Self-intersections of immersions and Steenrod operations

Mark Grant (joint with Peter Eccles)

mark.grant@durham.ac.uk

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- **Problem** Can we find a formula relating these two types of operation?

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- If f has no r-fold self-intersections, then $\psi_r(f)$ represents the zero class
- So we may exploit our formula to find obstructions to a given cohomology class containing an embedding (immersion without double points)/immersion without triple points/...
- Regularly homotopic immersions represent the same cohomology class, so we may find cohomological obstructions to an immersion being regularly homotopic to an embedding

Immersions

- Immersions are smooth maps $f\colon\ M^{n-k} \looparrowright X^n$ with $df_x\colon\ TM_x \to TX_{f(x)}$ injective for each $x \in M$ (integer $k \geq 0$ is called the *codimension* of f)

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- The *normal bundle* of $f\colon\ M^{n-k} \looparrowright X^n$ is k-dimensional bundle ν_f over M defined by $\nu_f \oplus TM = f^*TX$ (uses a metric on X)

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- Let E^* be a generalised cohomology theory. We say $f\colon M \looparrowright X$ is E^**-oriented* if ν_f has a preferred Thom class $t \in \tilde{\mathsf{E}}^k(T\nu_f)$, giving a Thom isomorphism $\mathsf{E}^*(M) \cong \tilde{\mathsf{E}}^{*+k}(T\nu_f)$

Classes represented by immersions

Let $f\colon\ M^{n-k} \looparrowright X^n$ be a proper immersion oriented with respect to some generalised cohomology theory E^*. Then f represents a class $[f] \in \mathsf{E}^k(X)$ as follows:

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- $g = (f, f')\colon M \hookrightarrow X \times \mathbb{R}^\ell$ regularly homotopic to $(f, 0)\colon M \looparrowright X \times \{0\} \hookrightarrow X \times \mathbb{R}^\ell$, so $\nu_g \cong \nu_f \oplus \varepsilon^\ell$ and $T\nu_g \simeq \Sigma^\ell T\nu_f$

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- $[f] \in \mathsf{E}^k(X)$ is image of $1 \in \mathsf{E}^0(M)$ under

$$\mathsf{E}^0(M) \cong \tilde{\mathsf{E}}^k(T\nu_f) \cong \tilde{\mathsf{E}}^{k+\ell}(\Sigma^\ell T\nu_f) \cong$$

$$\tilde{\mathsf{E}}^{k+\ell}(T\nu_g) \to \tilde{\mathsf{E}}^{k+\ell}\big((X \times \mathbb{R}^\ell)_+\big) \cong \tilde{\mathsf{E}}^{k+\ell}(\Sigma^\ell X_+) \cong \mathsf{E}^k(X)$$

Self-intersections of immersions

- $f\colon M \looparrowright X$ is *self-transverse* if whenever $x_1, \ldots, x_n \in M$ are distinct points mapping to $y \in X$, the $df_{x_i}TM_{x_i}$ are in general position in TX_y

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- A self-transverse immersion f has r*-fold multiple point manifolds*

$$\overline{\Delta}_r(f) = \{(x_1, \ldots, x_r) \in M^{(r)} \mid x_i \text{ distinct, } f(x_i) = f(x_j)\}$$

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- f induces r*-fold self-intersection immersions*

$$\overline{\psi}_r(f)\colon \overline{\Delta}_r(f) \looparrowright X, \quad \psi_r(f)\colon \Delta_r(f) \looparrowright X,$$

$$(x_1, \ldots, x_r) \mapsto f(x_1), \quad [x_1, \ldots, x_r] \mapsto f(x_1)$$

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- The normal fibre of $\psi_r(f)$ at $[x_1, \ldots, x_r]$ is the unordered direct sum of the normal fibres of f at $x_1, \ldots, x_r$
- If f is E*-oriented, then $\psi_r(f)$ *may* be E*-oriented (related to existence of Steenrod operations in E*).

  However this does *not* give a well-defined cohomology operation $[f] \mapsto [\psi_r(f)]$

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$$f\colon\ M \xhookrightarrow{\ i\ } E \xrightarrow{\ p\ } X,$$

where $p\colon\ E \to X$ is a smooth vector bundle, $i\colon\ M \hookrightarrow E$ an embedding and ν_i is $M\Gamma^*$-orientable

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- A Γ-*map* is a map of manifolds $f\colon\ M \to X$ together with an equivalence class of Γ-orientations of f

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- **Examples** The identity $\mathrm{id}\colon X \to X$ is trivially a Γ-map. A Γ-orientation of the map $f\colon M \to \{\mathrm{pt}\}$ is a Γ-orientation of M

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- $f\colon M \looparrowright X$ is $M\Gamma^*$-oriented $\implies f\colon M \to X$ is Γ-oriented, since $g = (f, f')\colon M \hookrightarrow X \times \mathbb{R}^\ell$ has a canonical Thom class $\Sigma^\ell t$, where $t \in M\Gamma^k(T\nu_f)$ is Thom class of ν_f. Converse is false.

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- **Proposition**(Quillen, after Thom-Pontrjagin) If X is a manifold, one may put a *cobordism* relation $\sim$ on the set of proper Γ-maps to X of codimension k such that

$$M\Gamma^k(X) \cong \{\text{proper } \Gamma\text{-maps } f\colon M^{n-k} \to X^n\}/(\sim)$$

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- **Covariance** Proper Γ-map $h\colon\ X^n \to Z^{n+m}$ induces $h_*\colon\ M\Gamma^*(X) \to M\Gamma^{*+m}(Z)$ by $[f] \mapsto [h \circ f]$.

  (Note that $[f] = f_*(1)$, where $1 = [\mathrm{id}\colon\ M \to M] \in M\Gamma^0(M)$)

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where $\triangle\colon X \to X \times X$ is diagonal $x \mapsto (x, x)$

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- **Poincaré-Atiyah duality** If X^n is a closed Γ-manifold, then $M\Gamma^k(X) \cong M\Gamma_{n-k}(X)$
- **Euler class** of a k-dimensional $M\Gamma^*$-oriented vector bundle is $e(\xi) = i^*[i] \in M\Gamma^k(X)$, where $i\colon\ X \to E$ is zero section

Steenrod operations

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- **Notation** Let Y be a $\mathbb{Z}_2$-space, S^ℓ sphere with antipodal involution, and define

$$S^\ell(Y) := S^\ell \times_{\mathbb{Z}_2} Y, \quad S(Y) := S^\infty \times_{\mathbb{Z}_2} Y = \bigcup_\ell S^\ell(Y)$$

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- **Examples** $S^\ell(X \times X)$, where $X \times X$ has a natural involution $T(x, y) = (y, x)$. If Y is trivial $\mathbb{Z}_2$-space, $S^\ell(Y) = P^\ell \times Y$

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- **Definition** An *external Steenrod operation* of type $(\mathbb{Z}_2, d)$ in E^* is a series of natural transformations

$$R = R^{dk} \colon \mathsf{E}^{dk}(X) \to \mathsf{E}^{2dk}\big(S(X \times X)\big), \quad k \in \mathbb{Z}$$

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- **Definition** Composing with the map induced by the *extended diagonal* $\triangle_2\colon\ P^\infty \times X \hookrightarrow S(X \times X)$, $([v], x) \mapsto [v, x, x]$ gives a *Steenrod operation*

$$\mathscr{R} = \mathscr{R}^{dk}\colon\ \mathsf{E}^{dk}(X) \to \mathsf{E}^{2dk}(P^\infty \times X), \quad k \in \mathbb{Z}$$

Steenrod operations

- **Example** Steenrod's original operation (of type $(\mathbb{Z}_2, 1)$)

$$R\colon\ H^k(X;\mathbb{Z}_2) \to H^{2k}(S(X\times X);\mathbb{Z}_2)$$

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- Since $H^*(P^\infty \times X;\mathbb{Z}_2) \cong \mathbb{Z}_2[\mu] \otimes H^*(X;\mathbb{Z}_2)$, can define *internal operations* $\mathrm{Sq}^i\colon\ H^k(X;\mathbb{Z}_2) \to H^{k+i}(X;\mathbb{Z}_2)$ by

$$\mathscr{R}(\alpha) = \sum_{i=0}^{\infty} \mu^{k-i} \otimes \mathrm{Sq}^i(\alpha), \quad \alpha \in H^k(X;\mathbb{Z}_2)$$

Steenrod-tom Dieck operations

Proposition(tom Dieck) For $\Gamma = O, U, Sp, SO, SU$ there is a Steenrod operation $\mathscr{R}$ of type $(\mathbb{Z}_2, d)$ in $M\Gamma^*$, where $d = 1, 2, 4, 2, 4$.

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- Let $f\colon\ M^{n-dk} \to X^n$ be a proper Γ-map of codimension dk, and consider the proper codimension $2dk$ map

$$S^\ell(f \times f) = 1 \times_{\mathbb{Z}_2} (f \times f)\colon\ S^\ell(M \times M) \to S^\ell(X \times X)$$

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- $S^\ell(f \times f)$ has a canonical Γ-orientation coming from Γ-orientation of f and $M\Gamma^*$-orientation of *extended power bundles* $S\big(\Gamma(d*) \times \Gamma(d*)\big)$

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- Then $\mathscr{R}_\ell = \iota_\ell^* \circ \mathscr{R}$, where $\iota_\ell \colon\ P^\ell \times X \to P^\infty \times X$ is inclusion

Statement of result

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Theorem(Eccles, G) For any natural number ℓ,

$$\mathscr{R}_\ell[f] = [S^\ell\big(\overline{\psi}_2(f)\big)] + (1 \times f)_* e(\gamma_\ell \otimes \nu_f) \in M\Gamma^{2dk}(P^\ell \times X)$$

Here γ_ℓ is the canonical line bundle over P^ℓ, and $\otimes$ denotes exterior tensor product of vector bundles

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Remark Quillen (1971) showed that when f is embedding,

$$\mathscr{R}_\ell[f] = (1 \times f)_* e(\gamma_\ell \otimes \nu_f) \in M\Gamma^{2dk}(P^\ell \times X)$$

Possible applications

- The class $[S^\ell(\overline{\psi}_2(f))] = \mathscr{R}_\ell[f] - (1 \times f)_* e(\gamma_\ell \otimes \nu_f)$ is an obstruction to f being cohomologous to an embedding

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- **Problem** Find an $M\Gamma^*$-oriented immersion f with $[S^\ell(\overline{\psi}_2(f))] \neq 0 \in M\Gamma^{2dk}(P^\ell \times X)$

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- It vanishes when mapped to $\mathbb{Z}_2$-cohomology, giving the relation

$$\mathrm{Sq}^i[f] = f_* w_i(\nu_f) \quad \text{for all } i,$$

where w_i denotes i-th Stiefel-Whitney class

Possible applications

- The class $[S^\ell(\overline{\psi}_2(f))] = \mathscr{R}_\ell[f] - (1 \times f)_* e(\gamma_\ell \otimes \nu_f)$ is an obstruction to f being cohomologous to an embedding
- **Problem** Find an $M\Gamma^*$-oriented immersion f with $[S^\ell(\overline{\psi}_2(f))] \neq 0 \in M\Gamma^{2dk}(P^\ell \times X)$
- It vanishes when mapped to $\mathbb{Z}_2$-cohomology, giving the relation

  $$\mathrm{Sq}^i[f] = f_* w_i(\nu_f) \quad \text{for all } i,$$

  where w_i denotes i-th Stiefel-Whitney class
- **Extensions** Use twisted integer cohomology to get more refined results? Steenrod operations at primes $p \neq 2$?

Sub-cartesian diagrams

Definition(Ronga) The diagram of proper immersions

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & B \\ \alpha\downarrow & & \downarrow f \\ A & \xrightarrow[g]{} & X \end{array}$$

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The bundle $\xi = \mathrm{coker}(dg - df)$ is called the *excess bundle*

Sub-cartesian diagrams

Proposition(Ronga) Suppose that in the sub-cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[g]{} & X \end{array}$$

the bundles $\xi = \operatorname{coker}(dg - df)$ and ν_f are E*-oriented.

Then for any $c \in$ E$^*(B)$ we have

$$g^* f_*(c) = \alpha_*\big(e(\xi) \cdot \beta^*(c)\big) \in \mathsf{E}^*(A)$$

Idea of proof

Lemma The following diagram is sub-cartesian:

$$\begin{array}{ccc} S^\ell\big(\overline{\Delta}_2(f)\big) \sqcup P^\ell \times M & \longrightarrow & S^\ell(M \times M) \\ \Big\downarrow{\scriptstyle S^\ell\left(\overline{\psi}_2(f)\right)\sqcup 1\times f} & & \Big\downarrow{\scriptstyle S^\ell(f\times f)} \\ P^\ell \times X & \xrightarrow[\triangle_2^\ell]{} & S^\ell(X \times X). \end{array}$$

The excess bundle is zero over $S^\ell\big(\overline{\Delta}_2(f)\big)$, and $S^\ell(\nu_f)$ over $S^\ell(M) = P^\ell \times M$, where ν_f is regarded as a $\mathbb{Z}_2$-bundle over the trivial $\mathbb{Z}_2$-space M via $v \mapsto -v$.

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The Theorem follows on applying the Proposition to this square, with $c = 1 \in M\Gamma^0\big(S^\ell(M \times M)\big)$, since $S^\ell(\nu_f) \cong \gamma_\ell \otimes \nu_f$