Lower bounds for the topological complexity of groups

(Joint with Greg Lupton and John Oprea)

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Plan

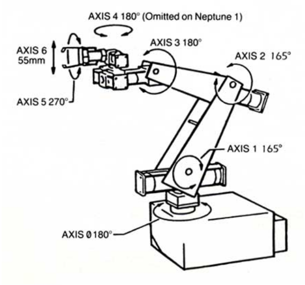
1. Topological complexity of robot motion planning
2. Topological complexity of groups
3. A new lower bound for $\mathrm{TC}(G)$
4. Examples
   - Pure braid groups
   - The Borromean rings
   - Higman's group
   - Further work

Topological complexity of robot motion planning

Topological complexity is a numerical homotopy invariant defined by Michael Farber in the early 2000s.

Its definition is motivated by the motion planning problem in Robotics.

Configuration spaces



Any mechanical system is parameterized by a topological space X, the configuration space of the system.

Points in X correspond to states or configurations of the system.

Paths in X correspond to motions of the system.

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More formally, consider the endpoint map

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A motion planning algorithm is a section of π_X, that is, a function $s\colon X \times X \to X^I$ such that $\pi_X \circ s = \mathrm{Id}_{X \times X}$.

The motion planning problem

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So motion planning algorithms in X often have essential discontinuities, due to the topology of X.

Topological complexity

Premise

It is desirable to find motion planning algorithms with fewest domains of continuity, since these will be optimally robust to changes in the input.

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Definition (Farber)

The topological complexity of a space X, denoted $\mathrm{TC}(X)$, is the least integer k such that $X \times X$ admits a cover by open sets $U_0, U_1, \ldots, U_k$, on each of which π_X admits a local section (a continuous map $s_i \colon U_i \to X^I$ such that $\pi_X \circ s_i = \mathrm{incl} \colon U_i \subseteq X \times X$). If no such integer exist, we set $\mathrm{TC}(X) = \infty$.

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Note that $\mathsf{TC}(X)$ is one less than the number of sets in the cover!

Topological complexity: basic properties

- If $X \simeq Y$ then $\mathsf{TC}(X) = \mathsf{TC}(Y)$ (homotopy invariance).
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Example

The topological complexity of the n-sphere $(n \geq 1)$ is given by

$$\mathrm{TC}(S^n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

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Theorem (Farber)

For any space X,

$$\mathrm{TC}(X) \geq \operatorname{nil}\ker(\cup).$$

Lusternik–Schnirelmann category

Definition

The (Lusternik–Schnirelmann) category of a space X, denoted $\mathrm{cat}(X)$, is the least integer k such that X admits a cover by open sets $U_0, U_1, \ldots, U_k$, with each inclusion $U_i \hookrightarrow X$ null-homotopic.

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Proposition

For any path-connected space X we have

$$\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq \mathrm{cat}(X \times X).$$

Topological complexity of groups

Recall that for any group G, one can construct a path-connected complex $K(G, 1)$ which has

$$\pi_i\big(K(G,1)\big) = \begin{cases} G & (i = 1), \\ 0 & (i > 1). \end{cases}$$

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Problem (Farber)

Describe $\mathsf{TC}(G) := \mathsf{TC}\big(K(G,1)\big)$ in terms of known algebraic invariants of the group G.

Cohomological dimension

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Theorem (Eilenberg–Ganea)

If $\mathrm{cd}(G) \geq 3$ then $\mathrm{cd}(G) = \mathrm{gd}(G)$, where $\mathrm{gd}(G)$ denotes the smallest possible dimension of a $K(G, 1)$ complex.

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If $\mathrm{cd}(G) = 1$ then G is a free group (and hence $\mathrm{cd}(G) = \mathrm{gd}(G)$).

The remaining question, of whether $\mathrm{cd}(G) = 2$ implies $\mathrm{gd}(G) = 2$, is known as the Eilenberg–Ganea conjecture.

Category of groups

Theorem (Eilenberg–Ganea, Stallings, Swan)

For any group G we have

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Examples

- If G is free then $\mathrm{cat}(G) = 1$.
- If G is an infinite surface group then $\mathrm{cat}(G) = 2$.
- If $G \cong \mathbb{Z}^n$ then $\mathrm{cat}(G) = n$.
- If G has torsion then $\mathrm{cat}(G) = \infty$.

Topological complexity of groups: a survey

Note that the inequalities

$$\mathrm{cd}(G) = \mathrm{cat}(G) \leq \mathrm{TC}(G) \leq \mathrm{cat}(G \times G) = \mathrm{cd}(G \times G)$$

show that $\mathrm{TC}(G) = \infty$ if G has torsion. So the problem is interesting mainly for torsion-free groups (of finite cohomological dimension).

Topological complexity of groups: a survey

Groups for which the exact value of $\mathsf{TC}(G)$ is known include:

Topological complexity of groups: a survey

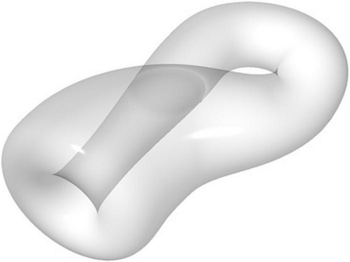
Groups for which the exact value of $\mathsf{TC}(G)$ is known include:

- Free abelian groups $\mathbb{Z}^n$ (Farber 2003)
- Orientable surface groups $\pi_1(\Sigma_g)$, $g \geq 1$ (Farber 2003)
- Free groups $\mathcal{F}_n$ (Farber 2004)
- Pure braid groups $\mathcal{P}_n = \pi_1\big(F_n(\mathbb{C})\big)$ (Farber–Yuzvinsky 2004)
- Pure braid groups of the punctured plane $\mathcal{P}_{n,m} = \ker(\mathcal{P}_n \to \mathcal{P}_m) = \pi_1\big(F_n(\mathbb{C} \setminus m \text{ points})\big)$ (Farber–G.–Yuzvinsky 2006)
- Right-angled Artin groups G_Γ (Cohen–Pruidze 2008)
- Basis-conjugating automorphism groups $P\Sigma_n$ and upper-triangular McCool groups $P\Sigma_n^+$ (Cohen–Pruidze 2008)
- Almost-direct products of free groups (Cohen 2010)
- Pure braid groups of surfaces $\pi_1\big(F_n(\Sigma_g)\big)$ (Cohen–Farber 2011)

Topological complexity of groups: a survey

Groups conspicuously missing from this list include:

- Finitely generated torsion-free nilpotent groups
- Non-orientable surface groups



A new lower bound for TC(G)

Theorem (G.–Lupton–Oprea)

Let A and B be subgroups of G such that $gAg^{-1} \cap B = \{1\}$ for every $g \in G$. Then

$$\operatorname{cd}(A \times B) \leq \operatorname{TC}(G).$$

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$$\mathrm{cd}(A \times B) \leq \mathrm{TC}(G).$$

Recall that A and B are complementary in G if $A \cap B = \{1\}$ and $AB = G$.

Corollary (G.–Lupton–Oprea)

Let A and B be complementary subgroups of G. Then

$$\mathrm{cd}(A \times B) \leq \mathrm{TC}(G).$$

Remarks

- The proof uses elementary homotopy theory together with properties of the sectional category under pullbacks.

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- This lower bound does not require knowledge of the cohomology ring structure of G, and can improve on the zero-divisors cup-length lower bound.
- It illustrates that $\mathrm{TC}(G)$ is related to the subgroup structure of G. For instance, upper bounds on $\mathrm{TC}(G)$ imply that certain pairs of subgroups have conjugate elements.

Pure braid groups

The pure braid group on n strands can be defined as

$$\mathcal{P}_n = \pi_1\big(F_n(\mathbb{C})\big),$$

where $F_n(\mathbb{C}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid i \neq j \implies z_i \neq z_j\}$ is the classical configuration space.

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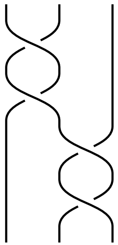
Theorem (Farber–Yuzvinsky)

We have

$$\mathrm{TC}(\mathcal{P}_n) = 2n - 3$$

for all $n \geq 2$.

Pure braid groups



Recall that elements of $\mathcal{P}_n$ can also be described geometrically as isotopy classes of braids, with the group operation given by concatenation.

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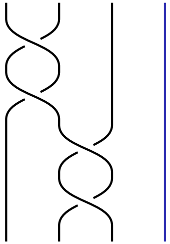
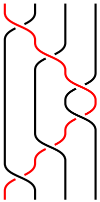


Recall that elements of $\mathcal{P}_n$ can also be described geometrically as isotopy classes of braids, with the group operation given by concatenation.

There is an inclusion $\mathcal{P}_{n-1} \hookrightarrow \mathcal{P}_n$ given by introducing an n-th non-interacting strand after the other strands.

Pure braid groups



For $j = 1, \ldots, n{-}1$, let α_j be the braid which runs the j-th strand in front of the last $n - j$ strands, then passes behind the last $n - j$ strands to its original position.

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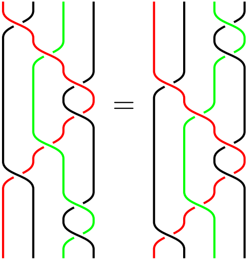

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The α_j's commute pairwise, so they generate a free abelian subgroup A of rank $(n - 1)$.

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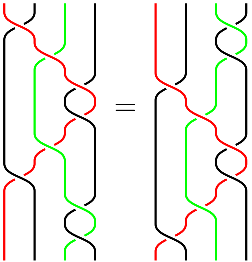

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Since conjugate braids close to isotopic links, one checks using linking numbers with the last strand that $gAg^{-1} \cap \mathcal{P}_{n-1} = \{1\}$ for all $g \in \mathcal{P}_n$.

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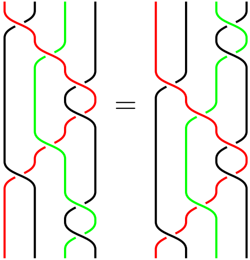

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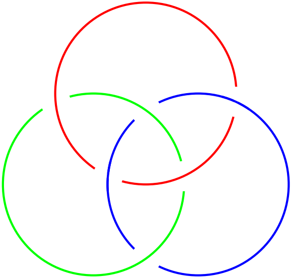
So the Theorem gives

$$\mathrm{TC}(\mathcal{P}_n) \geq \mathrm{cd}(A \times \mathcal{P}_{n-1}) = (n-1) + (n-2) = 2n-3.$$

The Borromean rings



The link complement X of the Borromean rings is a compact aspherical 3-manifold with fundamental group

$$G \cong \left\langle a, b, c \;\middle|\; [a, [b^{-1}, c]], [b, [c^{-1}, a]] \right\rangle.$$

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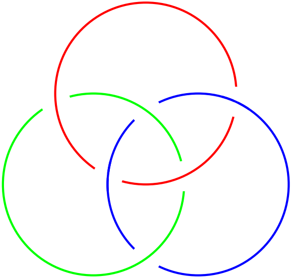

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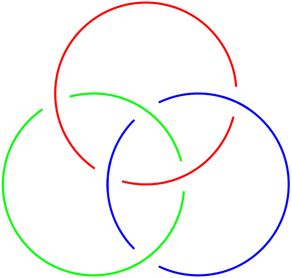

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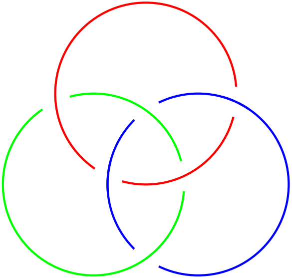
Using Massey products in $H^*(X; \mathbb{Q})$ and sectional category weight, we can show that $\mathrm{TC}(X) \geq 3$ (G., 2009).

The Borromean rings



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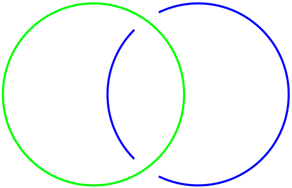

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There results a split extension

$$K \rightarrowtail G \overset{p}{\twoheadrightarrow} \mathcal{F}_2\langle \alpha, \beta \rangle,$$

$$p\colon \begin{cases} a \mapsto \alpha \\ b \mapsto \beta \\ c \mapsto 1 \end{cases}$$

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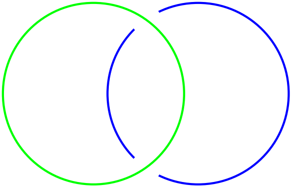

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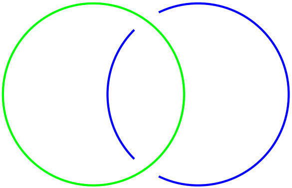

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Letting $A = \langle a \rangle$ and $B = p^{-1}\langle \beta \rangle$, one can show algebraically that $gAg^{-1} \cap B = \{1\}$ for all g in G.

Since B is not free, the Theorem gives

$$\mathrm{TC}(G) \geq \mathrm{cd}(A \times B) = 1 + 2 = 3.$$

Higman's group

Higman's group $\mathcal{H}$ is a finitely presented group with presentation

$$\left\langle x, y, z, w \mid xyx^{-1}y^{-2}, yzy^{-1}z^{-2}, zwz^{-1}w^{-2}, wxw^{-1}x^{-2} \right\rangle$$

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It has no non-trivial finite quotients. It follows that $H^*(\mathcal{H}; M)$ is trivial for any $\mathbb{Z}[\mathcal{H}]$-module M which is finitely generated as a $\mathbb{Z}$-module.

Higman's group

Higman's group $\mathcal{H}$ is a finitely presented group with presentation

$$\left\langle x, y, z, w \mid xyx^{-1}y^{-2}, yzy^{-1}z^{-2}, zwz^{-1}w^{-2}, wxw^{-1}x^{-2} \right\rangle$$

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The above presentation is aspherical, and so $\mathrm{cd}(\mathcal{H}) = 2$.

Higman's group

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We claim that $gH_{xy}g^{-1} \cap H_{zw} = \{1\}$ for all $g \in \mathcal{H}$. Hence

$$\mathrm{TC}(\mathcal{H}) \geq \mathrm{cd}(H_{xy} \times H_{zw}) = 4.$$

Higman's group

The proof of the claim (communicated to us by Yves Cornulier) uses Bass–Serre theory, and the following Lemmas:

Lemma

In an amalgam $G = A *_C B$, if an element of A is conjugate in G to an element of B, then it is conjugate in G to an element of C.

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□

Further work

- Obtain a more general result about $\mathrm{TC}(G)$ for $G = A *_C B$.
- Can our result be extended to deal with non-orientable surfaces?

Thanks for listening!