Lusternik-Schnirleumann Category and Moment-Angle Complexes

Piotr Beben (with Jelena Grbić)

Aberdeen

May 2016
Polyhedral Products

$K$-simplicial complex on $n$ vertices, 

$$(X, A) = ((X_1, A_1), \ldots (X_n, A_n)), \ A_i \subseteq X_i,$$

$$(X, A)^K = \bigcup_{\sigma \in K} Y_1^\sigma \times \cdots \times Y_n^\sigma \subseteq X_1 \times \cdots \times X_n,$$

where

$$Y_i^\sigma = \begin{cases} X_i, & \text{if } i \in \sigma \\ A_i, & \text{if } i \notin \sigma, \end{cases}$$

example

- $K = \Delta^n \Rightarrow (X, A)^K = X_1 \times \cdots \times X_n.$
- $K$ disjoint points $\Rightarrow (X, \ast)^K = X_1 \lor \cdots \lor X_n.$
Moment-angle complexes

Davis-Januszkiewicz spaces

$$\text{DJ}(K) = (\mathbb{C}P^\infty, *)^K$$

Moment-angle complexes

$$Z_K = (D^2, \partial D^2)^K$$

$$\simeq (\mathbb{C}^n, \mathbb{C}^n - \{0\})^K$$

$$= \mathbb{C}^n - \bigcup_{\sigma \notin K} \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = 0 \text{ if } i \in \sigma\}.$$ 

More generally, $$Z^\ell_K = (D^{\ell+1}, \partial D^{\ell+1})^K$$ and $$\mathbb{R}Z_K = Z^0_K = (D^1, \partial D^1)^K.$$

- The orbit space of a certain free torus action on $$Z_K$$ for $$K$$ boundary of dual simple polytope is a quasi-toric manifold $$M = Z_K/T^{f_0-n}$$.
- A subspace arrangement $$A$$ is a finite set of affine subspaces of $$\mathbb{R}^n$$ (or $$\mathbb{C}^n$$), $$V_A$$ is their union, and $$M_A = \mathbb{R}^n - V_A$$ (or $$M_A = \mathbb{C}^n - V_A$$) its complement.

Thus $$Z_K$$ is homotopy equivalent to the complement $$M_A$$ of the arrangement

$$\bigcup_{\sigma \notin K} \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = 0 \text{ if } i \in \sigma\}.$$ Similarly for $$\mathbb{R}Z_K$$. 
**k-equal Problem**

Given a set of real numbers \( \{x_1, \ldots, x_n\} \) and \( k \geq 2 \), how many comparisons \( x_i - x_j \geq 0 \) are needed (by the best algorithm in the worst case) to determine if any \( k \) of them are equal?

**More Generally**

Given an arrangement \( A \) and any \( x \) in \( \mathbb{R}^n \), how many comparisons of the form \( \ell(x) \geq 0 \) (for each linear equation \( \ell \) associated to each affine subspace in \( A \)) are needed to determine if \( x \in V_A \)?

The arrangement associated to the \( k \)-equal problem is the \( k \)-equal arrangement

\[
A = \{ (x_1, \ldots, x_n) \mid x_{i_1} = \cdots = x_{i_k} \text{ for each } \{i_1, \ldots, i_k\} \subseteq [n] \}.
\]
Theorem (Björner, Lovász, Yao,...)

The minimum number of comparisons \( \ell(x) \geq 0 \) are given asymptotically in terms of the Betti numbers of \( M_A \) and the intersection lattice of \( A \).

Consequently, the minimum number of comparisons needed for the \( k \)-equal problem are asymptotically

\[ \Theta(n \log \frac{2n}{k}). \]
Let $S$ be a **simplicial $n$-polytope** (a polytope whose codimension 1 faces are $(n-1)$-simplices, i.e. the dual of a simple polytope).

The **$f$-vector**

$$f(S') = (f_0, \ldots, f_{n-1})$$

where $f_i$ is the number of $i$-dimensional faces of $S'$.

Its **$h$-vector**

$$h(S') = (h_0, h_1, \ldots, h_n)$$

is given by

$$h_0 t_n + \cdots + h_{n-1} t + h_n = (t - 1)^n + f_0(t - 1)^{n-1} + \cdots + f_{n-1}$$
The \textit{g-Theorem (Stanley, Birella, Lee)}

$$(f_0, \ldots, f_{n-1})$$ is $f$-vector of some simplicial (or simple) polytope $S$ iff $h$-vector 

$$(h_0, \ldots, h_n)$$ satisfies:

1. \( h_i = h_{n-i} \) for \( i = 0, \ldots, n \) \text{(the Dehn-Sommerville relations)};

2. \( 1 = h_0 \leq h_1 \leq h_{\lfloor \frac{n}{2} \rfloor} \);

3. \( h_{i+1} - h_i \leq (h_i - h_{i-1})^{(i)} \) \text{(the \( i^{th} \) pseudopower) for \( i = 0, \ldots, \lfloor \frac{n}{2} \rfloor - 1 \).}

\[ \square \]

- Proof of the necessity part of \textit{g}-theorem involves cohomology of quasitoric varieties.
- Betti numbers of quasitoric manifold

\[ M = \mathbb{Z}_{\partial S}/T^{f_0-n} \]

satisfy \( \beta^{2i}(M) = h_i(S) \), so Dehn-Sommerville relations follow by Poincaré Duality.

- \textit{g}-Theorem is still a conjecture for general triangulations of spheres.
- But Dehn-Sommerville equations hold also for arbitrary triangulations $K$ of 
closed compact manifolds, and this can be shown using the bigraded betti numbers of \( \mathbb{Z}_K \).
Chemistry

A fullerene spherical molecules of carbon, each carbon atom belongs to three carbon rings, each carbon ring is either a pentagon or hexagon.

Mathematically, a fullerene is a simple 3-polytope all of whose 2-faces are pentagons and hexagons.
Belts

- $p_k$ is the number of $k$-gonal 2-faces of 3-polytope $P$.
- A $k$-belt of a 3-polytope $P$ is a sequence of 2-faces $(F_0, \ldots, F_{k-1})$ s.t. $F_{i-1} \cap F_{(i \mod k)}$ is an edge, and all other intersections are $\emptyset$.

Figure: A 4-belt
Dual $K = \partial P^*$ of simple polytope $P$ is a simplicial complex. $\exists$ bigrading on $H^*(\mathbb{Z}_K)$ defining:

$$\beta^{i,j}(P)$$ the **bigraded Betti number** of $H^*(\mathbb{Z}_{\partial P^*})$ (more on this later).

**Buchstaber, Erokhovets**

**Theorem**

*For a fullerene $P$*

- $\beta^{-1,6} = 0$ – the number of 3-belts.
- $\beta^{-2,8} = 0$ – the number of 4-belts.
- $\beta^{-3,10} = 12 + k$, $k \geq 0$ – the number of 5-belts. If $k > 0$, then $p_6 = 5k$.

Thus fullerenes have no 3-belts and 4-belts.

**Theorem**

*For any fullerene*

- $\beta^{-1,4} = \frac{(8 + p_6)(9 + p_6)}{2}$
- $\beta^{-2,6} = \frac{(6 + p_6)(8 + p_6)(10 + p_6)}{3}$
- $\beta^{-3,8} = \frac{(4 + p_6)(7 + p_6)(9 + p_6)(10 + p_6)}{8}$

For any simple 3-polytope, $p_k$ for $k \neq 6$ satisfy

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k \geq 7} (k - 6)p_k$$ (Eberhard).
Chemists are interested in symmetries of molecules such as fullerenes $P$, e.g. $\text{Aut}(P)$.

*Fan, Ma, Wang*

If $P$ and $Q$ are fullerenes:

$$P \approx Q \iff H^*(\mathbb{Z}_{\partial P^*}) \cong_{\text{rings}} H^*(\mathbb{Z}_{\partial Q^*})$$

**Corollary**

$$\text{Aut}(P) \approx \text{Aut}(H^*(\mathbb{Z}_{\partial P^*}))$$

(*polytope automorphisms and ring automorphisms, respectively*).
Question

- To what extent does $H^*(\mathbb{Z}_G)$ determine a graph $G$? What other invariants, if any, are needed?
- How does commutative algebra in the form of $H^*(\mathbb{Z}_G)$ fit into algebraic graph theory?
Commutative and Combinatorial Algebra

(Buchstaber, Panov)

For $k$ a field or $\mathbb{Z}$:

- $H^*(\text{DJ}(K); k) \cong k[K] = \frac{k[v_1,\ldots,v_n]}{I_K}$ (Stanley-Reisner Ring),

where ideal

$$I_K = \langle \text{square-free monomials } v_{i_1}\ldots v_{i_k} \text{ s.t. } \{i_1,\ldots,i_k\} \notin K \rangle,$$

- $H^*(\Omega \text{DJ}(K); k) \cong \text{Tor}_{k[K]}(k,k)$ (Homology ring of $k[K]$),

- $H^*(\mathbb{Z}_K; k) \cong \text{Tor}_{k[v_1,\ldots,v_n]}(k[K],k).$
\( (\text{Buchstaber, Baskakov, Franz, Panov, Hochster}):\)

\[ H^*(\mathcal{Z}_K ; k) \cong \text{Tor}_{k[v_1,\ldots,v_n]}(k[K], k) \cong \bigoplus_{I \subseteq [n]} \tilde{H}^{*-|I|-1}(|K_I| ; k). \]

cup product multiplication induced by canonical inclusions for disjoint \( I \) and \( J \):

\[ \iota_{I,J} : |K_{I \cup J}| \longrightarrow |K_I \ast K_J| \cong |K_I| \ast |K_J| \cong \Sigma |K_I| \wedge |K_J|. \]

**Bigraded Betti number**:

\[ \beta^{i,2j}(\mathcal{Z}_K) = \sum_{|I|=j} \text{rank}(\tilde{H}^{i+j-1}(|K_I|)). \]
The Golod Property

**Golod: algebraic definition**

A ring $R = k[v_1, \ldots, v_n]/\mathcal{I}$, $\mathcal{I}$ a homogenous ideal, is **Golod** over field $k$ if $\text{Tor}^+_k(v_1, \ldots, v_n)(R, k)$ has trivial multiplication and all Massey products vanish.

**Golod: combinatorial definition**

A simplicial complex $K$ is **Golod** over field $k$ if Stanley-Reisner ring $k[K]$ is Golod.

**Golod: topological definition**

A simplicial complex $K$ is **Golod** over ring $F$ if all cup products (of positive degree elements) and Massey products vanish in $H^+(\mathbb{Z}_K; F) \cong \text{Tor}^+_F(v_1, \ldots, v_n)(F[K], F)$. 

**Long-standing problem:**
Compute the Poincaré series

\[ P(R) = \sum b_i t^i \]

of a commutative ring \( R \), where

\[ b_i = \text{Tor}^i_R(k, k). \]

**Conjecture (Kaplansky, Serre)**
\( R = k[v_1, \ldots, v_n]/\mathcal{I} \) (\( \mathcal{I} \) a homogenous ideal) \( \implies \) then \( P(R) \) is a rational function.

**Theorem (Golod)**
\( R = k[v_1, \ldots, v_n]/\mathcal{I} \) is Golod \( \implies \) \( P(R) \) is a rational function.

**Theorem (Grbić, Theriault)**
\( k[K] \) is Golod (i.e. \( K \) is Golod) \( \implies \) \( P(k[K]) = \frac{t(1-t)^n}{t-P(H^*(\mathbb{Z}_K))} \).

**Problem:**
Characterize Golod rings \( R \). In particular, characterize Golod complexes \( K \).
Homotopy Theory

**Theorem (Grbić, Theriault, Panov, Wu, Berglund, Jöllenbeck)**

If $K$ is a flag complex, then the following are equivalent:

1. $K$ is Golod;
2. $\prod$ vanishes in $H^+(\mathbb{Z}_K) \cong \text{Tor}^+_{\mathbb{Z}[v_1,\ldots,v_n]}(\mathbb{Z}[K],\mathbb{Z})$;
3. $1$-skeleton of $K$ is a chordal graph;
4. $\mathbb{Z}_K$ is a co-$H$-space (wedge of spheres).

(4) $\Rightarrow$ (1), since in general:

$$X \text{ a co} - H - \text{space} \implies \text{cup prod. and Massey prod. vanish in } H^+(X)$$

co-$H$-spaces are precisely LS-category 0 and 1 spaces...
The \textbf{LS-category} \( \text{cat}(X) \) is smallest number \( m \) of open sets \( U_1, \ldots, U_{m+1} \) that cover \( X \) and are contractible in \( X \).

The \textbf{cup length} \( \text{cup}(X) \) of \( X \) is largest number \( m \) s.t. there is a nonzero cup product \( u_1 \cdots u_m \) for some \( u_i \in \tilde{H}^*(X) \).

- \( \text{cat}(X) \leq m \implies \text{cup}(X) \leq m; \)
- \( \text{cat}(X) \leq 1 \implies \text{Massey prod. vanish in } H^+(X) \)
Theorem (Rudyak,...)

If $\text{cat}(X) \leq m$ then:

(i) $\text{cup}(X) \leq m$;

(ii) Massey products $\langle v_1, \ldots, v_k \rangle \in H^+(X)$ vanish whenever for some odd $i$, even $j$, and $m_i + m_j > m$ we have:

$$v_i = a_1 \cdots a_{m_i} \text{ and } v_j = b_1 \cdots b_{m_j} \text{ for some } a_s, b_t \in H^+(X).$$

$$\square$$

Definition

Say $K$ is $m$-Golod if (i) and (ii) above hold for $X = \mathbb{Z}_K$.

Thus:

- $\text{cat}(X) \leq m \implies K$ is $m$-Golod.
**Problem 1**
Characterize those $K$ for which $K$ is $m$-Golod.

and:

**Problem 2**
Characterize those $K$ for which $\text{cat}(\mathcal{Z}_K) = m$.

- As we saw, both these problems are solved when $m = 1$ and $K$ is a flag complex.
- Problem 2 is solved for $(X, \ast)^K$ for certain nice $X$ (Felix, Tanre).
**LS-category ≤ 1 (co-$H$-spaces) and 1-Golod $K$**

- Since $\text{cat}(X) \leq 1 \implies K$ is Golod, knowing homotopy types of $\mathcal{Z}_K$ tells us when $K$ is Golod.
- This has had some success, e.g. shifted complexes and chordal flag complexes.
- In the other direction, since $H^*(\mathcal{Z}_K)$ is simplest when $K$ is Golod, homotopy type of $\mathcal{Z}_K$ should be simplest here as well.

Homotopy type $\mathcal{Z}_K$ known for the following Golod $K$ (increasing generality, due to Iriye, Kishimoto, Grbić, Theriault, Panov, Wu,...):

- $K$ is $n$ disjoint vertices;
- $K$ is shifted;
- $K$ is chordal flag complex or graph;
- $K$ is a $\frac{n}{2}$-neighbourly or a 1-neighbourly 2-dim surface;
- $K$ is Alexander dual of shellable or sequential Cohen-Macaulay;
- $K$ extractible (most general);

In all cases they are co-$H$-spaces, often a wedge of spheres.
**Conjecture A**

$K$ is Golod $\iff \mathcal{Z}_K$ is a co-$H$-space ($\text{cat} (\mathcal{Z}_K) \leq 1$).

**(equivalently) Conjecture A**

Massey products vanish and $\iota_{I,J}: |K_{I \cup J}| \longrightarrow |K_I \ast K_J|$ trivial on cohomology for all $I \cap J = \emptyset$ $\iff \mathcal{Z}_K$ is a co-$H$-space.

**Theorem (Iriye, Kishimoto)**

$\mathcal{Z}_K$ is a co-$H$-space if and only if $\mathcal{Z}_K \simeq \bigvee_{I \subseteq [n]} \Sigma^{\lvert I \rvert + 1} \lvert K_I \rvert$.  

This is a desuspension of a general splitting of $\Sigma \mathcal{Z}_K$ due to Bahri, Bendersky, Cohen, Gitler (the BBCG splitting).

Conjecture is true for flag complexes (Grbić, Panov, Theriault, Wu), and rationally (Berglund)
Conjecture A seems to be false (Iriye, Yano).

**Question A**

*To what extent is it true?*
**Question A - Main Idea**

**Ganea**

A space $Y$ is a co-$H$-space ($\text{cat}(Y) \leq 1$) $\iff$ the evaluation map $\Sigma \Omega Y \xrightarrow{\text{ev}} Y$ has a right homotopy inverse.

- To see if $\mathcal{Z}_K$ is a co-$H$-space, we construct what looks like a configuration space model

  $$\gamma : C(\mathbb{R}\mathcal{Z}_K) \to \Omega \mathcal{Z}_K.$$

  and use $\Sigma \gamma$ to help us find right homotopy inverses for $\Sigma \Omega \mathcal{Z}_K \xrightarrow{\text{ev}} \mathcal{Z}_K$. 
Labelled Configuration Spaces

Let $M$ be any path connected space, $N \subseteq M$ a subspace, and $Y$ a basepointed space with basepoint $\ast$.

$$SP((M/N) \wedge Y) \cong \bigsqcup_{i=0}^{k} M^\times i \times Y^\times i / \sim$$

where the equivalence relation $\sim$ is given by

- $(z_1, \ldots, z_i; y_1, \ldots, y_i) \sim (z_{\sigma(1)}, \ldots, z_{\sigma(i)}; y_{\sigma(1)}, \ldots, y_{\sigma(i)})$ for permutations $\sigma \in \Sigma_i$;

- $(z_1, \ldots, z_i; y_1, \ldots, y_i) \sim (z_1, \ldots, z_{i-1}; y_1, \ldots, y_{i-1})$ for $y_i = \ast$ or $z_i \in N$.

The pairs $(z_j, y_j)$ are called particles, the $y_j$'s their labels.

Classical Labelled Configuration Space

$C(M, N; Y) \subseteq SP((M/N) \wedge Y)$ subspace of all configurations $(z_1, \ldots, z_i; y_1, \ldots, y_i)$ such that $z_1 \neq \cdots \neq z_i$. 
A Classical Result

Segal, B"odigheimer, McDuff,...

$M$ smooth parallelizable $\ell$-manifold, there exists a (weak) homotopy equivalence

$\gamma: C(M, \emptyset; Y) \longrightarrow \text{map}(Z, Z - M; \Sigma^\ell Y)$.

where $Z = M \cup (\partial M \times [0, 1))$. 
• For $X_1 \times \cdots \times X_n$:

$$\Omega \prod_i \Sigma X_i \cong \Omega \Sigma X_1 \times \cdots \times \Omega \Sigma X_n \cong C(\mathbb{R}, \emptyset; X_1) \times \cdots \times C(\mathbb{R}, \emptyset; X_n).$$

• For $X_1 \vee \cdots \vee X_n$:

$$\Omega \bigvee_i \Sigma X_i \cong \Omega \Sigma (X_1 \vee \cdots \vee X_n) \cong C(\mathbb{R}, \emptyset; X_1 \vee \cdots \vee X_n).$$

Polyhedral products tend to sit between these two outliers.
Suppose $W \subseteq X_1 \times \cdots \times X_n$.

**Represented sets**

- A multiset of points $S = \{x_1, \ldots, x_s\} \subset \bigsqcup_i X_i$ is **represented** by a point in $W$ if there exists a point in $p \in W$ such that some subset of coordinates of $p$ is equal to $S$.

Let $V = X_1 \uplus \cdots \uplus X_n$.

$C(M, N; W)$ is the subspace of $SP((M/N) \land V)$ of configurations

$$y = (z_1, \ldots, z_k; x_1, \ldots, x_k) \in SP((M/N) \land V)$$

such that

- $y$ non-degenerate, $i_1 \neq \cdots \neq i_s$, and $z_{i_1} = \cdots = z_{i_s}$, $\implies \{x_{i_1}, \ldots, x_{i_s}\}$ is represented by some point in $W$. 
**Lemma**

$C(D^\ell, \partial D^\ell; \mathbb{R} Z_K)$ deformation retracts onto $Z^\ell_K$.

1. Inclusion $Z^\ell_K \xrightarrow{i} C(D^\ell, \partial D^\ell; \mathbb{R} Z_K)$

$$((t_1, x_1), \ldots, (t_n, x_n)) \mapsto (t_1, \ldots t_n; x_1, \ldots x_n)$$

homeomorphic onto its image;

2. Two step radial expansion $R_t(x) = tx$:

**Figure:** (1) Radially expand particles towards $\partial D^\ell$ to annihilation until there are no more than $n$ particles left, each with a label in a distinct coordinate of $(D^1)^{\times n}$, at the same time attract labels near the basepoint towards it; (2) continue radially expanding until the remaining particles are a point in $Z^\ell_K \subseteq C(D^\ell, \partial D^\ell; \mathbb{R} Z_K)$, meanwhile coordinate-wise attract the label set towards $\mathbb{R} Z_K$. 
Scanning Map

**Theorem**

*M* smooth parallelizable ℓ-manifold, exists a natural scanning map

\[ \gamma : C(M, \emptyset; \mathbb{R}Z_K) \to \text{map}(Z, Z - M; Z^\ell_K). \]

where \( Z = M \cup (\partial M \times [0, 1]) \).

**Proof.**

\[ \gamma : C(M, \emptyset; \mathbb{R}Z_K) \quad \to \quad \text{map}(Z, Z - MC(D^\ell, \partial D^\ell; \mathbb{R}Z_K)) \quad \to \quad \text{map}(Z, Z - M; Z^\ell_K) \]

When \( \partial M = \emptyset \) (\( Z = M \)):
In particular (taking $M = D^1$), we have a map

$$\gamma: C(\mathbb{R}, \emptyset; \mathbb{R}\mathcal{Z}_K) \xrightarrow{\sim} C(D^1, \emptyset; \mathbb{R}\mathcal{Z}_K) \longrightarrow \Omega\mathcal{Z}_K.$$
Splittings

Let

\[ C(\mathbb{R} \mathbb{Z}_K) = C(\mathbb{R}, \emptyset; \mathbb{R} \mathbb{Z}_K). \]

\[ D_k(\mathbb{R} \mathbb{Z}_K) = \frac{C_i(\mathbb{R} \mathbb{Z}_K)}{C_{i-1}(\mathbb{R} \mathbb{Z}_K)} \]

Theorem B

\[ \sum C(\mathbb{R} \mathbb{Z}_K) \simeq \bigvee_{i \geq 1} \sum D_i(\mathbb{R} \mathbb{Z}_K). \]
Moreover,

\[ D_k(\mathbb{R} \mathbb{Z}_K) \cong \bigvee_{(k,n)\text{-partitions } S} D_S(\mathbb{R} \mathbb{Z}_K) \]

where \( S = (s_1, \ldots, s_n) \) non-negative integers s.t. \( s_1 + \cdots + s_n = k \), and \( D_S(\mathbb{R} \mathbb{Z}_K) \subseteq D_k(\mathbb{R} \mathbb{Z}_K) \) subspace where \( s_i \) particles have labels in a \( i^{th} \)-coordinate.
Let $\mathcal{A} = (1, \ldots, 1) \in \mathbb{N}^n$.

**Proposition**

(i) *homotopy commutative diagram of (homotopy) cofibration sequences*

\[
\begin{align*}
C(i) \xrightarrow{\partial} \Sigma D_A(\mathbb{R} \tilde{Z}_K) & \xrightarrow{\Sigma \hat{i}} \Sigma D_A(\mathbb{R} Z_K) \xrightarrow{\hat{q}} \Sigma C(i) \\
\Sigma^n |K| \xrightarrow{\sim} \tilde{Z}_K & \xrightarrow{\text{include}} Z_K \xrightarrow{q} \tilde{Z}_K \cong \Sigma^{n+1}|K|,
\end{align*}
\]

(where $\Sigma^n |K| \cong C(i)$ similar to 2-step radial expansion argument, this time particles vanishing on tip of mapping cone).

(ii) $Z_K$ is a co-$H$-space when quotient map $Z_{K_I} \xrightarrow{q} \tilde{Z}_{K_I}$ has a right homotopy inverse for each $I \subseteq [n]$.

Quotiented config. spaces $D_A(\mathbb{R} Z_K)$ have nice combinatorial description (in analogy to quotiented moment-angle complexes $\tilde{Z}_K = Z_K/\tilde{Z}_K \cong \Sigma^{n+1}|K|$. having an obvious nice combinatorial description).
Homotopy Golod Condition

(1) *Homotopy version of trivial cup products:* 
Require the inclusions 

$$\Sigma\iota_{I,J} : \Sigma |K_{I \cup J}| \to \Sigma |K_I \ast K_J|$$

to be nullhomotopic, for each $I \cap J = \emptyset$.

(2) *Coherence of these nullhomotopies:* 
consider that there is a commutative diagram

\[
\begin{array}{ccc}
|K_{I \cup J_1 \ast K_{J_2}}| & \xrightarrow{(\iota_{I,J_1}) \ast \iota_{I,J_2}} & |K_I \ast K_{J_1} \ast K_{J_2}| \\
\downarrow^{\iota_{I \cup J_1,J_2}} & & \downarrow^{\iota_{I \ast (\iota_{J_1,J_2})}} \\
|K_{I \cup J_1 \cup J_2}| & \xrightarrow{\iota_{I,J_1 \cup J_2}} & |K_I \ast K_{J_1 \cup J_2}| \\
\end{array}
\]

Then we require the composites

$$Cone(\Sigma |K_{I \cup J_1 \cup J_2}|) \xrightarrow{\hat{\iota}_{I \cup J_1,J_2}} \Sigma |K_{I \cup J_1} \ast K_{J_2}| \to \Sigma |K_I \ast K_{J_1} \ast K_{J_2}|$$

$$Cone(\Sigma |K_{I \cup J_1 \cup J_2}|) \xrightarrow{\hat{\iota}_{I,J_1 \cup J_2}} \Sigma |K_I \ast K_{J_1 \cup J_2}| \to \Sigma |K_I \ast K_{J_1} \ast K_{J_2}|$$

to be homotopic to to each other via a homotopy that is fixed on the base $\Sigma |K_{I \cup J_1 \cup J_2}|$ of the cone $Cone(\Sigma |K_{I \cup J_1 \cup J_2}|)$, where $\hat{\iota}_{I \cup J_1,J_2}$ and $\hat{\iota}_{I,J_1 \cup J_2}$ are the extensions given by the nullhomotopies of $\Sigma \iota_{I \cup J_1,J_2}$ and $\Sigma \iota_{I,J_1 \cup J_2}$. One then continues this process for longer joins.
Figure: Skeleton of an order 4 permutohedron

Take the dual of the order $n$ permutohedron: the delta set

$$\mathcal{K}_n = \{\mathcal{F}_0, \ldots, \mathcal{F}_{n-1}\}$$

with $(m - 2)$-faces the set

$$\mathcal{F}_{m-2} = \{\text{all length } m \text{ ordered partitions } (I_1, \ldots, I_m) \text{ of } [n]\}$$

and face maps $d_i: \mathcal{F}_{m-2} \to \mathcal{F}_{m-3}$ given by

$$d_i((I_1, \ldots, I_m)) = (I_1, \ldots, I_{i-1}, I_i \cup I_{i+1}, I_{i+2}, \ldots, I_m).$$

$\mathcal{K}_n$ is a simplicial complex of dimension $n - 2$, $|\mathcal{K}_n| \cong S^{n-2}$. 
A simplicial complex $K$ on vertex set $[n]$ is **homotopy Golod** if $K$ is a single vertex, or (recursively) $K\setminus \{i\}$ is homotopy Golod for each $i \in [n]$ and there is a map

$$\Psi_K : |\mathcal{K}_n| \times Cone(\Sigma|K|) \longrightarrow \Sigma|\Delta^{n-1}|$$

such that for any $\gamma \in |\mathcal{K}_n|$:

1. $\Psi_K(\gamma, *)$ is the basepoint $\ast_{-1} \in \Sigma|\Delta^{n-1}|$
2. the restriction

$$\left(\Psi_K\right)|_{\{\gamma\} \times \{0\} \times \Sigma|K|} : \Sigma|K| \longrightarrow \Sigma|\Delta^{n-1}|$$

is the suspended inclusion;
3. if $\gamma \in |S|$ for some $S = (I_1, \ldots I_m) \in \mathcal{F}_{m-2}$, then

$$\Psi_K(\{\gamma\} \times Cone(\Sigma|K|)) \subseteq \Sigma|K_{I_1} \ast \cdots \ast K_{I_m}| \subseteq \Sigma|\Delta^{n-1}|.$$
\textbf{Theorem D}

$K$ is homotopy Golod $\implies Z_K$ is a co-$H$-space (i.e. $\text{cat}(Z_K) \leq 1$).
**Extractible complexes are homotopy Golod:**

$K$ is **extractible** if $K\setminus\{i\}$ is a simplex for some $i$, or (recursively) each $K\setminus\{i\}$ is extractible and

$$\bigvee_i \Sigma|K\setminus\{i\}| \xrightarrow{\text{include}} \Sigma|K|$$

has a right homotopy inverse. This is one of the most general classes of Golod complexes for which $\mathcal{Z}_K$ is a co-$H$-space.

**$\left\lfloor \frac{n}{2} \right\rfloor$-neighbourly complexes are homotopy Golod:**

$K$ is **$\left\lfloor \frac{n}{2} \right\rfloor$-neighbourly** if every subset of at most $\left\lfloor \frac{n}{2} \right\rfloor$ vertices is a face of $K$, where $n$ is the number of vertices of $K$.

**$\left\lfloor \frac{n}{2} \right\rfloor$-neighbourly but not extractible:**

Let

$$K = (\partial \Delta^2 \ast \partial \Delta^1) \cup_{\partial \Delta^1} \Delta^1.$$  

$K$ has 5 vertices, and every pair of vertices is connected by an edge, so $K$ is highly connected, therefore homotopy Golod. But $H_2(|K|) \cong \mathbb{Z}$, while $H_2(|K\setminus\{i\}|) = 0$ for each $i \in [5]$, so $K$ cannot be extractible.
$K$ is **homotopy Golod (without coherence)** if each suspended inclusion

$$
\sum |I \cup J| t_{I,J} : \sum |I \cup J| K_{I \cup J} \longrightarrow \sum |I \cup J| K_I * K_J
$$

is nullhomotopic for each $I \cap J = \emptyset$.

**Theorem E**

$Z_K$ is a co-$H$-space $\implies K$ is homotopy Golod without coherence.
$K$ is \textit{k-neighbourly} if every $k$ vertices in $K$ span a $(k-1)$-simplex in $K$.

\textbf{Corollary}

If $K$ is $\frac{n}{3}$-neighbourly then:

$\mathcal{Z}_K$ is a co-H-space $\iff$ each $\Sigma^{|I \cup J|} K_{I \cup J} \longrightarrow \Sigma^{|I \cup J|} K_I \ast K_J$ nullhomotopic.
Question

What about $\text{cat}(\mathcal{Z}_K) > 1$. 

**LS-Category > 1**
Flag complexes

Problem
Repeat the classification of \( \text{cat}(\mathcal{Z}_K) = 1 \) (equiv. 1-Golod) flag complexes \( K \) for \( \text{cat}(\mathcal{Z}_K) = m > 1 \) and \( m \)-Golod \( K \).

- The \( m = 1 \) case involved knowledge about topology of \( \mathcal{Z}_L \) for 1-dimensional cycles \( L \).
- Thus we need to know more about \( \mathcal{Z}_L \) for higher dimensional cycles.
- Start with the simplest cycles: triangulated spheres \( K \) (still difficult). Here \( \mathcal{Z}_K \) is a manifold.
Spheres

The above can be used to show:

**Theorem**

- If $K$ on vertex set $[n] = \{1, \ldots, n\}$ is any triangulated $d$-sphere for $d = 0, 1, 2$,
- or (under some conditions) $K$ is obtained from these spheres via join, connected sum, and vertex double operations,

then the following are equivalent.

1. $K$ is $m$-Golod over $\mathbb{Z}$;
2. $\text{Nil}((\text{Tor}_{\mathbb{Z}[v_1, \ldots, v_n]}(\mathbb{Z}[K], \mathbb{Z}))) \leq m + 1$ (equivalently $\text{cup}(\mathbb{Z}_K) \leq m$);
3. for any filtration of full subcomplexes
   \[ \partial \Delta^{d+2-\ell} = K_{I_\ell} \not\subset K_{I_{\ell-1}} \not\subset \cdots \not\subset K_{I_1} = K \]
   such that $|K_{I_i}| \cong S^{d+1-i}$, we have $\ell \leq m$;
4. $\text{cat}(\mathbb{Z}_K) \leq m$.

Moreover, $1 \leq m \leq d + 1$; that is, $K$ satisfies any of the above for some $m$ which cannot be greater than $d + 1$.

**Question:** General sphere triangulations? Triangulations of manifolds?
Gluings

- **Geometric category:** $gcat(X) \leq k$ if open cover $X = \bigcup_{i=0}^{k} U_i$ s.t. $U_i$ (self) contractible.

- **Strong category:** $Cat(X) = \min \{gcat(Y) \mid Y \simeq X\}$.

- $Cat(X) - 1 \leq cat(X) \leq Cat(X) \leq gcat(X)$.

**Proposition**

If $C$ is a (possibly empty) full subcomplex common to $K_1, \ldots, K_m$,

$$cat(\mathcal{Z}_{K_1 \cup_C \ldots \cup_C K_m}) \leq \max\{1, cat(\mathcal{Z}_{K_1}), \ldots, cat(\mathcal{Z}_{K_m})\} + Cat(\mathcal{Z}_C).$$
Connected sums

Let \( \dim L_i = d \), \( \sigma_i \) is a \( d \)-face common to \( L_i \) and \( L_{i+1} \), and \( \sigma_i \cap \sigma_j = \emptyset \) when \( i \neq j \).

**Proposition**

Let

\[
K = L_1 \#_{\sigma_1} L_2 \#_{\sigma_2} \cdots \#_{\sigma_{k-1}} L_k
\]

Then

\[
\text{cat}(\mathcal{Z}_K) \leq \max\{1, \text{cat}(\mathcal{Z}_{L_1^{(d-1)}}), \ldots, \text{cat}(\mathcal{Z}_{L_k^{(d-1)}})\} + 1.
\]
Dimension and Coord. Suspension

**Proposition**

- \( \text{cat}((\Sigma X, \Sigma A)^K) \leq \text{cat}((X, A)^K) \).
- \( \Rightarrow \) for \( \ell \geq 1 \), \( \text{cat}(\mathcal{Z}_K^\ell) \leq \text{cat}(\mathcal{Z}_K) \leq \text{cat}(\mathbb{R}\mathcal{Z}_K) \).
- \( \text{cat}(\mathcal{Z}_K) \leq \dim K + 1 \).

**example**

If \( G \) is a graph,

\[
\text{cat}(\mathcal{Z}_G) \leq 2.
\]

Thus, \( G \) is (at most) 2-Golod, and all Massey products of decomposable elements in \( \text{Tor}^+_{\mathbb{Z}[v_1,\ldots,v_n]}(\mathbb{Z}[G],\mathbb{Z}) \) vanish.
Many interesting spaces found amongst (or derived from) moment-angle complexes

\[ \mathcal{Z}_K = (D^2, \partial D^2)^K \simeq (ES^1, S^1)^K \]

and other polyhedral products of the form \((BG, \ast)^K\) and \((EG, G)^K\):

- Intersections of quadrics, complements of arrangements.
- \(K\) a sphere \(\implies\) \(\mathcal{Z}_K\) a smooth closed manifold.
- \(K\) a triangulated manifold \(\implies\) \(\mathcal{Z}_K - \ast\) open manifold (\(\mathcal{Z}_K\) has 1 singularity).
- \((B\mathbb{Z}, \ast)^K\) and \((E\mathbb{Z}, \mathbb{Z})^K\) are classifying spaces for the right-angled Artin group \(A(K)\) and its commutator subgroup.
- \((B\mathbb{Z}_2, \ast)^K\) and \((E\mathbb{Z}_2, \mathbb{Z}_2)^K\) are classifying spaces for the right-angled Coxeter group \(C(K)\) and its commutator subgroup.
- Every complex cobordism class has as a representative a quasitoric manifold \(M = \mathcal{Z}_{\partial P^\ast}/T^{n-\dim P}\) (Panov).

So extending above bounds might answer certain questions about these spaces, e.g.

- group cohomology of commutators subgroups of Right-angled Artin/Coxeter groups;
- counting minimal number of critical points of a Morse function, since

\[ 1 + \text{cat}(M) \leq \text{crit}(M) = \min \{ \# \text{ critical points } f \mid f: M \longrightarrow \mathbb{R} \text{ smooth } \} . \]

for smooth manifolds \(M\).
**Question**

Given $m$-Golod $K$, when is $\text{cat}(\mathcal{Z}_K) = m$? Is $\text{cat}(\mathcal{Z}_K)$ even bounded? If not, how fast does it grow asymptotically w.r.t. vertex set $[n]$?

- If such a bound exists, we get a generalization of the basic and well-known fact:

  the identity map $|K| \xrightarrow{1} |K|$ induces trivial map on cohomology

  $\iff \Sigma|K| \xrightarrow{\Sigma 1} \Sigma|K|$ is nullhomotopic.

(e.g. for $m = 1$, the inclusions $|K_{I \cup J}| \xrightarrow{\iota_{I,J}} |K_I \ast K_J|$ in place of $|K| \xrightarrow{1} |K|$, together with vanishing Massey products, since $\text{cat}(\mathcal{Z}_K) = 1$ implies they are nullhomotopic).
Rational Homotopy

Theorem (Avramov, Berglund)
Localized at the rationals $\mathbb{Q}$, the following are equivalent:

1. $K$ is Golod over $\mathbb{Q}$;
2. $H^+ (\mathbb{Z}_K; \mathbb{Q}) \cong \text{Tor}^+_{\mathbb{Q}[v_1, \ldots, v_n]} (\mathbb{Q}[K], \mathbb{Q})$ trivial mult./vanishing Massey prod.;
3. $H^+ (\mathbb{Z}_K; \mathbb{Q})$ has trivial mult. and $(\Lambda (u_1, \ldots, u_n) \otimes \mathbb{Q}[K], d)$ is formal;
4. $\mathbb{Z}_K \simeq$ a wedge of spheres;
5. $\text{cat} (\mathbb{Z}_K) = 1$.

Question
How does this generalize to $\text{cat} (\mathbb{Z}_K) > 1$? In particular, $(3) \iff (2)$? Note: there is a generalization of formality due to Stasheff and Halperin.
**Related Invariants**

**Problem:** Compute $TC(\mathcal{Z}_K)$, sectional category, and related invariants. What are the implications of these in commutative algebra?

- $TC((S^n, \ast)^K)$ is known (González, Gutiérrez, Yuzvinsky).
- $\hat{\text{cat}}(X) \leq TC(X) \leq 2\hat{\text{cat}}(X) - 1$.
- $\mathcal{Z}_K$ can be thought of as configuration space of $n$ points in $(D^2, \partial D^2)^K$, where tuples of points are allowed in $\text{interior}(D^2)$ depending on $K$.

Thus $\mathcal{Z}_K$ (or $\mathcal{Z}_K \cap \text{Config}(D^2, n)$) are simple models for motion planning for air traffic control, or robotic $n$-arm motion with restraints.

- An $n$-arm 2-leg **arachnoidal linkage** in $\mathbb{R}^d$ is precisely $\mathcal{Z}_K^d = (D^d, \partial D^d)^K$ for $K$ the $n$-gon. A similar result for certain planar $m$-arm linkages. (Kamiyama, Tsukuda)
**Connections to Motion Planning on Graphs**

Take a graph $G$ with $m$ edges, a collection of simplicial complexes

$$\mathcal{L} = \{L_1, \ldots, L_m\}$$

each on $n$ vertices. We have a configuration space $C_n(G; \mathcal{L})$:

modelling factory-floor motion planning: $n$ robots, certain tuples of robots (of right or shape size) can bypass one another in $i^{th}$ corridor, depending on $L_i$. 
• When each \( L_i \) is \( n \) disjoint points: there are no collisions, and (usually) \( C_n(G; \mathcal{L}) = K(\pi_1, 1) \) (Abrams, Ghrist).

• When \( m = 1 \), \( C_n(G; L_1) \) is special case (\( K = G \)) of certain diagonal arrangements \( C_K(\mathbb{R}; I) \), appearing in summands of a stable splitting

\[
\Sigma^\infty \Omega(\Sigma X, *) \simeq \bigvee_{I \in \mathbb{N}^n} C_K(\mathbb{R}; I)_+ \wedge X^I
\]

and in the homology of \( \Omega(\mathbb{C}P^\infty, *)^K \) (Dobrinskaya).

• When \( m = 1 \), \( I = (1, \ldots, 1) \in \mathbb{N}^n \), we have Euler characteristic

\[
\chi(C_K(\mathbb{C}P^{\ell-1}; I)) = \ell^{th} \text{ chromatic number of } G
\]

where \( K \) is the flag complex of complement graph \( \bar{G} \) of \( G \) (Eastwood, Huggett).
Anders Björner - a Number Theoretic Complex:

Let $P(k)$ be the set of prime factors of integer $k$. Simplicial complex:

$$K(m) = \{ P(k) \mid k \text{ is square-free and } k \leq m \}.$$

**Theorem**

Euler characteristic $\chi(K(m))$ gives:

- **Prime Number Theorem true** $\iff |\chi(K(m))| \leq \epsilon m$ all $\epsilon > 0$, large $m$.
- **Riemann hypothesis true** $\iff |\chi(K(m))| \leq m^{\frac{1}{2}} + \epsilon$ all $\epsilon > 0$, large $m$.

- Thus, one should study Betti numbers of $\beta_i(K(m))$.
- Björner estimated $\beta_i(K(m))$ by showing $K(m)$ is a shifted complex, and using properties of shifted complexes.

**Problem**

What can stronger topological/combinatorial invariants such as $H^*(\mathbb{Z}_{K(m)})$ tell us about number theory? Do they raise the right questions?

- Since shifted complexes are Golod, cup/Massey prod. vanish in $H^+(\mathbb{Z}_{K(m)})$, and $|K_{I \cup J}| \rightarrow |K_I \ast K_J|$ trivial on $H^*(\cdot)$. What does this tell us?
Thank you!