Turning spheres inside-out

Mark Grant

mark.grant@durham.ac.uk

Smale's paradox

- **Theorem** There is a 1-1 correspondence between regular homotopy classes of immersions $S^2 \looparrowright \mathbb{R}^n$ and $\pi_2(V_{n,2})$.
- **Corollary** Any two immersions of S^2 in $\mathbb{R}^3$ are regularly homotopic

Stephen Smale, "A classification of immersions of the two-sphere", Trans. Amer. Math. Soc. 90 (1958)

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...the set of points at distance 1 from the origin in some Euclidean space

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$$S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\} \subseteq \mathbb{R}^n$$

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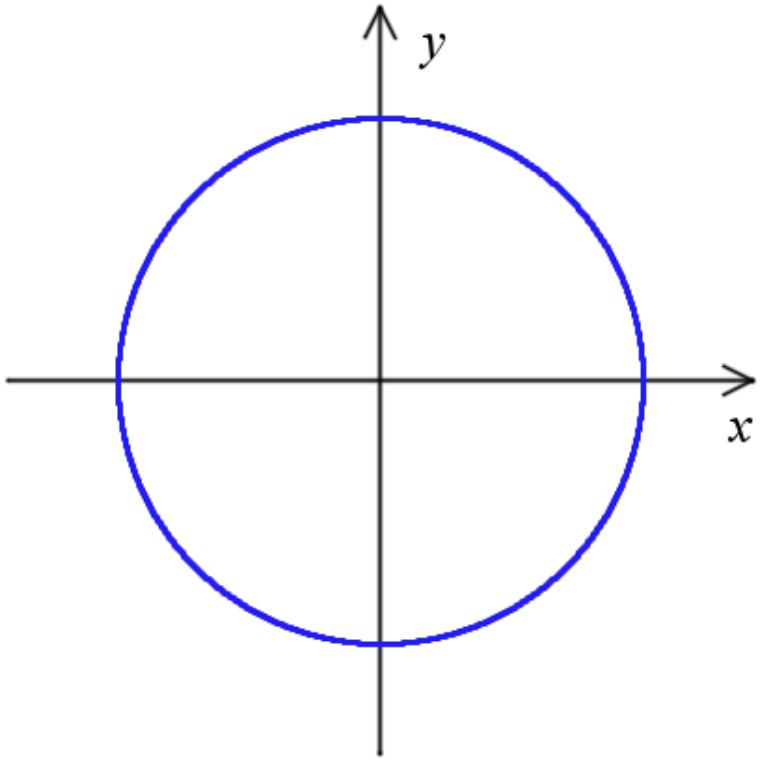

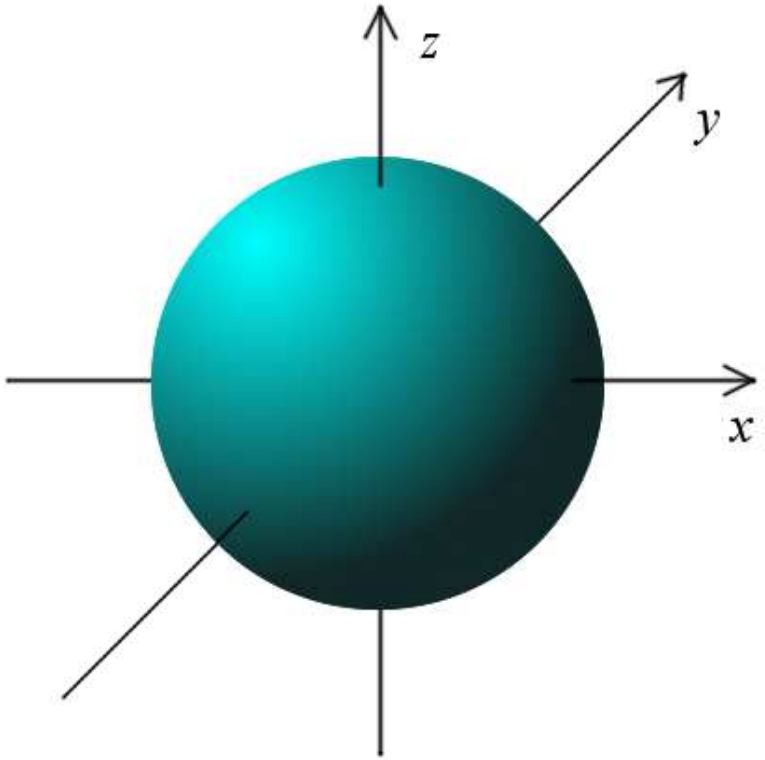

$S^1 \subseteq \mathbb{R}^2$

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Immersions of circles

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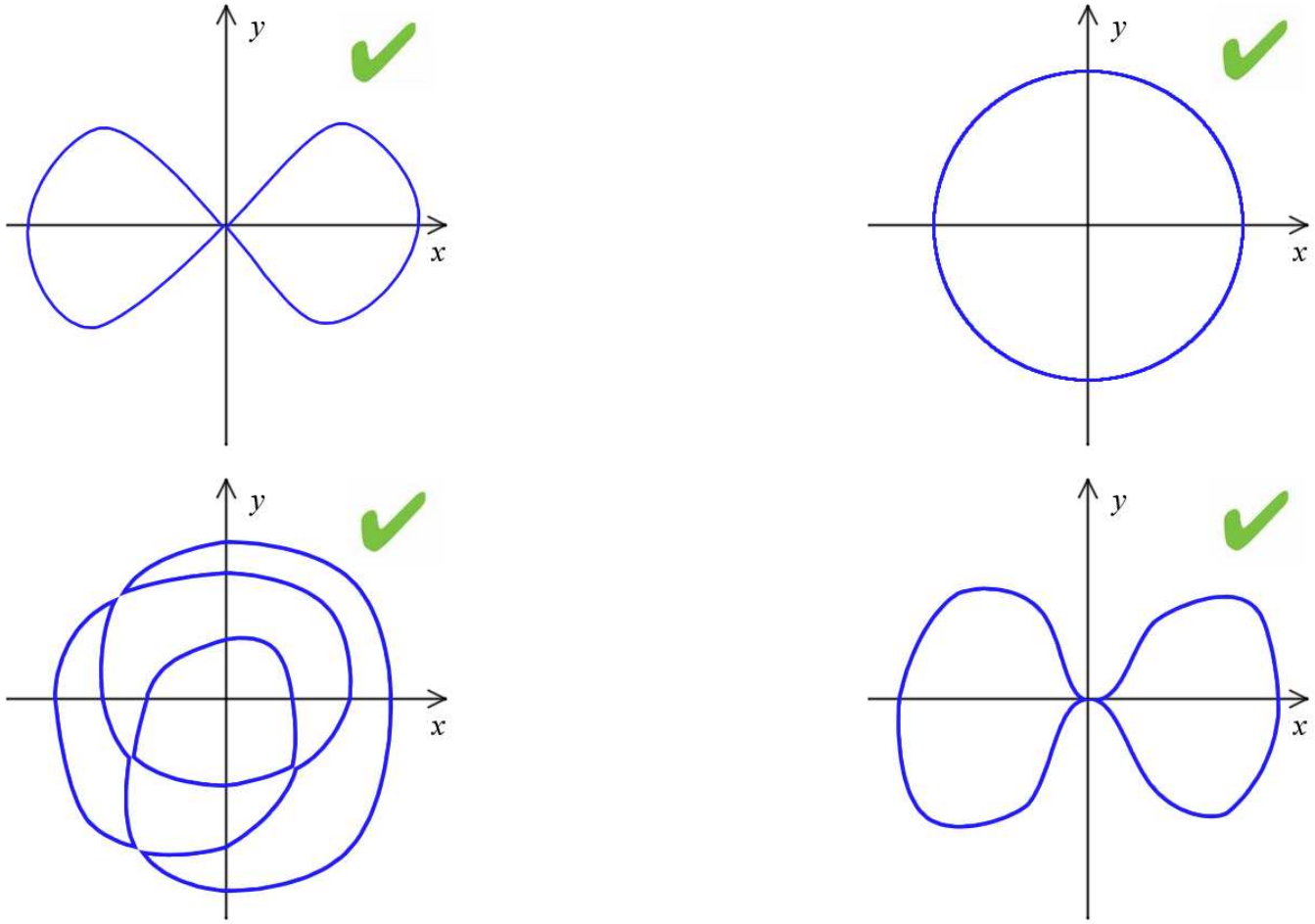
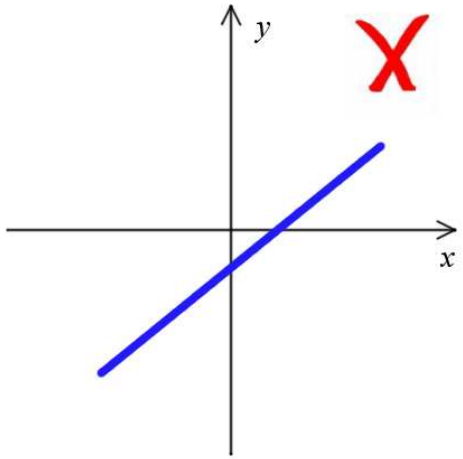
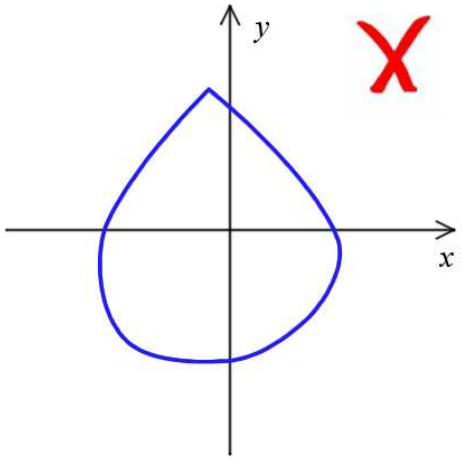
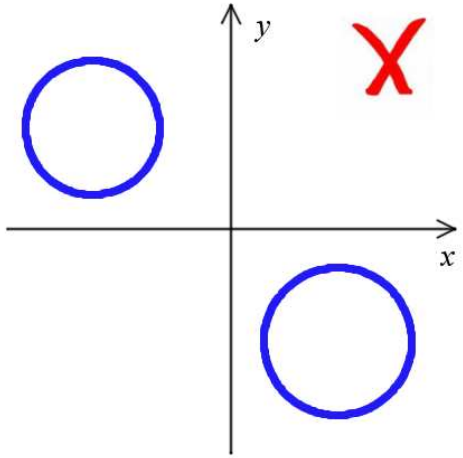
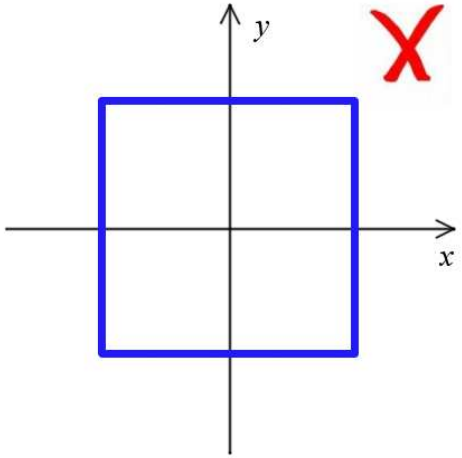


Non-immersions of circles









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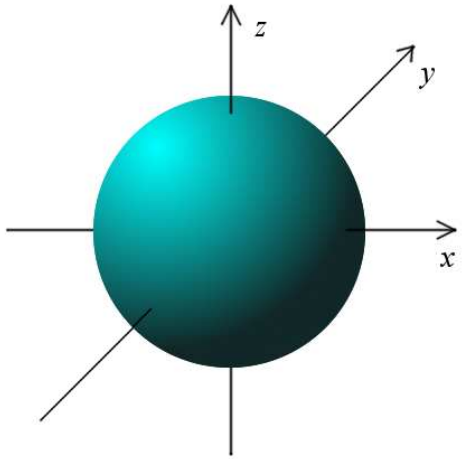
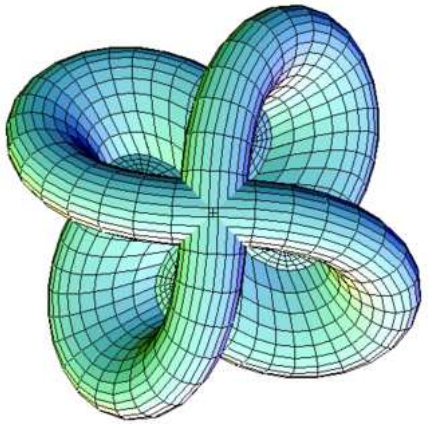
- Mathematically: f is a C^2 function such that the image of

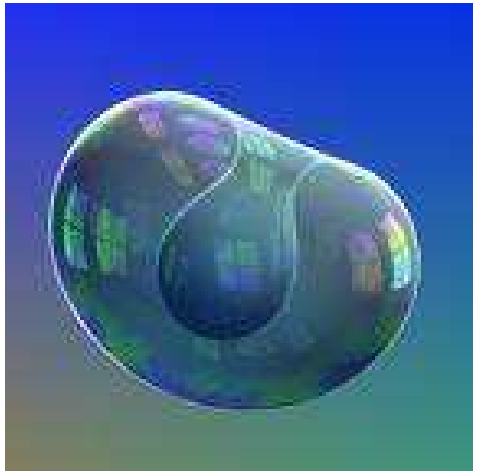
$$df_x\colon\ TS^2_x \to T\mathbb{R}^3_{f(x)} \approx \mathbb{R}^3$$

is two-dimensional, for all points $x \in S^2$

Immersions of spheres







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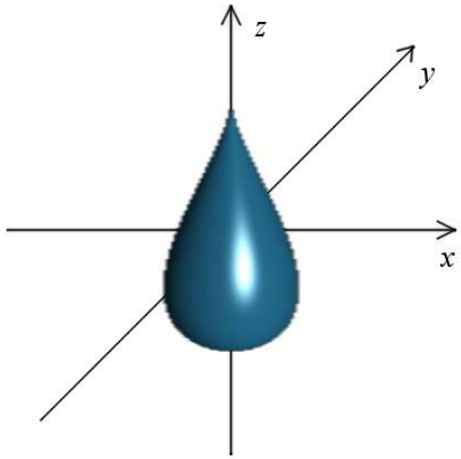

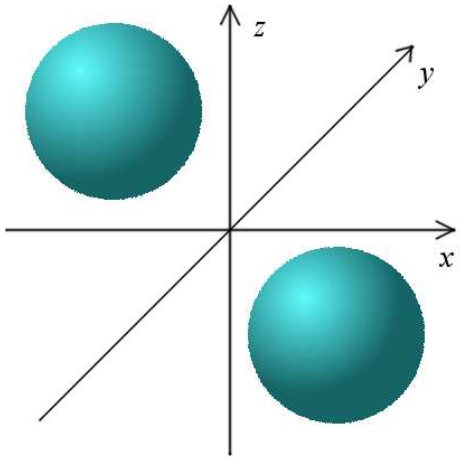

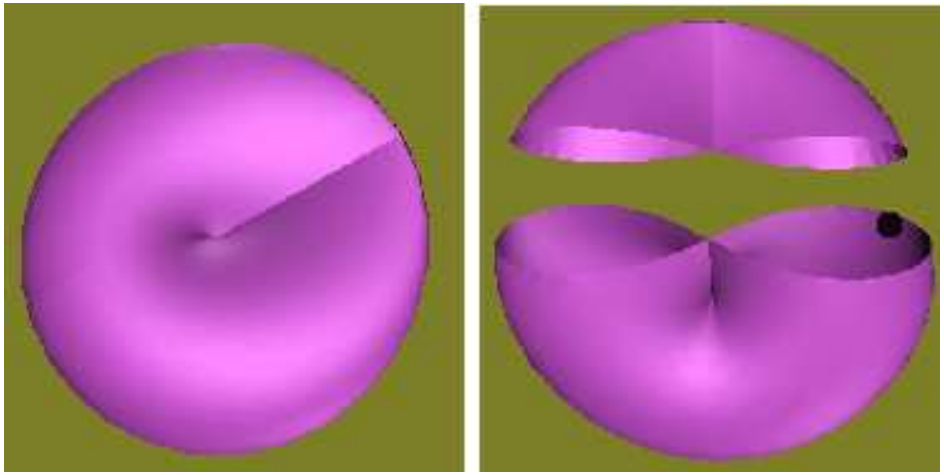

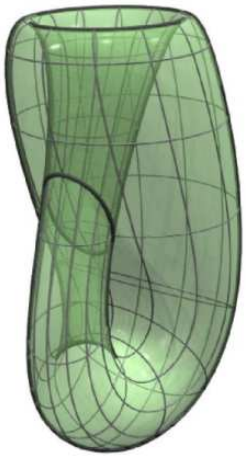

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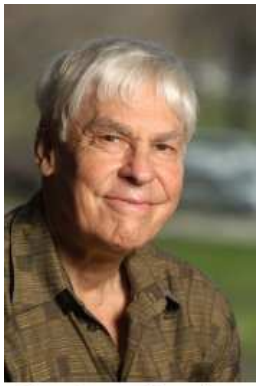
- "One can be deformed into another through immersions"

What Smale proved...



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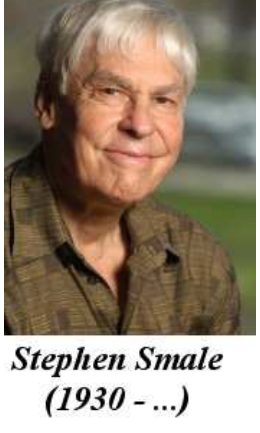

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In particular we can turn the sphere inside-out - a mathematical process known as *eversion*

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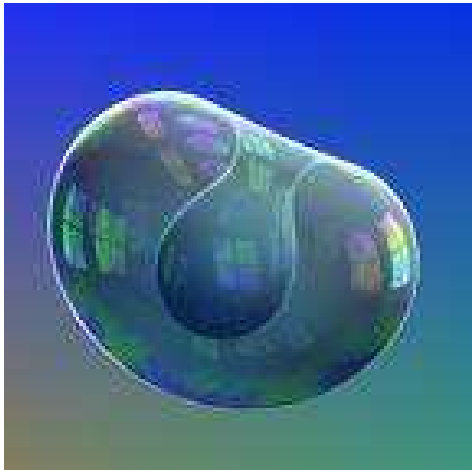

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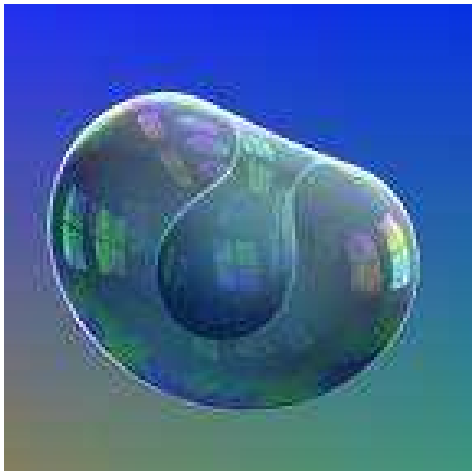

Attempt to 'push top through bottom' introduces a crease

- Corresponding result is false for $S^1 \looparrowright \mathbb{R}^2$ - curves are classified by their 'winding number' $\in \mathbb{Z}$ (Whitney-Graustein Thm 1937)

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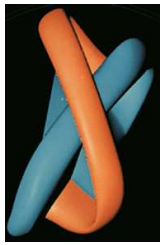
- 1977 N. Max produces first animated eversion



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- 1998 ‘The Optiverse’ Computer animated inversion in which the eversion is computed automatically by minimising Willmore energy