Symmetrized topological complexity

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Topological complexity of motion planning

Definition

The sectional category (or Schwarz genus) of a fibration $p : E \to B$, denoted $\operatorname{secat}(p)$, is the minimum k such that B admits a cover by open sets $U_0, U_1, \ldots, U_k$, each of which admits a local section $s_i : U_i \to E$ of p.

For any space X, consider the free path fibration

$$\pi_X : PX \to X \times X, \qquad \pi_X(\gamma) = \big(\gamma(0), \gamma(1)\big).$$

Definition (Farber, 2003)

The topological complexity of X, denoted $\mathrm{TC}(X)$, is

$$\mathrm{TC}(X) := \operatorname{secat}(\pi_X).$$

Motivation

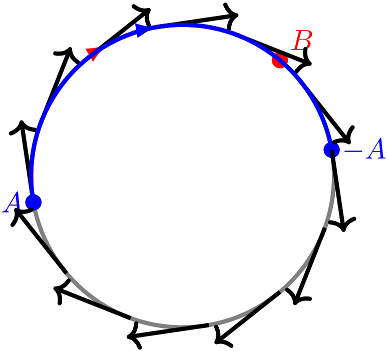
If X is the configuration space of a mechanical system, then sections of π_X are motion planning algorithms for that system.

The simplest[1] motion planning algorithms are continuous (this requires X to be contractible).

The number $\mathrm{TC}(X)$ quantifies the minimum complexity[1] of motion planning algorithms in systems whose configuration space is homotopy equivalent to X.

[1] From the topological viewpoint.

Example: odd spheres



$$U_0 = \{(A, B) \mid A \neq -B\}$$

$s_0(A, B) =$ shortest path, unit speed

$$U_1 = \{(A, -A)\}$$

$s_1(A, -A) =$ geodesic arc with initial velocity $v(A)$

This shows $\mathrm{TC}(S^{\mathrm{odd}}) \leq 1$. Then $S^{\mathrm{odd}} \not\simeq *$ implies $\mathrm{TC}(S^{\mathrm{odd}}) = 1$.

Example: even spheres

A similar construction (using a vector field on S^{even} with one singularity) gives $\mathrm{TC}(S^{\text{even}}) \leq 2$. The lower bound $\mathrm{TC}(S^{\text{even}}) \geq 2$ comes from

Theorem (Farber)

Suppose there exist cohomology classes

$$u_1, \ldots, u_k \in \ker\left(\Delta^* : H^*(X \times X) \to H^*(X)\right),$$

(where $\Delta : X \to X \times X$ is the diagonal) such that $u_1 \cdots u_k \neq 0$.
Then $\mathrm{TC}(X) \geq k$.

Symmetric motion planning

We may impose additional conditions on our motion planning algorithms, such as that they are:

Symmetric The motion from B to A is the reverse of the motion from A to B;

Monoidal The motion from A to A is constant at A.

These lead to several variants of $\mathrm{TC}(X)$.

Symmetric topological complexity

Restricting the path fibration $\pi_X : PX \to X \times X$ results in a fibration

$$\pi'_X : P'X \to F(X,2),$$

where $P'X$ denotes the space of paths in X with distinct endpoints, and $F(X,2) = \{(x,y) \in X \times X \mid x \neq y\}$.

Both $P'X$ and $F(X,2)$ are free $\mathbb{Z}_2$-spaces, and π'_X is equivariant.

Definition (Farber, 2005, Farber–G, 2006)

The symmetric topological complexity of X, denoted $\mathrm{TC}^S(X)$, is

$$\mathrm{TC}^S(X) := \mathrm{secat}\left(\pi'_X/\mathbb{Z}_2 : P'X/\mathbb{Z}_2 \to F(X,2)/\mathbb{Z}_2\right) + 1.$$

Immersion and embedding dimensions

Given a smooth manifold M, define

$$\mathrm{Imm}(M) = \min\{k \in \mathbb{Z} \mid M \text{ immerses in } \mathbb{R}^k\},$$

$$\mathrm{Emb}(M) = \min\{k \in \mathbb{Z} \mid M \text{ embeds in } \mathbb{R}^k\}.$$

Theorem (Farber–Tabachnikov–Yuzvinsky, 2003)

If $n \neq 1, 3, 7$ then

$$\mathrm{TC}(\mathbb{R}P^n) = \mathrm{Imm}(\mathbb{R}P^n).$$

Theorem (González–Landweber, 2009)

If $n \neq 6, 7, 11, 12, 14, 15$ then

$$\mathrm{TC}^S(\mathbb{R}P^n) = \mathrm{Emb}(\mathbb{R}P^n).$$

$\mathrm{TC}^S(-)$ is not a homotopy invariant

Convention: If $E = \varnothing = B$, then $\mathrm{secat}(p : E \to B) = -1$.

With this convention,

$$\mathrm{TC}^S(*) = -1 + 1 = 0,$$

whereas a contractible space X with $|X| > 1$ has

$$\mathrm{TC}^S(X) \geq 0 + 1 = 1.$$

Symmetrized topological complexity

We can consider

$$\pi_X : PX \to X \times X, \qquad \pi_X(\gamma) = \big(\gamma(0), \gamma(1)\big)$$

as a $\mathbb{Z}_2$-equivariant map.

Definition (Basabe–González–Rudyak–Tamaki, 2014)

The symmetrized topological complexity of X, denoted $\mathrm{TC}^{\Sigma}(X)$, is the minimum k such that $X \times X$ admits a cover by invariant open sets $U_0, U_1, \ldots, U_k$, each of which admits an equivariant local section $\sigma_i : U_i \to PX$ of π_X.

$\mathrm{TC}^{\Sigma}(-)$ has the following properties:

(1) $\mathrm{TC}^{\Sigma}(X) = \mathrm{TC}^{\Sigma}(Y)$ if $X \simeq Y$;

(2) $\mathrm{TC}(X) \leq \mathrm{TC}^{\Sigma}(X)$;

(3) $\mathrm{TC}^{S}(X) - 1 \leq \mathrm{TC}^{\Sigma}(X) \leq \mathrm{TC}^{S}(X)$ for X an ENR.

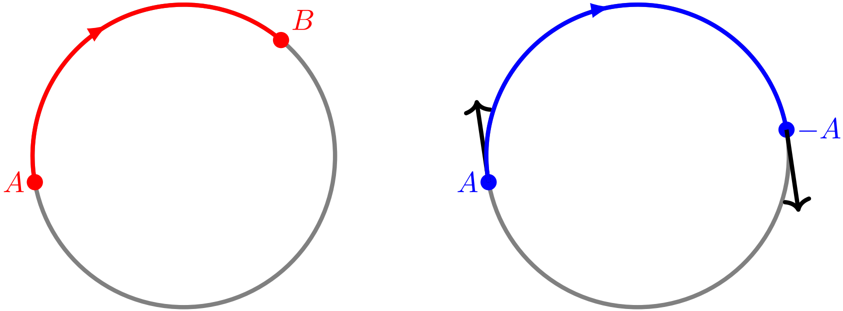
From (2), (3) and the fact that $\mathrm{TC}^{S}(S^n) = 2$ for all n, we have

$$\mathrm{TC}^{\Sigma}(S^{\mathrm{even}}) = 2, \quad \text{and}$$

$$1 \leq \mathrm{TC}^{\Sigma}(S^{\mathrm{odd}}) \leq 2.$$

Recall the proof that $\mathrm{TC}(S^{\mathrm{odd}}) \leq 1$.

Although $U_0 = \{(A, B) \mid A \neq -B\}$ and $U_1 = \{(A, -A)\}$ are invariant, and s_0 is equivariant, s_1 is not equivariant.



Equivariant sectional category

Let G be a compact Lie group.

Definition (Colman–G, 2012)

The equivariant sectional category of a G-fibration $p : E \to B$, denoted $\mathrm{secat}_G(p)$, is the minimum k such that B admits a cover by invariant open sets $U_0, \ldots, U_k$, each of which admits an equivariant local section $s_i : U_i \to E$ of p.

In particular,

$$\mathrm{TC}^{\Sigma}(X) = \mathrm{secat}_{\mathbb{Z}_2}(\pi_X).$$

The $(k+1)$-fold fibred join of a fibration $p : E \to B$ is a fibration

$$p_k : J^k_B(E) \to B$$

with fibre $J^k(F)$, the $(k+1)$-fold join of the fibre of p.

If p is a G-fibration, then so is p_k for $k \geq 0$.

The following generalizes a result of Schwarz.

Proposition (G, 2017)

Let $p : E \to B$ be a G-fibration over a paracompact base space. Then $\mathrm{secat}_G(p) \leq k$ if and only if $p_k : J^k_B(E) \to B$ admits a (global) G-section.

The obstructions to finding a G-section of $p_k : J^k_B(E) \to B$ live in Bredon cohomology groups

$$H^{i+1}_G(B; \pi_i(J^k \mathscr{F}))$$

where the local coefficients are given by

$$\pi_i(J^k \mathscr{F})(G/H) = \pi_i(J^k(F)^H) = \pi_i(J^k(F^H)).$$

Corollary

Let $p : E \to B$ be a G-fibration, with B a G-CW complex of $\dim B \geq 2$. Assume $\pi_i(F^H) = 0$ for all subgroups $H \leq G$ and all $i < s$, some $s \geq 0$. Then

$$\mathrm{secat}_G(p) \leq \frac{\dim B}{s+1}.$$

Theorem (G, 2017)

Let X be an s-connected polyhedron. Then

$$\mathrm{TC}^{\Sigma}(X) \leq \frac{2\dim X}{s+1}.$$

Proof: Apply previous Corollary to $\pi_X : PX \to X \times X$, and note:

- $X \times X$ can be given a $\mathbb{Z}_2$-CW structure;
- Fibre of π_X is based loop space ΩX, and $\Omega X^{\mathbb{Z}_2} \approx P_0 X$, the based path space.

□

Compare Farber's (2004) upper bound for $\mathrm{TC}(X)$.

Lower bounds for $\mathrm{TC}^{\Sigma}(X)$

Lower bounds for $\mathrm{TC}^{\Sigma}(X)$ are given by 'zero-divisors cup-length' in $\mathbb{Z}_2$-equivariant cohomology.

More precisely, if h^* is any $\mathbb{Z}_2$-equivariant cohomology theory with products, and there exist

$$u_1, \ldots, u_k \in \ker\big(\Delta^* : h^*(X \times X) \to h^*(X)\big)$$

with $u_1 \cdots u_k \neq 0$, then $\mathrm{TC}^{\Sigma}(X) \geq k$.

Can this be used to prove $\mathrm{TC}^{\Sigma}(S^{\mathrm{odd}}) \geq 2$?

With $h^*(-) = H^*(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} -)$ (Borel cohomology, constant coefficients), there are insufficient products.

With $h^*(-) = H^*_{\mathbb{Z}_2}(-)$ (Bredon cohomology), very few computations of cup products are known.

Finally we tried the most naïve thing, $h^*(-) = H^*(-/\mathbb{Z}_2)$, the cohomology of the orbit space.

Let $SP^2(X) = (X \times X)/\mathbb{Z}_2$, the symmetric square of X.

Denote by $dX \subseteq SP^2(X)$ the image of the diagonal $\Delta X \subseteq X \times X$.

Theorem (G, 2017)

Suppose there are $u_1, \ldots, u_k \in H^*(SP^2(X))$ which restrict to zero in $H^*(dX)$, such that $u_1 \cdots u_k \neq 0$. Then $\mathrm{TC}^{\Sigma}(X) \geq k$.

Theorem (G, 2017)

We have $\mathrm{TC}^{\Sigma}(S^n) = 2$ for $n > 1$ odd.

Proof: Nakaoka (1956) made extensive computations of the cohomology rings of symmetric powers.

In particular, his work shows there is an element $x \in H^n(SP(S^n); \mathbb{Z}_2)$ which restricts to zero in $H^n(dS^n; \mathbb{Z}_2)$ and has $x^2 \neq 0$.

Therefore $\mathrm{TC}^{\Sigma}(S^n) \geq 2$. $\square$

Remarks

- González (2017) has applied symmetric squares and Nakaoka's results to prove that

$$\mathrm{TC}^{\Sigma}(\mathbb{R}P^{2^e}) = \mathrm{TC}^{S}(\mathbb{R}P^{2^e}) = \mathrm{Emb}(\mathbb{R}P^{2^e}) = 2^{e+1} \qquad \text{for } e \geq 1.$$

- Since $SP^2(S^1) \approx \mathrm{Möb} \simeq S^1$, we cannot deduce that $\mathrm{TC}^{\Sigma}(S^1) \geq 2$.

Monoidal topological complexity

The definition of $\mathrm{TC}^{\Sigma}(X)$ does not incorporate the condition

Monoidal The motion from A to A is constant at A.

Definition (Iwase–Sakai, 2010)

The monoidal topological complexity of X, denoted $\mathrm{TC}^{M}(X)$, is the minimum k such that $X \times X$ admits a cover by open sets $U_0, U_1, \ldots, U_k$, each of which contains the diagonal ΔX and admits a local section $s_i : U_i \to PX$ of π_X satisfying $s_i(A, A) = \mathrm{const}_A$.

When X is an ENR, Iwase–Sakai showed that

$$\mathrm{TC}(X) \leq \mathrm{TC}^M(X) \leq \mathrm{TC}(X) + 1.$$

Conjecture (Iwase–Sakai, 2012)

For any locally finite simplicial complex X, we have $\mathrm{TC}^M(X) = \mathrm{TC}(X)$.

Theorem (Dranishnikov, 2014)

Let X be an s-connected simplicial complex such that

$$(s+1)(\mathrm{TC}(X)+1) > \dim X + 1.$$

Then $\mathrm{TC}^M(X) = \mathrm{TC}(X)$.

Monoidal symmetrized topological complexity

Definition

The monoidal symmetrized topological complexity of X, denoted $\mathrm{TC}^{M,\Sigma}(X)$, is the minimum k such that $X \times X$ admits a cover by invariant open sets $U_0, U_1, \ldots, U_k$, each of which contains the diagonal ΔX and admits a local equivariant section $s_i : U_i \to PX$ of π_X satisfying $s_i(A, A) = \mathrm{const}_A$.

Theorem (G, 2017)

Let X be a paracompact ENR. Then $\mathrm{TC}^{M,\Sigma}(X) = \mathrm{TC}^{\Sigma}(X)$.

Proof: Using relative $\mathbb{Z}_2$-homotopy lifting, deform an equivariant section $\sigma : X \times X \to J^k_{X\times X}(PX)$ to another such σ' which has $\sigma'|_{\Delta X}$ given by constant paths.

This requires:

- $\Delta X \hookrightarrow X \times X$ is a $\mathbb{Z}_2$-cofibration.
- $(\pi_X)_k : J^k_{X\times X}(PX) \to X \times X$ is a $\mathbb{Z}_2$-fibration.
- $J^k(\Omega X^{\mathbb{Z}_2}) \simeq *$.

□

Corollary

Let X be a paracompact ENR. Suppose there are relative classes $u_1, \ldots, u_k \in H^*(SP^2(X), dX)$ such that $u_1 \cdots u_k \neq 0$.
Then $\mathrm{TC}^{\Sigma}(X) = \mathrm{TC}^{M,\Sigma}(X) \geq k$.

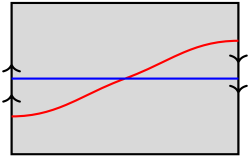
Proof uses the long exact sequence of a triple $(SP^2(X), \bar{U}_i, dX)$.

Theorem (G, 2017)

$$\mathrm{TC}^{\Sigma}(S^1) = 2.$$

Proof: Since $(SP^2(S^1), dS^1) \approx (\mathrm{M\ddot{o}b}, \partial\,\mathrm{M\ddot{o}b})$, it is sufficient to find $u \in H^1(\mathrm{M\ddot{o}b}, \partial\,\mathrm{M\ddot{o}b}; \mathbb{Z}_2)$ such that $u^2 \neq 0$.

We may take u to be the Poincaré dual of the core circle.



□

Remarks

- Don Davis (2017) has proved $\mathrm{TC}^{\Sigma}(S^1) = 2$ using theorems from general topology.
- Jesús González (2017) has used Nakaoka's results to show that $\mathrm{TC}^{\Sigma}(S^1 \times S^1) \geq 3$, which combined with the product inequality

$$\mathrm{TC}^{\Sigma}(X \times Y) \leq \mathrm{TC}^{\Sigma}(X) + \mathrm{TC}^{\Sigma}(Y)$$

also implies the above result.

Further work

- Analogues of all our results hold for symmetrized higher topological complexity $\mathrm{TC}_m^{\Sigma}(X)$. However, we do not know if

$$\mathrm{TC}_m^{\Sigma}(S^{\mathrm{odd}}) = m \qquad \text{for all } m > 2.$$

- Find a homotopically interesting space X with $\mathrm{TC}^{\Sigma}(X) < \mathrm{TC}^{S}(X)$.

- Define rational symmetrized topological complexity $\mathrm{TC}_{\mathbb{Q}}^{\Sigma}(X)$, and describe it in terms of equivariant minimal models.

Reference

M. Grant, *Symmetrized topological complexity*, arXiv:1703.07142

Thanks for listening!