On immersions in homology classes

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Steenrod's problem

A map $f\colon M^m \to N^n$ of closed (smooth) manifolds represents both $f_*[M] \in H_m(N)$ and its Poincaré dual cohomology class $x \in H^{n-m}(N)$ (where $[M] \in H_m(M)$ is a fundamental class)

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Thom (1954)

- Every homology class in $H_*(N;\mathbb{Z}_2)$ is representable
- The corresponding result for $H_*(N;\mathbb{Z})$ is false (may exist classes which cannot be represented by oriented manifolds)
- Given $a \in H_*(N;\mathbb{Z})$ there exists a positive integer k such that $k \cdot a \in H_*(N;\mathbb{Z})$ is representable

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Thom (1950)

If $x \in H^k(N;\mathbb{Z}_2)$ is represented by $f\colon M^{n-k} \looparrowright N^n$, then

$$Sq^i(x) = f_!(w_i(\nu_f))$$

- "Steenrod squares are the pushforwards of the SW-classes of the normal bundle of a representing immersion"

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- Each $O(k)$-bundle ν comes equipped with a map

$$T\nu \to MO(k) \qquad \text{(the universal Thom space)}$$

  and hence a unique Thom class $U_\nu \in \tilde{H}^k(T\nu)$, the image in cohomology of the universal Thom class

$$U_k \in \tilde{H}^k(MO(k))$$

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- Then $g\colon M^{n-k} \hookrightarrow N^n$ represents the cohomology class

$$x = G^*(U_k) \in \tilde{H}^k(N_+) \cong H^k(N)$$

Thom (1954)

A class $x \in H^k(N)$ is representable by an embedding if, and only if, there is a map $G\colon N_+ \to MO(k)$ such that $G^*(U_k) = x$

Representing classes by embeddings

Corollary

In every dimension $k > 1$ there exist mod 2 cohomology classes $x \in H^k(N)$ (where $\dim N > 2k$) which cannot be represented by embeddings

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Sketch proof:

$$H^*(K(\mathbb{Z}_2, k)) \cong \mathbb{Z}_2\left[Sq^I(\iota_k) \mid I \text{ admissible of excess } e(I) < k\right] \qquad \text{(Serre)}$$

$$H^*(MO(k)) \cong w_k\mathbb{Z}_2[w_1, \ldots, w_k] \qquad \text{(Thom)}$$

Here $\iota_k \in H^k(K(\mathbb{Z}_2, k))$ is the fundamental class represented by the identity map $1\colon K(\mathbb{Z}_2, k) \to K(\mathbb{Z}_2, k)$

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Comparing ranks of cohomology groups, we find there can be no map $V\colon K(\mathbb{Z}_2, k) \to MO(k)$ such that $U_k \circ V \simeq 1$ (ie such that $V^*(U_k) = \iota_k$)

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(Obtain smooth examples by the method of thickenings:
Embed some skeleton $K^{(r)} \subseteq K(\mathbb{Z}_2, k)$ in $\mathbb{R}^n$ and take the boundary of a regular neighbourhood)

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Remarks:

- $\dim N_k$ can be chosen to be $4k + 3$ if k even and $4k + 15$ if k odd
- By contrast, every class $x \in H^1(N)$ can be represented by an embedding (viz. the zeroes of a generic section of a line bundle η with $w_1(\eta) = x$)

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A class $x \in H^k(N)$ is representable by an immersion if, and only if, there exists a stable map $F\colon \Sigma^i N_+ \to \Sigma^i MO(k)$ such that $\Sigma^i(x) = F^*(\Sigma^i U_k) \in \tilde{H}^{k+i}(\Sigma^i N_+)$

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Corollary

If P is a stable cohomology operation such that $P(U_k) = 0$, and x is represented by an immersion, then $P(x) = 0$

Representing classes by immersions

Steenrod squares are stable operations $Sq^i\colon H^*(-) \to H^{*+i}(-)$
Let $I = (i_1, \ldots, i_r)$ be a sequence of non-negative indices. Then

- $Sq^I = Sq^{i_1} \circ \cdots \circ Sq^{i_r}$
- I is admissible if $i_j \geq 2i_{j+1}$ for $j = 1, \ldots, r$
- The degree and excess of I are given by

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Bockstein operator $\beta\colon H^*(-) \to H^{*+1}(-;\mathbb{Z})$ associated to short exact coefficient sequence (where ρ is reduction mod 2)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\rho} \mathbb{Z}_2 \longrightarrow 0$$

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- All torsion in $\tilde{H}^*(MO(k);\mathbb{Z}) \cong H^*(BO(k), BO(k-1);\mathbb{Z})$ is of order 2

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- Since $e(I) = k$ we have $Sq^I(U_k) = \left(Sq^{i_2} \cdots Sq^{i_r}(U_k)\right)^2$, and Sq^1 is a derivation, so $Sq^1(a \cup a) = 0$ for any a □

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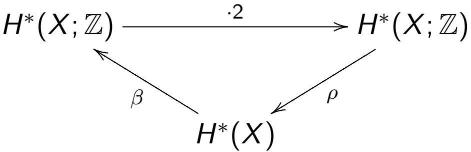
Proof uses the **Bockstein spectral sequence** associated to the exact couple



which has

$$E^*_{(1)} = H^*(X), \qquad E^*_{(2)} = H\left(H^*(X), Sq^1\right)$$

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Browder (1974)

If $X = K(\mathbb{Z}_2, k)$ then

$$E^*_{(2)}(X) = \mathbb{Z}_2\left[\{G^2\}, d_2\{G^2\}\right]$$

where G runs over the $Sq^J(\iota_k)$ such that J is admissible, $e(J) < k$, $j_1 \neq 1$ and $k + |J|$ is even

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If $k = 2m$ is even, then set $I = (2m)$, and

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If $k = 2m + 1$ is odd, set $I = (2m + 4, 2, 1)$, and

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Corollary

If $x \in H^k(N)$ is represented by an immersion, then $\xi(x)$ is the reduction of an integral class, where

$$\xi(x) = \begin{cases} x^2 & (k \text{ even}) \\ \left(Sq^2 Sq^1(x)\right)^2 & (k \text{ odd}) \end{cases}$$

Representing classes by maps with restricted singularities

Let τ be a finite set of multi-singularities (= finite set of singularity types)

Definition

A smooth map $f\colon M^{n-k} \to N^n$ is a τ-map if for each $y \in N$ the pre-image $f^{-1}(y)$ has singularity type in τ

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Theorem (G, Szűcs)

In every dimension $k > 1$ there exists a closed manifold N_k and a mod 2 cohomology class $x \in H^k(N_k)$ which cannot be represented by a τ-map

This is proved by comparing ranks of $H^*(X_\tau)$ and $H^*(K(\mathbb{Z}_2, k))$, where X_τ is classifying space of τ-maps (Rimanyi-Szűcs)