Turning spheres inside-out

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Smale’s paradox

- **Theorem** There is a 1-1 correspondence between regular homotopy classes of immersions $S^2 \hookrightarrow \mathbb{R}^n$ and $\pi_2(V_{n,2})$.

- **Corollary** Any two immersions of $S^2$ in $\mathbb{R}^3$ are regularly homotopic

A sphere is...

...the set of points at distance 1 from the origin in some Euclidean space.
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\[ S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\} \subseteq \mathbb{R}^n \]
A sphere is...

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\[ S^1 \subseteq \mathbb{R}^2 \]

\[ S^2 \subseteq \mathbb{R}^3 \]
Immersions of circles

An immersion $f : S^1 \hookrightarrow \mathbb{R}^2$ is a function from the circle to the plane whose image has no tears or sharp corners, but may intersect itself.
Immersions of circles

- An *immersion* \( f : S^1 \hookrightarrow \mathbb{R}^2 \) is a function from the circle to the plane whose image has no tears or sharp corners, but may intersect itself.

- Mathematically: \( f \) is a \( C^2 \) function with non-vanishing first derivative.
Immersions of circles
Non-immersions of circles
An immersion $f : S^2 \hookrightarrow \mathbb{R}^3$ is a function from the 2-sphere to 3-space whose image is always two-dimensional, and has no sharp creases or tears (but may intersect itself).
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Mathematically: $f$ is a $C^2$ function such that the image of

$$df_x : T_{f(x)}S^2 \rightarrow T_{f(x)}\mathbb{R}^3 \approx \mathbb{R}^3$$

is two-dimensional, for all points $x \in S^2$
Immersions of spheres
Non-immersions of spheres
Regular homotopy

- A way of *classifying* immersions
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- A *regular homotopy* is a continuous family of immersions $h_t : S^2 \looparrowright \mathbb{R}^3$ parameterised by a real variable $t \in [0, 1]$
Regular homotopy

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- Two immersions $f, g : S^2 \hookrightarrow \mathbb{R}^3$ are regularly homotopic if there is a regular homotopy $h_t$ with $h_0 = f$ and $h_1 = g$
Regular homotopy

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- “One can be deformed into another through immersions”
What Smale proved...

...was that *any* two immersions of the sphere into 3-space are regularly homotopic.
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...was that any two immersions of the sphere into 3-space are regularly homotopic.

In particular we can turn the sphere inside-out - a mathematical process known as eversion.
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- Because it’s so difficult to visualize an eversion
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Attempt to ‘push top through bottom’ introduces a crease

- Corresponding result is false for $S^1 \hookrightarrow \mathbb{R}^2$
  - curves are classified by their ‘winding number’ $\in \mathbb{Z}$ (Whitney-Graustein Thm 1937)
History of sphere eversions

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- 1977 N. Max produces first animated eversion
History of sphere eversions

- 1998 ‘The Optiverse’ Computer animated inversion in which the eversion is computed automatically by minimising Willmore energy