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Bordism Classes Represented by Multiple Point Manifolds of Immersed Manifolds

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Abstract—We present a geometrical version of Herbert's theorem determining the homology classes represented by the multiple point manifolds of a self-transverse immersion. Herbert's theorem and generalizations can readily be read off from this result. The simple geometrical proof is based on ideas in Herbert's paper. We also describe the relationship between this theorem and the homotopy theory of Thom spaces.

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1. INTRODUCTION

Suppose that $f\colon M \looparrowright N$ is a self-transverse immersion of M, a smooth compact closed $(n-k)$-dimensional manifold, into N, a smooth n-dimensional manifold without boundary. For integers $r \geq 1$, we define r-fold *self-intersection sets* of f in N and in M:

$$I_r(f) = \big\{y \in N \;\big|\; |f^{-1}(y)| \geq r\big\} \subseteq N, \qquad \widetilde{I}_r(f) = f^{-1}(I_r(f)) \subseteq M.$$

It follows from the self-transversality of f that each of these sets is the image of an immersion.

We recall the details in order to establish notation. Let

$$\overline{\Delta}_r(f) = \big\{(x_1, x_2, \dots, x_r) \in M^r \;\big|\; f(x_1) = f(x_2) = \ldots = f(x_r),\ i \neq j \Rightarrow x_i \neq x_j\big\}.$$

By the self-transversality of f, this is a submanifold of the Cartesian product M^r of codimension $(r-1)n$, i.e. of dimension $r(n-k)-(r-1)n = n-rk$. The symmetric group Σ_r acts freely on $\overline{\Delta}_r(f)$ by permuting the coordinates. Factoring out by this action (and by the action of the subgroup Σ_{r-1} acting on the last $r-1$ coordinates) gives two compact $(n-kr)$-manifolds:

$$\Delta_r(f) = \overline{\Delta}_r(f)/\Sigma_r, \qquad \widetilde{\Delta}_r(f) = \overline{\Delta}_r(f)/\Sigma_{r-1}.$$

These are the *multiple point manifolds* of f in N and in M, respectively.

We may define immersions

$$\theta_r(f)\colon \Delta_r(f) \looparrowright N, \qquad \widetilde{\theta}_r(f)\colon \widetilde{\Delta}_r(f) \looparrowright M$$

by $\theta_r(f)[x_1, x_2, \dots, x_r] = f(x_1)$ and $\widetilde{\theta}_r(f)(x_1, [x_2, \dots, x_r]) = x_1$, respectively. By definition, the image of $\theta_r(f)$ is $I_r(f)$ and the image of $\widetilde{\theta}_r(f)$ is $\widetilde{I}_r(f)$. Furthermore, there is a commutative

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diagram

$$\begin{array}{ccc} \widetilde{\Delta}_r(f) & \xrightarrow{\widetilde{\theta}_r(f)} & M \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle f} \\ \Delta_r(f) & \xrightarrow{\theta_r(f)} & N \end{array}$$

where π is the r-sheeted covering given by $\pi(x_1, [x_2, \dots, x_r]) = [x_1, \dots, x_r]$.

Ralph Herbert in his thesis [3] obtained a formula relating the homology classes in M and N represented by these immersions: the images of the fundamental classes of the multiple point manifolds in M and in N. This formula describes how the classes in M are determined by those in N together with the Euler class of the normal bundle of f. An alternative proof was provided by Felice Ronga [6] based on some ideas of Daniel Quillen's. However, we wished to understand the geometrical ideas behind Herbert's proof and to relate these to the first author's work on multiple points of immersions and the homotopy of Thom complexes (see, e.g., [1]). Herbert's proof appears long and complicated, but the basic ideas are quite simple.

2. BORDISM GROUPS OF IMMERSIONS

We write $\mathrm{Imm}_{n-k}(N) = \mathrm{Imm}^k(N)$ for the bordism group of immersions of compact closed $(n-k)$-manifolds in N, a manifold without boundary. The group structure is induced by the union of immersions. This is both a covariant functor on the category of smooth manifolds and immersions and also a contravariant functor, which is the reason for the dual notation.

For an immersion $g\colon N_1 \looparrowright N_2$, the homomorphism $\mathrm{Imm}_{n-k}(N_1) \to \mathrm{Imm}_{n-k}(N_2)$ is induced by composition with g. On the other hand, the homomorphism $g^*\colon \mathrm{Imm}^k(N_2) \to \mathrm{Imm}^k(N_1)$ is given by the pullback construction on a representing immersion that is transverse to g. When g is an embedding, the pullback construction is just a restriction to the submanifold mapping to $g(N_1)$.

There is a natural product

$$\mathrm{Imm}^{k_1}(N) \times \mathrm{Imm}^{k_2}(N) \to \mathrm{Imm}^{k_1+k_2}(N),$$

also given by a pullback construction, under which $\mathrm{Imm}^*(N) = \bigoplus_k \mathrm{Imm}^k(N)$ has the structure of a graded ring, so that Imm^* is a contravariant functor to the category of graded rings.

These definitions can be extended to manifolds N with boundary. In this case $\mathrm{Imm}^k(N)$ denotes the bordism group of immersions $f\colon M \looparrowright N$ of compact $(n-k)$-manifolds M in N such that $f(\partial M) \subseteq \partial N$ and f is transverse to ∂N. The definition of a bordism between such immersions involves the idea of a manifold with corners as described, for example, in [5]. The functor Imm_{n-k} extends to the category of smooth manifolds with boundary and immersions as above. However, the functor Imm^k extends also to immersions that do not respect the boundaries (such as the tubular neighborhood of an immersion, which we will use).

For smooth manifolds N without boundary, the dual covariant and contravariant view of this functor corresponds to Poincaré duality. To state this precisely, observe that an immersed $(n-k)$-manifold in N is a singular manifold and so represents an element of the bordism group $MO_{n-k}(N)$. This induces a homomorphism $\mathrm{Imm}_{n-k}(N) \to MO_{n-k}(N)$. On the other hand, by the Pontrjagin–Thom–Wells theorem [7],

$$\mathrm{Imm}^k(N) \cong \big[\Sigma^K N_+, \Sigma^K MO(k)\big]$$

for sufficiently large K, where N_+ denotes the one-point compactification of N, $MO(k)$ is the Thom space of the universal k-dimensional bundle, $\gamma^k\colon EO(k) \to BO(k)$, and Σ^K denotes the

K-fold suspension functor. Now the maps in the MO-spectrum induce a homomorphism

$$\bigl[\Sigma^K N_+, \Sigma^K MO(k)\bigr] \to \bigl[\Sigma^K N_+, MO(k+K)\bigr]$$

and $[\Sigma^K N_+, MO(k+K)]$ is isomorphic to $MO^k(N)$ since K is large.

Now the following commutative diagram gives the relationship with Poincaré duality:

$$\begin{array}{ccc}
\mathrm{Imm}_{n-k}(N) & \longrightarrow & MO_{n-k}(N) \\
\big\downarrow{=} & & \\
\mathrm{Imm}^k(N) & & \cong \big\downarrow \text{Poincaré duality} \\
\big\downarrow{\cong} & & \\
\bigl[\Sigma^K N_+, \Sigma^K MO(k)\bigr] & & \\
\big\downarrow & & \big\downarrow \\
\bigl[\Sigma^K N_+, MO(k+K)\bigr] & \xrightarrow{\ \cong\ } & MO^k(N)
\end{array}$$

Poincaré duality is used in the statement of Herbert's theorem. This diagram may be used to deduce Herbert's theorem from Theorem 2 stated in the next section.

3. THE FORMULA

3.1. The double point case. We first give the formula for double point manifolds from which we will deduce the general case.

Proposition 1. *Given a self-transverse immersion $f\colon M \looparrowright N$ of a smooth compact closed $(n-k)$-dimensional manifold into a smooth n-dimensional manifold without boundary, then*

$$f^*[f] = e + [\widetilde{\theta}_2(f)] \in \mathrm{Imm}^k(M),$$

where e is the Euler class $e = i^*[i]$, where $i\colon M \to E(\nu_f)$ is the zero section of the normal bundle of f.

Herbert observes (in the introduction to [3]) that the idea of this result goes back to Whitney [8]. The proposition states that the intersection of M with a deformation of M in N is given by the intersection of M with a deformation of M in $E(\nu_f)$ (the "near intersections") plus the double point set of f in M (the "distant intersections"). To achieve this result, we factorize f through a tubular neighborhood of f.

Let $F\colon E(\nu_f) \looparrowright N$ be an immersed tubular neighborhood of f which is injective on each fiber of ν_f, where $E(\nu_f)$ is the total space of the closed unit ball normal bundle of f. Then

$$f^*[f] = (F \circ i)^*[f] = i^*F^*[f].$$

Since F is automatically transverse to f, the class $F^*[f] \in \mathrm{Imm}^k(E(\nu_f))$ is represented by the pullback F^*f, an immersion $g\colon L \looparrowright E(\nu_f)$ of an $(n-k)$-dimensional manifold which is given by

$$L = \bigl\{(x,v) \in M \times E(\nu_f) \bigm| f(x) = F(v)\bigr\}$$

and

$$g(x,v) = v.$$

Notice that $L = L_1 \sqcup L_2$, where $L_1 = M = i(M) \subseteq M \times E(\nu_f)$ and $L_2 = \{(x,v) \in M \times F_{x'}(\nu_f) \mid x \neq x' \text{ and } f(x) = F(v)\}$, where $F_x(\nu_f)$ denotes the fiber over x. Then $i^*[g|L_1] = i^*[i] = e$ and a calculation shows that the pullback of $g|L_2$ is $\widetilde{\theta}_2(f)$, as required to prove the proposition.

To illustrate the proof, consider the special case of the figure-eight immersion $f\colon S^1 \looparrowright \mathbb{R}^2$ of the circle $S^1 \subseteq \mathbb{C}$ into the plane with a single double point $\mathbf{0} = f(1) = f(-1)$. For an appropriate choice of F, the image of $g|L_2$ is $\{\pm 1\} \times [-1,1] \subset S^1 \times [-1,1] = E(\nu_f)$. So in this case the pullback $i^*(g|L_2)$ is just a pair of points mapping to $\{\pm 1\} \subseteq S^1$, as we expect.

3.2. The general result. Our main result is as follows.

Theorem 2. *Given a self-transverse immersion $f\colon M \looparrowright N$ of a smooth compact closed $(n-k)$-dimensional manifold into a smooth n-dimensional manifold without boundary, then*

$$f^*[\theta_r(f)] = e.[\widetilde{\theta}_r(f)] + [\widetilde{\theta}_{r+1}(f)] \in \mathrm{Imm}^{rk}(M).$$

To prove this, observe that we may define a natural mapping $\theta_r\colon \mathrm{Imm}^k(N) \to \mathrm{Imm}^{rk}(N)$ by

$$\theta_r[f] = [\theta_r(f)].$$

Lemma 3.

$$\theta_r([f_1] + [f_2]) = \theta_r[f_1] + \sum_{i=1}^{r-1} \theta_{r-i}[f_1].\theta_i[f_2] + \theta_r[f_2].$$

This result is immediate from the definitions by choosing suitable representatives f_1 and f_2.

Lemma 4. *There is a natural diffeomorphism $\Delta_r(\widetilde{\theta}_2(f)) \cong \widetilde{\Delta}_{r+1}(f)$. Under this identification, $\theta_r(\widetilde{\theta}_2(f)) = \widetilde{\theta}_{r+1}(f)$.*

This is also immediate from the definitions.

The theorem now follows immediately from Proposition 1 as follows:

$$\begin{aligned}
f^*[\theta_r(f)] &= f^*\theta_r[f] && \text{(by definition)}\\
&= \theta_r f^*[f] && \text{(by naturality)}\\
&= \theta_r(e + [\widetilde{\theta}_2(f)]) && \text{(from Proposition 1)}\\
&= \theta_r e + \sum_{i=1}^{r-1} \theta_{r-i} e.\theta_i[\widetilde{\theta}_2(f)] + \theta_r[\widetilde{\theta}_2(f)] && \text{(from Lemma 3)}\\
&= e.\theta_{r-1}[\widetilde{\theta}_2(f)] + \theta_r[\widetilde{\theta}_2(f)] && \text{(since e is representable by an embedding and θ_1 is the identity)}\\
&= e.[\widetilde{\theta}_r(f)] + [\widetilde{\theta}_{r+1}(f)] && \text{(from Lemma 4),}
\end{aligned}$$

as required.

It is also possible to prove Theorem 2 directly by the same method as Proposition 1.

3.3. Comments on the theorem. Herbert's formula for nonoriented manifolds may be obtained from Theorem 2 using the ring homomorphism

$$\mathrm{Imm}^*(M) \to MO^*(M) \to H^*(M; \mathbb{Z}/2),$$

Poincaré duality, and the diagram at the end of Section 2.

We can work with immersions that are oriented with respect to some homology theory and then get more refined results, which will be explored elsewhere. In particular, in the case of immersions oriented with respect to integral homology, we may obtain Herbert's result for this case.

4. THE HOMOTOPY THEORY OF THOM SPACES

By the Pontrjagin–Thom–Wells theorem [7], $\mathrm{Imm}^k(N) \cong [N_+, QMO(k)]$. Here Q is the functor $\Omega^\infty\Sigma^\infty = \lim \Omega^K\Sigma^K$, where Ω denotes the based loop space functor. Under this isomorphism, the homomorphism $f^*\colon \mathrm{Imm}^k(N) \to \mathrm{Imm}^k(M)$ is induced by the composition of a map representing the homotopy class with $f_+\colon M_+ \to N_+$. It is natural to look for a formulation of Theorem 2 in terms of the representing objects.

Under the Pontrjagin–Thom–Wells isomorphism, the mapping

$$\theta_r\colon \mathrm{Imm}^k(N) \to \mathrm{Imm}^{rk}(N)$$

is induced by a map

$$QMO(k) \xrightarrow{h^r} QD_rMO(k) \xrightarrow{Q\zeta} QMO(rk)$$

(see, for example, [4]). Here $D_rMO(k)$ denotes the Thom space of

$$(\gamma^k)^r \times_{\Sigma_r} 1\colon EO(k)^r \times_{\Sigma_r} W\Sigma_r \to BO(k)^r \times_{\Sigma_r} W\Sigma_r,$$

the universal $(O(k) \wr \Sigma_r)$-bundle ($W\Sigma_r$ is a contractible space on which Σ_r acts freely), h^r denotes the stable James–Hopf map (see [2]), and ζ is the induced map of Thom spaces induced by the classifying map of the bundle. The normal bundles of the immersions $\theta_r(f)$ and $\widetilde{\theta}_{r+1}(f)$ each have a natural $((\gamma^k)^r \times_{\Sigma_r} 1)$-structure, and the formula of Theorem 2 remains true in the bordism group of immersions with such a structure on the normal bundles.

By an abuse of notation, we denote the map $N_+ \to QMO(k)$ corresponding to an immersion $f\colon M \looparrowright N$ also by f.

With this notation, Theorem 2 corresponds to the following homotopy commutative diagram:

$$\begin{array}{ccc}
M_+ & \xrightarrow{\quad f_+ \quad} & N_+ \\
\Big\downarrow{\scriptstyle \left(e\wedge h^{r-1}\circ\widetilde{\theta}_2(f)\right)\times h^r\circ\widetilde{\theta}_2(f)} & & \Big\downarrow{\scriptstyle h^r\circ f} \\
\big(MO(k) \wedge QD_{r-1}MO(k)\big) \times QD_rMO(k) & & QD_rMO(k) \\
\Big\downarrow{\scriptstyle (1\wedge Q\zeta)\times Q\zeta} & & \Big\downarrow{\scriptstyle Q\zeta} \\
\big(MO(k) \wedge QMO((r-1)k)\big) \times QMO(rk) & & QMO(rk) \\
\searrow & & \nearrow \\
 & QMO(rk) \times QMO(rk) &
\end{array}$$

In the diagram, the composition of the left-hand vertical maps corresponds to $e.[\widetilde{\theta}_r(f)]+[\widetilde{\theta}_{r+1}(f)]$ (using Lemma 4) and the composition of the right-hand vertical maps corresponds to $[\theta_r(f)]$.

We may prove the commutativity of this diagram by mimicking the proof of Theorem 2: first proving it in the case $r = 2$ and then deducing the general case by using an analogue of Lemma 3 for the maps h^r.

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