=1}^{n} \sum_{y\in\pi^{-1}(x_i)} \Big(\mathrm{ind}_v^{\perp}(y) + 1\Big) \\
&= k + (2n-k) + \sum_{i=1}^{n} \sum_{y\in\pi^{-1}(x_i)} \mathrm{ind}_v^{\perp}(y) \\
&= 2n + \sum_{i=1}^{n} \mathrm{p\,ind}_{\xi}^{\perp}(x_i) \\
&= 2n + \sum_{i=1}^{n} \big(\mathrm{p\,ind}_{\xi}(x_i) - 2\big)
\end{aligned}$$

We now apply Riemann–Hurwitz and the classical Poincaré–Hopf formula.

$$\begin{aligned}
2\chi(M) = k + \chi(\widetilde{M}) &= k + \sum_{i=1}^{n} \sum_{y\in\pi^{-1}(x_i)} \operatorname{ind}_v(y) \\
&= k + \sum_{i=1}^{n} \sum_{y\in\pi^{-1}(x_i)} \Big(\operatorname{ind}_v^{\perp}(y) + 1\Big) \\
&= k + (2n-k) + \sum_{i=1}^{n} \sum_{y\in\pi^{-1}(x_i)} \operatorname{ind}_v^{\perp}(y) \\
&= 2n + \sum_{i=1}^{n} \operatorname{p\,ind}_\xi^{\perp}(x_i) \\
&= 2n + \sum_{i=1}^{n} \big(\operatorname{p\,ind}_\xi(x_i) - 2\big) \\
&= \sum_{i=1}^{n} \operatorname{p\,ind}_\xi(x_i). \qquad \square
\end{aligned}$$

Further problems

- Extend Hopf's differential-geometric proof to higher dimensions using the higher-dimensional Gauss–Bonnet Theorem (Allendoerfer–Weil, Chern).

Further problems

- Extend Hopf's differential-geometric proof to higher dimensions using the higher-dimensional Gauss–Bonnet Theorem (Allendoerfer–Weil, Chern).
- A projective k-frame assigns k pairwise orthogonal lines in the tangent space at each point. Give complete obstructions to the existence of a projective k-frame on M.