Polyhedral Products and Configuration Spaces

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Polyhedral Products

Definition: Polyhedral products

K-simplicial complex on n vertices,
$(\underline{X}, \underline{A}) = ((X_1, A_1), \dots (X_n, A_n))$, $A_i \subseteq X_i$,

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} Y_1^\sigma \times \cdots \times Y_n^\sigma \quad \subseteq \quad X_1 \times \cdots \times X_n,$$

where

$$Y_i^\sigma = \begin{cases} X_i, & \text{if } i \in \sigma \\ A_i, & \text{if } i \notin \sigma, \end{cases}$$

example

- $K = \Delta^{n-1} \Rightarrow (\underline{X}, \underline{A})^K = X_1 \times \cdots \times X_n$.
- K *disjoint points* $\Rightarrow (\underline{X}, *)^K = X_1 \vee \cdots \vee X_n$.

Moment-angle complexes

Davis-Januszkiewicz spaces

$$\mathrm{DJ}(K) = (\mathbb{C}P^\infty, *)^K$$

Moment-angle complexes

$$\begin{aligned}
\mathcal{Z}_K &\simeq \text{ homotopy fiber of inclusion } \mathrm{DJ}(K) \longrightarrow (\mathbb{C}P^\infty)^{\times n} \\
&\simeq \text{ compl. coord. subspace arrangement} \\
&\qquad \mathbb{C}^n - \bigcup_{\sigma \notin K} \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i = 0 \text{ if } i \in \sigma\} \\
&= (D^2, S^1)^K \quad \text{ where } S^1 = \partial D^2.
\end{aligned}$$

The quotient of a certain free torus action on $\mathcal{Z}_K$ for K dual to a simple polytope is a *quasi-toric manifold*.

Commutative and Combinatorial Algebra

Geometrically realize important algebraic objects. For F a field or $\mathbb{Z}$:

- (Buchstaber, Panov):

$$H^*(\mathrm{DJ}(K);F) \cong F[K] \cong \frac{F[v_1,\ldots,v_n]}{I_K} \quad \text{(\textit{Stanley-Reisner Ring})},$$

  where ideal

$$I_K = \langle \text{square-free monomials } v_{i_1}\cdots v_{i_k} \text{ s.t. } \{i_1,\ldots,i_k\} \notin K \rangle\,,$$

- (Buchstaber, Baskakov, Franz, Panov):

$$H^*(\mathcal{Z}_K;F) \cong \mathrm{Tor}_{F[v_1,\ldots,v_n]}(F[K],F) \cong \bigoplus_{I\subseteq[n]} \tilde{H}^*(|K_I|;F).$$

  cup product multiplication induced by canonical inclusions

$$\iota_{I,J}\colon |K_{I\cup J}| \longrightarrow |K_I * K_J| \cong |K_I| * |K_J| \simeq \Sigma|K_I| \wedge |K_J|.$$

  for disjoint I and J.

The Golod Property

Golod: algebraic definition

A local ring R is **Golod** over field $\mathbf{k}$ if the multuplication and all Massey products in $\mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_n]}(R,\mathbf{k})$ vanish.

Golod: combinatorial definition

A simplicial complex K is **Golod** over field $\mathbf{k}$ if Stanley-Reisner ring $\mathbf{k}[K]$ is Golod.

Golod: topological definition

A simplicial complex K is **Golod** over ring F if all cup products in $H^*(\mathcal{Z}_K;F)$ vanish.

The Golod Property over fields

We will use topological definition. These are all related for fields F (Berglund, Jollenbeck):

$F[K]$ is Golod over F $\Longleftrightarrow$ cup products and Massey products vanish in $H^*(\mathcal{Z}_K; F)$

$\Longleftrightarrow$ cup products vanish in $H^*(\mathcal{Z}_K; F)$

$\Longleftrightarrow$ $\iota_{I,J}$ induce trivial maps on cohomology, for all disjoint $I, J \subseteq [n]$

$\Longleftrightarrow$ $H_*(\Omega\mathcal{Z}_K; F)$ is a free graded associative algebra.

If K is Golod over all fields $\mathbf{k}$, we simply say K is Golod.

Homotopy Theory

- Knowing homotopy types of $\mathcal{Z}_K$ tells us when K is Golod in particular.
- Since $H^*(\mathcal{Z}_K)$ is simplest when K is Golod, homotopy type of $\mathcal{Z}_K$ should be simplest here as well.

Homotopy type $\mathcal{Z}_K$ known for the following Golod K (increasing generality, due to Iriye, Kishimoto, Grbić, Theriault, Panov, Wu,...):

- K is n disjoint vertices;
- K is *shifted*;
- K is *chordal flag*;
- K is Alexander dual of *shellable* or *sequential Cohen-Macaulay*;
- K *extractible* (most general);

In all cases they are co-H-spaces, often a wedge of spheres.

Conjecture

K is Golod $\iff$ $\mathcal{Z}_K$ is a co-H-space.

Irije, Kishimoto

$\mathcal{Z}_K$ is a co-H-space if and only if $\mathcal{Z}_K \simeq \bigvee_{I \subseteq [n]} \Sigma^{|I|+1} |K_I|$. $\square$

This is a desuspension of a general splitting of $\Sigma\mathcal{Z}_K$ due to Bahri, Bendersky, Cohen, Gitler (the BBCG splitting).

Conjecture is true for flag complexes, or localized at $\mathbb{Q}$ (Berglund, Grbić, Panov, Theriault, Wu)

(K is a Golod flag complex $\Leftrightarrow$ K is a chordal flag complex $\Leftrightarrow$ K is flag and $\mathcal{Z}_K$ a co-H-space).

Main Idea

Fact: A space Y is a co-H-space if and only if the evaluation map $\Sigma\Omega Y \xrightarrow{ev} Y$ has a right homotopy inverse.

- Therefore, to see if Y is a co-H-space, start by trying to find the finest possible splitting of $\Sigma\Omega Y$.
- To do this for $Y = \mathcal{Z}_K$, construct configuration space models for $\Omega\mathcal{Z}_K$.

Labelled Configuration Spaces

Let M be any path connected space, $N \subseteq M$ a subspace, and Y a basepointed space with basepoint $*$. Let $D_0(M, N; Y) = *$ and take the quotient space

$$D_k(M, N; Y) = \coprod_{i=0}^{k} M^{\times i} \times Y^{\times i} / \sim$$

where the equivalence relation $\sim$ is given by

- $(z_1, \ldots, z_i; x_1, \ldots, x_i) \sim (z_{\sigma(1)}, \ldots, z_{\sigma(i)}; x_{\sigma(1)}, \ldots, x_{\sigma(i)})$ for permutations $\sigma \in \Sigma_i$;
- $(z_1, \ldots, z_i; x_1, \ldots, x_i) \sim (z_1, \ldots, z_{i-1}; x_1, \ldots, x_{i-1})$ for $x_i = *$ or $z_i \in N$.

Then

$$D(M, N; Y) = \bigcup_{k=0}^{\infty} D_k(M, N; Y) \quad (\cong SP((M/N) \wedge Y).$$

Classical Labelled Configuration Space:

$C(M, N; Y) \subseteq D(M, N; Y)$ subspace of all configurations $(z_1, \ldots, z_i; x_1, \ldots, x_i)$ such that $z_1 \neq \cdots \neq z_i$.

Classical Results

Bödigheimer, McDuff

X a CW-complex, M smooth compact parallelizable ℓ-manifold, N a submanifold, M/N or X is connected. Then there exists a homotopy equivalence

$$\gamma : \mathcal{C}(M, N; X) \longrightarrow map(Z - N, Z - M; \Sigma^{\ell} X),$$

where $Z = M \cup (\partial M \times [0,1))$.

Segal

If X is path connected, then

$$C(\mathbb{R}^n; X) \simeq \Omega^n \Sigma^n X.$$

Coordinate Suspensions

Coordinate suspensions

- $X = X_1 \times \cdots \times X_n$; each X_i connected basepointed;
- $W \subset X$

Definition: ℓ-fold coordinate suspension

$$W^\ell = \left\{ ((t_1, x_1), \ldots, (t_n, x_n)) \mid t_i \in D^\ell,\ (x_1, \ldots, x_n) \in W \right\}$$

as a subspace of $X^\ell = \Sigma^\ell X_1 \times \cdots \times \Sigma^\ell X_n$.

Definition: coordinate smash

Let $Y = \prod_{i=1}^n Y_i$, take

$$Y \wedge_X W = \{ ((y_1, x_1), \ldots, (y_n, x_n)) \mid y_i \in Y_i,\ (x_1, \ldots, x_n) \in W \}$$

as a subspace of $Y \wedge_X X = (Y_1 \wedge X_1) \times \cdots \times (Y_n \wedge X_n)$.

Properties:

- $W^0 = W, \qquad (W^{l_1})^{l_2} = W^{l_1+l_2}$;
- If W basepointed connected, and $n = 1$ (or $W \subseteq X_1 \vee \ldots \vee X_n \subseteq X$), then
$$W^\ell = \Sigma^\ell W.$$
- If $W = \bigcup_{\mathcal{J}} A_{j_1} \times \cdots \times A_{j_n}$, then
$$W^\ell = \bigcup_{\mathcal{J}} \Sigma^\ell A_{j_1} \times \cdots \times \Sigma^\ell A_{j_n}$$
- Let $(\underline{X}, \underline{A}) = ((X_1, A_1), \ldots (X_n, A_n))$ and K a simplicial complex on n vertices.
$$W = (\underline{X}, \underline{A})^K \Longrightarrow W^\ell = (\Sigma^\ell \underline{X}, \Sigma^\ell \underline{A})^K.$$
where $(\Sigma^\ell \underline{X}, \Sigma^\ell \underline{A}) = ((\Sigma^\ell X_1, \Sigma^\ell A_1), \ldots (\Sigma^\ell X_n, \Sigma^\ell A_n))$.

Config. Space Models for Spaces of Maps to Coordinate Suspensions

- If $W = X_1 \times \cdots \times X_n$, then

$$\Omega W^1 \cong \Omega\Sigma X_1 \times \cdots \times \Omega\Sigma X_n \simeq C(\mathbb{R}, \varnothing; X_1) \times \cdots \times C(\mathbb{R}, \varnothing; X_n).$$

- If $W = X_1 \vee \cdots \vee X_n$, then

$$\Omega W^1 \cong \Omega\Sigma(X_1 \vee \cdots \vee X_n) \simeq C(\mathbb{R}, \varnothing; X_1 \vee \cdots \vee X_n).$$

Definition: Represented multisets

- A submultiset of points $\mathcal{S} = \{x_1, \ldots, x_s\} \subset \coprod_i X_i$ is **represented** by a point in W if there exists a $(\bar{x}_1, \ldots, \bar{x}_n) \in W$ and some injective function $f_{\mathcal{S}}: \{1, \cdots, s\} \longrightarrow \{1, \cdots, n\}$ such that $x_i = \bar{x}_{f_{\mathcal{S}}(i)}$.
- A configuration y is **represented** if its multiset of labels is represented.

Let $X^{\vee} = X_1 \vee \cdots \vee X_n$ be the wedge at basepoints $* \in X_i$.

Allow represented labelled particles to collide

$\mathcal{C}(M, N; W)$ is the subspace of $D(M, N; X^{\vee})$ of configurations $y = (z_1, \ldots, z_k; x_1, \ldots, x_k) \in D(M, N; X^{\vee})$ such that

- y non-degenerate, $i_1 \neq \cdots \neq i_s$, and $z_{i_1} = \cdots = z_{i_s}$, $\Longrightarrow \{x_{i_1}, \ldots, x_{i_s}\}$ is represented by some point in W.

There are other generalizations that allow collisions under various rules (Kallel, Salvatore, Dobrinskaya).

Theorem A

Each X_i a connected basepointed CW complexes, W a connected basepointed subcomplex of $X = \prod X_i$, M-smooth compact parallelizable ℓ-manifold, N-submanifold, M/N connected.
Then there is a homotopy equivalence

$$\gamma : \mathcal{C}(M, N; W) \longrightarrow map(Z - N, Z - M; W^{\ell})$$

where $Z = M \cup (\partial M \times [0,1))$.

Examples

- free loop spaces $N = \varnothing \quad Z = M = S^1 \qquad \Lambda W^1 \simeq \mathcal{C}(S^1; W)$
- based loop spaces $N = \varnothing \quad M = D^1 \quad Z \simeq \mathbb{R}$

$$\Omega W^1 = \mathrm{map}(\mathbb{R}, \mathbb{R} - D^1; W^1) \simeq \mathcal{C}(D^1, \varnothing; W) \simeq \mathcal{C}(\mathbb{R}^1, \varnothing; W)$$

- We can take $W = (D^1, S^0)^K$. Then

$$W^1 = (\Sigma D^1, \Sigma S^0)^K = (D^2, S^1)^K = \mathcal{Z}_K.$$

The Simplest Case: $M = D^\ell$, $N = \partial D^\ell$

Lemma
$\mathcal{C}(D^\ell, \partial D^\ell; W) \simeq map(D^\ell, \varnothing, W^\ell) \simeq W^\ell$

Proof.

(1) Inclusion $W^\ell \longrightarrow \mathcal{C}(D^\ell, \partial D^\ell; W)$;

$$((t_1, x_1), \ldots, (t_n, x_n)) \mapsto (t_1, \ldots t_n; x_1, \ldots x_n)$$

(2) $\mathcal{C}^\varepsilon \subseteq \mathcal{C}(D^\ell, \partial D^\ell; W)$ subspace of configurations $y = (z_1, \ldots, z_k; x_1, \ldots, x_k)$ such that:

- if y non-degenerate, $i_1 \neq \cdots \neq i_s$, and $\{\{x_{i_1}, \ldots, x_{i_s}\}\}$ is <u>not</u> represented by point in W then

$$var(z_{i_1}, \ldots, z_{i_s}) = \frac{1}{s(s-1)} \sum_{i \neq j} |z_i - z_j| \quad \geq \quad \varepsilon.$$

(3) $\mathcal{C}^\epsilon \simeq \mathcal{C}(D^\ell, \partial D^\ell; W)$

(4) W^ℓ and $\mathcal{C}^\epsilon$ have same homeomorphic image in $\mathcal{C}(D^\ell, \partial D^\ell; W)$ when $\epsilon \geq 2$.

Section Spaces

Z - a smooth ℓ-manifold without boundary, $M \subseteq Z$ - a smooth compact codimension 0 submanifold, $T(Z)$ - the S^ℓ-bundle over Z obtained from tangent bundle. Pullback:

$$\begin{tikzcd} T_{\Delta(Z)} \arrow[r] \arrow[d] & Z \arrow[d, "\Delta"] \\ \prod^n T(Z) \arrow[r] & \prod^n Z. \end{tikzcd}$$

Construct the $W^\ell = (\prod^n S^\ell) \wedge_X W$-bundle $\mathcal{T}(Z;W) \xrightarrow{\pi} Z$. Let $\kappa_\infty\colon Z \longrightarrow \mathcal{T}(Z;W)$ be the section of π that sends a point in $z \in Z$ to the basepoint $*_z$ at infinity on the fiber at z.

Definition: Section space $\Upsilon(Z;A,B;W)$

For any subspaces $B \subset A \subset Z$, $\Upsilon(Z;A,B;W)$ space of sections of π defined on A, that agree with κ_∞ on B.

- Z is parallelizable, $\Upsilon(Z;A,B;W) \cong map(A,B;W^\ell)$.

Proof of Theorem A

Identical to Bödingheimer, McDuff:

(1) (when Z parallelizable) there is a *scanning map*

$$\gamma\colon \mathcal{C}(M,N;W) \longrightarrow map(Z-N, Z-M; \mathcal{C}(D^\ell, \partial D^\ell; W)) \xrightarrow{\simeq} map(Z-N, Z-M; W^\ell)$$

(2) (quasi)fibration diagram

$$\begin{tikzcd} \mathcal{C}(L, L\cap N; W) \arrow[r] \arrow[d, "\gamma"] & \mathcal{C}(M,N;W) \arrow[r] \arrow[d, "\gamma"] & \mathcal{C}(M, L\cup N; W) \arrow[d, "\gamma"] \\ \Upsilon(Z-L\cap N, Z-L; W) \arrow[r] & \Upsilon(Z-N, Z-M; W) \arrow[r] & \Upsilon(Z-L\cup N, Z-M; W). \end{tikzcd}$$

(3) Induction for progressively more general (M, N):

- *(a)* Handles $(D^k \times D^{\ell-k}, D^k \times S^{\ell-k-1})$ of index k, induction on k (base case $(D^\ell, \partial D^\ell)$ done);
- *(b)* $(M, \partial M)$ (induction on handle decomposition of M)
- *(c)* (M, N) ...

Splittings

Let $\mathcal{C}(W) = \mathcal{C}(\mathbb{R}, \varnothing; W) \simeq \Omega W^1$. There is a map

$$\zeta \colon \mathcal{C}(W) \longrightarrow C(\mathbb{R}, \varnothing; V_\infty(W)) \simeq \Omega\Sigma V_\infty(W),$$

where $C(\mathbb{R}, \varnothing; V_\infty(W))$ is a classical labelled config. space, and

$$V_k(W) = \bigvee_{i=1}^{k} \Sigma^i \frac{\mathcal{C}_i(W)}{\mathcal{C}_{i-1}(W)},$$

where $\mathcal{C}_i(W) = \{j\text{-particle configurations in } \mathcal{C}_i(W) \mid j \leq i\}$.
Adjointing, there is a homotopy equivalence

$$\zeta' \colon \Sigma\mathcal{C}(W) \longrightarrow \Sigma V_\infty(W).$$

Theorem B

$$\Sigma\Omega W^1 \simeq \bigvee_{i\geq 1} \Sigma \frac{\mathcal{C}_i(W)}{\mathcal{C}_{i-1}(W)} \simeq \bigvee_{i\geq 1} \Sigma\mathcal{D}_i(W).$$

Take $W = (D^1, S^0)^K$. Then $W^1 = (D^2, S^2)^K = \mathcal{Z}_K$. Recall:

Conjecture

K is Golod $\Longleftrightarrow$ $\mathcal{Z}_K$ is a co-H-space.

<u>In other words</u>:

Conjecture

$\iota_{I,J} : |K_{I\cup J}| \longrightarrow |K_I * K_J|$ trivial on cohomology for all $I \cap J = \varnothing$ $\Longleftrightarrow$ $\mathcal{Z}_K \simeq \bigvee_{I\subseteq[n]} \Sigma^{|I|+1}|K_I|$.

Theorem B

Conjecture is true localized at a sufficiently large prime p.

Proof.

$D_i(W) \simeq$ finite CW-complex $\Rightarrow$ $\Sigma D_i(W) \simeq$ wedge of spheres at large primes, and if $H^*(W^1; \mathbb{Z}_p)$ has trivial cup products, then $\Sigma\Omega W^1 \xrightarrow{eval} W^1$ induces surjection on $H_*(-;\mathbb{Z}_p)$, so using splitting $\Sigma\Omega W^1 \simeq \bigvee \Sigma D_i(W)$ and picking appropriate spheres, can construct right homotopy inverse of $eval$. $\square$

Some Useful Spaces

For any $I = \{i_1, \ldots, i_k\} \subseteq [n]$, let

$$W_I^\ell = image(W^\ell \xrightarrow{include} \Sigma^\ell X_1 \times \cdots \times \Sigma^\ell X_n \xrightarrow{project} \Sigma^\ell X_{i_1} \times \cdots \times \Sigma^\ell X_{i_k}),$$

$$W_j^\ell = \bigcup_{I \subseteq [n], |I| = j} W_I^\ell \quad \cong \quad (W_j)^\ell.$$

Filtration

$$* = W_0^\ell \subseteq \cdots \subseteq W_{n-1}^\ell \subseteq W_n^\ell = W^\ell$$

Let

$$\hat{W}_j^\ell = \frac{W_j^\ell}{W_{j-1}^\ell}$$

Then $\hat{W}_j^\ell \cong \bigvee_{|I|=j} \hat{W}_I^\ell$ where

$$\hat{W}_I^\ell \cong \Sigma^{|I|\ell+1} |K_I|.$$

Proposition

(i) homotopy commutative diagram of (homotopy) cofibration sequences

$$\begin{tikzcd}
\mathbb{C}(\hat{\imath}) \arrow[r, "\partial"] \arrow[d, "\simeq"] & \Sigma\mathcal{D}_n(W_{n-1}) \arrow[r, "\Sigma\hat{\imath}", "include"'] \arrow[d] & \Sigma\mathcal{D}_n(W_n) \arrow[r, "\hat{q}"] \arrow[d] & \Sigma\mathbb{C}(\hat{\imath}) \arrow[d, "\simeq"] \\
\Sigma^{n-1}\hat{W}_n \arrow[r] & W^1_{n-1} \arrow[r, "include"] & W^1_n \arrow[r, "q"] & \hat{W}^1_n,
\end{tikzcd}$$

(ii) W^1 is a co-H-space $\Longleftrightarrow$ quotient map $W^1_I \xrightarrow{q} \hat{W}^1_I$ has a right homotopy inverse for each $I \subseteq [n]$.

(iii) (quotient map) q has a right homotopy inverse
$\Longleftrightarrow$ (homotopy cofiber inclusion) $\hat{q}$ has a right homotopy inverse
$\Longleftrightarrow$ ∂ is nullhomotopic.

A Necessary and Sufficient Condition

Let $\triangle_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = \cdots = x_n\}$, and consider the following subspace of $\mathcal{P}_n = (\mathbb{R}^n - \triangle_n) \wedge \Sigma\Delta^n$:

$$\mathcal{Q}_K = \bigcup_{\substack{y \in (\mathbb{R}^n - \triangle_n) \\ (I_1, \ldots, I_m) = [n]_y}} \{y\} \wedge \Sigma|K_{I_1} * \cdots * K_{I_m}|.$$

Then $\mathcal{D}_n(W_{n-1}) \cong \mathcal{Q}_K \vee$ (something else).

K on vertex set $[n]$ is **weakly coherently homotopy Golod** if K is a single vertex, or (recursively) $K \backslash \{i\}$ is weakly coherently homotopy Golod for each $i \in [n]$, and the map

$$\psi_K : \Sigma^n|K| \longrightarrow \Sigma\mathcal{Q}_K$$

given for any $z \in |K|$, $t_1, \ldots t_{n-1}, t \in [-1, 1]$ by

$$\psi_K(t_1, \ldots, t_{n-1}, t, z) = (2\beta - 1, (t_1, \ldots, t_{n-1}, 0), (t, z))$$

is nullhomotopic, where $\beta = \max\{|t_1|, \ldots, |t_{n-1}|, 0\}$.

Theorem C

$\mathcal{Z}_K$ is a co-H-space $\iff$ K is weakly coherently homotopy Golod.

The Coherent Homotopy Golod Condition

(1) *Homotopy Version of Golod:*

Require the inclusions

$$\Sigma\iota_{I,J} \colon \Sigma|K_{I\cup J}| \longrightarrow \Sigma|K_I * K_J|$$

to be nullhomotopic, for each $I \cap J = \varnothing$.

(2) *Coherence of Nullhomotopies:*

Moreover, we will require transitions between these null-homotopies. For example, there is a commutative diagram of inclusions

$$\begin{tikzcd}
 & {|K_{I\cup J_1} * K_{J_2}|} \arrow[dr, "\iota_{I,J_1} * \iota_{J_2}"] & \\
{|K_{I\cup J_1\cup J_2}|} \arrow[ur, "\iota_{I\cup J_1, J_2}"] \arrow[dr, "\iota_{I, J_1\cup J_2}"'] & & {|K_I * K_{J_1} * K_{J_2}|} \\
 & {|K_I * K_{J_1\cup J_2}|,} \arrow[ur, "\iota_I * \iota_{J_1,J_2}"'] &
\end{tikzcd} \tag{1}$$

for disjoint I, J_1, and J_2, and the null-homotopy of $\Sigma\iota_{I,J}$ gives a map

$$\hat{\iota}_{I,J} \colon Cone(\Sigma|K_{I\cup J}|) \longrightarrow \Sigma|K_I * K_J|,$$

we require the composite $\Sigma(\iota_{I,J_1} * \iota_{J_2}) \circ \hat{\iota}_{I\cup J_1,J_2}$ to be homotopic to $\Sigma(\iota_I * \iota_{J_1,J_2}) \circ \hat{\iota}_{I,J_1\cup J_2}$ via a homotopy that is fixed on the base $|K_{I\cup J_1\cup J_2}|$ of the cone $Cone(\Sigma|K_{I\cup J_1\cup J_2}|)$, and so on for longer joins.

Coherently Homotopy Golod: Precise Definition

Delta set

$$\mathcal{K}_n = \{\mathcal{F}_0, \ldots, \mathcal{F}_{n-1}\}$$

- $(m-2)$-face

$$\mathcal{F}_{m-2} = \{\text{set of all ordered partitions } (I_1, \ldots, I_m) \text{ of } [n]\};$$

- face maps $d_i \colon \mathcal{F}_{m-2} \longrightarrow \mathcal{F}_{m-3}$ given by

$$d_i((I_1, \ldots, I_m)) = (I_1, \ldots, I_{i-1}, I_i \cup I_{i+1}, I_{i+2}, \ldots, I_m).$$

$\mathcal{K}_n$ is a simplicial complex of dimension $n-2$,

$$|\mathcal{K}_n| \cong S^{n-2}.$$

A simplicial complex K on vertex set $[n]$ is **coherently homotopy Golod** if K is a single vertex, or (recursively) $K\backslash\{i\}$ is coherently homotopy Golod for each $i \in [n]$ and there is a map

$$\bar{\Psi}_K : |\mathcal{K}_n| \times Cone(\Sigma|K|) \longrightarrow \Sigma|\Delta^{n-1}|$$

such that for any $\gamma \in |\mathcal{K}_n|$, $\bar{\Psi}_K(\gamma, *)$ is the basepoint $*_{-1} \in \Sigma|\Delta^{n-1}|$, and:

(1) the restriction of $\bar{\Psi}_K$ to $\{\gamma\} \times (\{0\} \times \Sigma|K|)$ is the suspended inclusion $\Sigma|K| \longrightarrow \Sigma|\Delta^{n-1}|$;

(2) if $\gamma \in |\mathcal{S}|$ for some $\mathcal{S} = (I_1, \dots I_m) \in \mathcal{F}_{m-2}$, then $\bar{\Psi}_K$ maps $\{\gamma\} \times Cone(\Sigma|K|)$ to a subspace of $\Sigma|K_{I_1} * \cdots * K_{I_m}| \subseteq \Sigma|\Delta^{n-1}|$.

Theorem D

K is coherently homotopy Golod $\Longrightarrow$ $\mathcal{Z}_K$ is a co-H-space.

Extractible complexes are coherently homotopy Golod:

K is **extractible** if $K\backslash\{i\}$ is a simplex for some i, or (recursively) each $K\backslash\{i\}$ is extractible and

$$\bigvee_i \Sigma|K\backslash\{i\}| \xrightarrow{include} \Sigma|K|$$

has a right homotopy inverse. This was the most general class of Golod complexes for which $\mathcal{Z}_K$ is known to be a co-H-space.

Highly complete complexes are coherently homotopy Golod:

K is **highly complete** if every subset of at most $\left\lfloor \frac{n}{2} \right\rfloor$ vertices is a face of K, where n is the number of vertices of K.

Highly complete but not extractible:

Let

$$K = (\partial\Delta^2 * \partial\Delta^1) \cup_{\partial\Delta^1} \Delta^1.$$

K has 5 vertices, and every pair of vertices is connected by an edge, so K is highly connected, therefore homotopy Golod. But $H_2(|K|) \cong \mathbb{Z}$, while $H_2(|K\backslash\{i\}|) = 0$ for each $i \in [5]$, so K cannot be extractible.

K is **homotopy Golod** if it satisfies a homotopy version of Golodness, i.e. the suspended inclusions

$$\Sigma^{n+1}\iota_{I,J} \colon \Sigma^{n+1}|K_{I\cup J}| \longrightarrow \Sigma^{n+1}|K_I * K_J|$$

are nullhomotopic for each $I \cap J = \varnothing$.

Theorem E

$\mathcal{Z}_K$ is a co-H-space $\Longrightarrow$ K is homotopy Golod.

weaker ↓	K is:	$\mathcal{Z}_K$ is a co-H-space?
	coherently homotopy Golod	sufficient
	weakly coherently homotopy Golod	necessary and sufficient
	homotopy Golod	necessary
	Golod	necessary

Question:

$$K \text{ Golod} \Longleftrightarrow \text{homotopy Golod}$$
$$\Longleftrightarrow K \text{ weakly coherently homotopy Golod?}$$

In particular:

If each $\iota_{I,J}$ induces trivial maps on cohomology, then is each $\Sigma^{n+1}\iota_{I,J}$ nullhomotopic?

A counterexample to this would be a counterexample to conjecture: K Golod iff $\mathcal{Z}_K$ co-H-space.

(Grbić, _, Iriye, Kishimoto) Sometimes direct topological proofs are easier:

Let Δ^I denote the $(|I|-1)$-simplex on vertex set $I \subseteq [n]$. Given a simplicial complex L on vertex set J and L' on vertex set J', let

$$K = (L * \Delta^J) \cup_{L * L'} (\Delta^{J'} * L').$$

Then $\mathcal{Z}_K$ is the homotopy pushout:

$$\begin{tikzcd} \mathcal{Z}_L \times \mathcal{Z}_{L'} \arrow[r, "include"] \arrow[d, "include"] & \mathcal{Z}_L \times (D^2)^{\times |J'|} \arrow[d] \\ (D^2)^{\times |J|} \times \mathcal{Z}_{L'} \arrow[r] & \mathcal{Z}_K. \end{tikzcd}$$

Since the inclusions are up to homotopy equivalence the projection maps onto the left and right factor of $\mathcal{Z}_L \times \mathcal{Z}_{L'}$,

$$\mathcal{Z}_K \simeq \mathcal{Z}_L * \mathcal{Z}_{L'} \simeq \Sigma \mathcal{Z}_L \wedge \mathcal{Z}_{L'}$$

i.e. $\mathcal{Z}_K$ is a co-H-space $\Rightarrow$ K Golod. Combinatoral proof K Golod seems more difficult, no direct proof known that K is weakly homotopy Golod.

Problem:

For general K, determine the homotopy type or CW-structure of $\mathcal{Z}_K$ combinatorially in terms of K.

Do this by describing the attaching maps in terms of the combinatorics of K (and perhaps familiar maps such as higher Whitehead products).

Say K on vertex set $[n]$ is ℓ-**Golod** if K_I is Golod whenever $|I| \leq n - \ell$. So Golod is the same as 0-Golod.

Filtration

$$\mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_\infty = \{\text{all finite complexes}\},$$

where $\mathcal{G}_\ell = \{\text{all } \ell - \text{Golod complexes}\}$.

minimally non-Golod complexes

Certain connected sums of sphere products

$$\mathcal{Z}_K = \#_\alpha (S^{i_\alpha} \times S^{j_\alpha})$$

correspond to certain sphere triangulations $K \in (\mathcal{G}_1 - \mathcal{G}_0)$ (McGavran, Bosio, Meersseman). $\mathcal{G}_1 - \mathcal{G}_0$ is the set **minimally non-Golod complexes**.

Homotopy types for K minimally non-Golod:

Let K be minimally non-Golod. Let $W = W_n = (D^k, S^{k-1})^K$, then

$$\begin{tikzcd} & \mathbb{C}(\hat{\imath}) \arrow[r, "\partial"] \arrow[d, "\simeq"] & \Sigma\mathcal{D}_n(W_{n-1}) \arrow[r, "\Sigma\hat{\imath}", "include"'] \arrow[d, "\phi"] & \Sigma\mathcal{D}_n(W_n) \arrow[r, "\hat{q}"] \arrow[d] & \Sigma\mathbb{C}(\hat{\imath}) \arrow[d, "\simeq"] \\ \Sigma^{nk}|K| \arrow[r, equal] & \Sigma^{n-1}\hat{W}_n \arrow[r] & W^1_{n-1} \arrow[r, "include"] & W^1_n \arrow[r, "q"] & \hat{W}^1_n \end{tikzcd}$$

If moreover we assume each $K\backslash\{i\}$ is weakly homotopy Golod, then

$$\Sigma\mathcal{D}_n(W_{n-1}) \simeq \bigvee_{\text{partitions } I_1\cup\cdots\cup I_N=[n],N>1} \Sigma^{nk-N+2}|K_{I_1} * \cdots * K_{I_N}|,$$

and if $k \geq 2$, then ∂ is a suspension, so ∂ is determined by each map

$$\Sigma^{nk}|K| \longrightarrow \Sigma^{nk-N+2}|K_{I_1} * \cdots * K_{I_N}|.$$

What are these maps?
The vertical map ϕ can be determined with respect to the above splitting.

In the interest of finding configuration space models for mapping spaces between manifolds:

Question:

When is a manifold a coordinate suspension? When is the coordinate suspension of a manifold embedded into a product of 1*-disks a manifold?*

This is true for connected sums of sphere products, or for the manifold $\mathcal{Z}_K$ when K is a sphere triangulation.

Thank you!