**EXAMPLES OF SYMPLECTIC MANIFOLDS**
**LECTURE NOTES FOR THE LMS SHORT COURSE**

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## 1. Definition and first examples

*"It is impossible to understand an unmotivated definition but this does not stop the criminal algebraists-axiomatisators..."* - V. I. Arnol'd.

**Definition 1.1.** Let $M$ be a $2n$-dimensional smooth manifold. A closed and non-degenerate two-form $\omega \in \Omega^2(M)$ is called a *symplectic form* on $M$. The pair $(M, \omega)$ is then called a *symplectic manifold*. The closedness means that $d\omega = 0$ and nondegeneracy means that $\omega^n$ is a volume form on $M$. ♠

It is a consequence of the nondegeneracy that the formula

$$X \mapsto i_X\omega$$

defines an isomorphism between the algebra of vector fields and the space of one-forms on a symplectic manifold $(M, \omega)$. Next, it follows from the closedness of the symplectic form that

$$L_X = 0 \quad \text{if and only if} \quad di_X\omega = 0.$$

In other words, the flow $f_t$ of a vector field $X$ preserves the symplectic form if and only if the corresponding one-form is closed. This generalises verbatim to time-dependent vector fields.

**Definition 1.2.** The flow $h_t$ induced by a family of exact one-forms $dH_t$ is called *Hamiltonian*. The function $H_t$ is called a *Hamiltonian function* generating the flow $h_t$. ♠

**Exercise 1.3.** Let $X, Y$ be symplectic vector fields. Show that the function $\omega(X, Y)$ is a Hamiltonian function for $[X, Y]$. ♢

**Definition 1.4.** A differentiable map $f : (M, \omega_M) \to (N, \omega_N)$ is called *symplectic* if $f^*\omega_N = \omega_M$. A symplectic diffeomorphism is sometimes called a *symplectomorphism*. The group of all symplectic diffeomorphism is denoted by $\mathrm{Symp}(M, \omega)$.

A symplectic diffeomorphism generated by a Hamiltonian flow is called a *Hamiltonian diffeomorphism*. The connected group of Hamiltonian diffeomorphisms is denoted by $\mathrm{Ham}(M, \omega)$. ♠

**Exercise 1.5.** Show that $\mathrm{Ham}(M, \omega)$ is indeed a group. ♢

**Exercise 1.6.** Show that a symplectic map is an immersion. ♢

## 1.A. **First examples of symplectic manifolds.**

**Example 1.7** (The standard Euclidean space)**.** The two-form $\omega_0 := dx^1 \wedge dy^1 + \ldots + dx^n \wedge dy^n$ is a symplectic form on $\mathbf{R}^{2n}$. It is called the standard symplectic form on $\mathbf{R}^{2n}$. Observe that it is exact. The Darboux theorem states that every symplectic form is localy diffeomorphic to $\omega_0$. In particular, there are no local invariants of symplectic manifolds. $\diamondsuit$

**Example 1.8** (The complex projective space)**.** Let $\mathbf{h}$ be the standard Hermitian form on $\mathbf{C}^n$ defined by $\mathbf{h}([v_1, \ldots, v_n], [w_1, \ldots, w_n]) := \sum v_i \overline{w}_i$. Observe that the real part of $\mathbf{h}$ is the standard real scalar product and the imaginary part is the standard symplectic form on $\mathbf{C}^n = \mathbf{R}^{2n}$.

Let $\mathbf{CP}^n$ be the complex projective space, that is the space of complex lines in $\mathbf{C}^{n+1}$. The tangent space $T_x\mathbf{CP}^n$ can be identified with the subspace of $\mathbf{C}^{n+1}$ orthogonal to the line $x$ with respect to the Hermitian form $\mathbf{h}$. Define $\omega \in \Omega^2(\mathbf{CP}^n)$ by

$$\omega_x(v, w) := \operatorname{Im} \mathbf{h}(v, w).$$

This is the standard symplectic form on the complex projective space. $\diamondsuit$

**Example 1.9** (The cotangent bundle of a smooth manifold)**.** Let $Q$ be a smooth manifold and let $\lambda \in \Omega^1(T^*Q)$ be a one-form on the cotangent bundle defined by

$$\lambda(X_\alpha) := \langle \alpha, d\pi(X_\alpha) \rangle$$

where $\pi \colon T^*Q \to Q$ is the projection and $\alpha \in T^*Q$. It is called the Liouville form. Observe that in local coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$ we have $\lambda = \sum p_i dq^i$ This implies that $d\lambda$ is a symplectic form. $\diamondsuit$

**Example 1.10.** An area form on an oriented surface is symplectic.

The Cartesian product of two symplectic manifolds $(M_1, \omega_1)$ and $(M_2, \omega_2)$ is symplectic with respect to the form $a \cdot p_1^*\omega_1 + b \cdot p_2^*\omega_2$ for nonzero real numbers $a, b \in \mathbf{R}$. $\diamondsuit$

**Exercise 1.11.** Let $(M, \omega)$ be a closed symplectic $2n$-manifold. Show that $[\omega]^k \neq 0$ in $H^{2k}(M; \mathbf{R})$ for $0 \leq k \leq n$. Produce examples of manifolds which do not admit a symplectic form. $\diamondsuit$

## 1.B. **Symplectic diffeomorphisms.**

**Exercise 1.12.** Draw several examples of functions on $\mathbf{R}^2$ and Hamiltonian flows they generate. Draw two examples of Hamiltonian diiffeomorphisms with intersecting supports. Try to imagine their composition and the complexity of Hamiltonian diffeomorphisms.

Prove that $\operatorname{Ham}(M, \omega)$ acts transitively on $k$-tuples of points. $\diamondsuit$

The above exercise is somewhat misleading because of the following result.

**Theorem 1.13** (Non-squeezing). *Let $(M, \omega)$ be a symplectic $2n$-manifold and let $B^{2n}(r) \subset \mathbf{R}^{2n}$ denote a closed Euclidean ball of radius $r > 0$ equipped with the standard symplectic form. If there exists a symplectic embedding $B^{2n+2}(r) \to B^2(R) \times M$ then $r < R$.* $\square$

If $H\colon M \to \mathbf{R}$ is a function then the associated Hamiltonian vector field has as many zeros as the function $H$ has critical points. Thus the induced flow has as many fixed points as above.

**Exercise 1.14.** Let $(M, \omega)$ be a symplectic manifold. Consider $M \times M$ equipped with a symplectic form $\Omega := p_1^*\omega - p_2^*\omega$. Let $f \in \mathrm{Symp}(M, \omega)$ be a symplectic diffeomorphism. Show that if $\phi\colon M \to M \times M$ is the inclusion of the graph of $f$ then $\phi^*\Omega = 0$. $\diamondsuit$

**Definition 1.15.** Let $(M, \omega)$ be a symplectic $2n$-manifold. An $n$-dimensional submanifold $i\colon L \subset M$ such that $i^*\omega = 0$ is called *Lagrangian*. ♠

**Exercise 1.16.** If $(M, \omega)$ is two dimensional then every one-dimensional submanifold is Lagrangian. If $L \subset \mathbf{S}^2$ is an equator then $f(L) \cap L \neq \emptyset$ for every symplectic diffeomorphism $f$. $\diamondsuit$

**Exercise 1.17.** A diffeomorphism $f\colon Q \to Q$ induces a symplectic diffeomorphism of $(T^*Q, d\lambda)$.

The inclusion of the zero section is Lagrangian.

If $\alpha\colon Q \to T^*Q$ is a one-form viewed as a section of the cotangent bundle then $\alpha^*(\lambda) = \alpha$. In particular, $\alpha^*(d\lambda) = 0$ if and only if $d\alpha = 0$. $\diamondsuit$

**Exercise 1.18.** Let $(Q, g)$ be a Riemannian manifold and let $H\colon T^*Q \to \mathbf{R}$ be defined by $H(p) = \|p\|^2$. Show that the flow induced by $H$ is the geodesic flow of the Riemannian metric $g$. $\diamondsuit$

**Theorem 1.19.** *Let $L \subset (M, \omega)$ be a compact Lagrangian submanifold. There exists a tubular neighbourhood of $L$ symplectically diffeomorphic to a neighbourhood of the zero section in the cotangent bundle $T^*L$.* $\square$

**Exercise 1.20.** Let $\Sigma \subset \mathbf{R}^4$ be a closed, connected and oriented Lagrangian submanifold. Show that it is a torus. Construct a Lagrangian torus in $\mathbf{R}^4$. $\diamondsuit$

**Exercise 1.21.** Suppose that $(M, \omega)$ is simply connected. Show that a symplectic diffeomorphism $f$ of $M$ that is $C^1$-close to the identity has at least as many fixed points as a smooth function has critical points.

If, moreover, all fixed points of $f$ are transverse (that is the graph of $f$ intersects the diagonal transversely) then there are as many of them as critical points of a Morse function. $\diamondsuit$

The generalisation of the above exercise to all Hamiltonian diffeomorphisms and all symlpectic manifolds is known as the *Arnol'd Conjecture*.

**Example 1.22.** The group $U(n+1)$ of unitary transformatons of $\mathbf{CP}^n$ preserves the Hermitian form $\mathbf{h}$. Thus the projective group $PU(n+1)$ acts on $\mathbf{CP}^n$ by symplectic (in fact Hamiltonian) diffeomorphisms. $\diamondsuit$

## 2. Two families of examples

2.A. **Kähler manifolds.** Let $\omega, g$ and $J$ be a nondegenerate two-form, a Riemannian metric and an almost complex structure. This triple is called compatible if

$$\omega(X, JY) = g(X, Y).$$

For every nondegenerate two-form there exists a compatible almost complex structure and the set of such structures is contractible. As a consequence we obtain that a symplectic manifold $(M, \omega)$ admits an almost complex structure $J$. Since the space of almost complex structures is contractible the Chern classes of a symplectic manifold are well defined. We will denote them by $c_k(M, \omega)$.

**Exercise 2.1.** Show that the connected sum $\mathbf{CP}^2 \# \mathbf{CP}^2$ of two complex projective planes does not admit an almost complex structure and hence it cannot be symplectic. $\diamondsuit$

Let $(M, \omega)$ be a symplectic manifold. If a compatible almost complex structure is integrable then the manifold is called Kähler and the compatible metric $g$ is called Kähler as well. A complex submanifod of a Kähler manifold is itself Kähler with respect to the induced structures.

**Exercise 2.2.** Let $(M, \omega)$ be a $2n$-dimensional closed Kähler manifold. The multiplication by the cohomology class of the $k$-th power of the symplectic form is an isomorphism

$$[\omega^k]\wedge \colon H^{n-k}(M; \mathbf{R}) \to H^{n+k}(M; \mathbf{R}).$$

Show that the odd Betti numbers of $M$ are even.

Show that the first Betti number of the fundamental group of a closed Kähler manifold is even. If, moreover, the first Betti number is nonzero then so is the second. $\diamondsuit$

**Exercise 2.3.** Let $M \subset \mathbf{CP}^3$ be a smooth projective hypersurface of degree $d$. That is a submanifold defined as the zero set of a homogeneous polynomial of degree $d$. Show that $M$ is simply connected, compute its Euler characteristic and the signature.

**Answer:** $\chi(M) = d^2 - 4d + 6$ and $\sigma(M) = \frac{1}{3}(4 - d^2)d$.

Generalise this observation to surfaces which are complete intersections, i.e. intersection of $n$ hypersurfaces in $\mathbf{CP}^{n+2}$. ♢

**Exercise 2.4.** Show that the complex projective space $\mathbf{CP}^3$ admits an almost complex structure with the first Chern class equal to zero.

Show that the four dimensional torus $\mathbf{T}^4$ admits an almost complex structure with nontrivial first Chern class.

Determine whether the above almost complex structures admit compatible symplectic forms. ♢

2.B. **Symplectically aspherical manifolds.** Symplectic spheres in symplectic manifolds cause a lot of analytical problems in Gromov-Witten or Floer theory. It is sometimes much easier to deal with manifolds which have no symplectic spheres.

**Definition 2.5.** A symplectic manifold $(M, \omega)$ is called *symplectically aspherical* if

$$\int_{\mathbf{S}^2} f^*\omega = 0$$

for every continuous map $f \colon \mathbf{S}^2 \to M$. ♠

**Exercise 2.6.** Observe that the above property depends only on the cohomology class of the symplectic form. An exact symplectic manifold and a manifold with trivial second homotopy groups are trivially symplectically aspherical.

Show that $(M, \omega)$ is symplectically aspherical if and only if the pullback of the symplectic form to the universal cover is exact.

Decuce that the fundamental group of a closed symplectically aspherical manifold is infinite. It is not known which finitely presented groups can be realised as fundamental groups of symplectically aspherical manifolds. ♢

**Example 2.7.** The following manifolds are symplectically aspherical: an exact symplectic manifold, a manifold with trivial second homotopy group, a torus, a product of oriented surfaces. ♢

**Exercise 2.8.** Show that symplectic submanifold of a symplectically aspherical manifold is symplectically aspherical. Use this to construct examples of Kähler symplectically aspherical manifolds by taking a hyperplane section of a projective manifold $M \subset \mathbf{CP}^n$ isomorphic to a product of surfaces (e.g. an Abelian variety). ♢

**Exercise 2.9.** Let $\Sigma$ be a closed oriented surface of negative Euler characteristic. Show that the total space of a symplectic bundle $\Sigma \to E \to M$ over a symplectically aspherical base is symplectically aspherical. ♢

## 3. Some constructions

### 3.A. Quotients.

**Exercise 3.1.** Let $\Gamma$ be a discrete group acting properly discontinuously on a symplectic manifold $(M, \omega)$. If the action preserves the symplectic form then it induces a symplectic form on the quotient $\Gamma \backslash M$. $\diamondsuit$

**Example 3.2.** The standard symplectic form on $\mathbf{R}^{2n}$ descends to a symplectic form on the torus $\mathbf{T}^{2n} = \mathbf{Z}^{2n} \backslash R^{2n}$.

Let $N_3(\mathbf{R})$ denotes the group of upper triangular matrices with real entries. It is diffeomorphic to $\mathbf{R}^3$ via the map

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, z).$$

The group multiplication defines a free action of $N_3(\mathbf{Z})$ on $N_3(\mathbf{R}) \cong \mathbf{R}^3$. The product action of $\Gamma := N_3(\mathbf{Z}) \times \mathbf{Z}$ on $\mathbf{R}^4 = \mathbf{R} \times \mathbf{R}^3$ preserves the standard symplectic form $dt \wedge dx + dy \wedge dz$. $\diamondsuit$

**Exercise 3.3.** Compute the homology of the *Kodaira-Thurston* manifold

$$KT := (N_3(\mathbf{Z}) \times \mathbf{Z}) \backslash (N_3(\mathbf{R}) \times \mathbf{R}).$$

Deduce that it cannot be Kähler. $\diamondsuit$

### 3.B. Blow-up.

Let $S$ be a compact symplectic submanifold of a symplectic manifold $(M, \omega)$. Remove a small open tubular neighbourhood of $S$ from $M$. The boundary of the remaining part is diffeomorphic to an $S^{2k+1}$-bundle over $S$. Divide it out by the circle action definitng the Hopf map. The resulting manifold is $M$ with a neighbourhood of $S$ replaced by a $\overline{\mathbf{CP}}^k$-bundle over $S$. It can be shown that this manifold admits a symplectic form equal to $\omega$ on the complement of a (slightly bigger) tubular neighbourhood of $S$. However, it depends on choices made and the careful details are quite involved. It is called a blow-up of $(M, \omega)$ along $S$ and we will denote it by $\widetilde{M}_S$.

**Exercise 3.4.** Let $\tilde{M}_k$ be the blow-up of $(M, \omega)$ at $k$-points. Compute the cohomology of $\tilde{M}_k$ assuming that the cohomology of $M$ is known.

Let $(M, \omega)$ be a closed 4-manifold with integral symplectic form. It can be symplectically embedded in $\mathbf{CP}^5$. Compute the cohomology of the blow-up $\widetilde{\mathbf{CP}}^5_M$. Apply this to the Kodaira-Thurston manifold and observe that the resulting manifold is simply connected and not Kähler. $\diamondsuit$

3.C. **Connected sum.** We saw in Example 2.1 that the connected sum of two projective planes is not symplectic. Notice that the orientation matters bacause the blow-up at a point is just the connected sum with $\overline{\mathbf{CP}}^n$.

Let $i, j\colon S \to (M,\omega)$ be two disjoint symplectic embeddedings of a compact submanifold of codimension two. Suppose that the normal bundles to these embeddings have opposite Euler classes. We remove disjoint open tubular neighbourhoods of the two embeddings and identify the boundaries of the remaining manifold. It is a theorem of Gompf that the resulting manifold admits a symplectic form equal to $\omega$ away from the gluing locus. This construction is called the symplectic connected sum.

**Exercise 3.5.** Let $E_1, E_2 \subset \mathbf{CP}^2$ be submanifolds given as zero sets of generic polynomials $p_1$ and $p_2$ both of degree three. Show that each $E_i$ is diffeomorphic to a torus and $E_1$ and $E_2$ intersect transversely at nine points $x_1, \ldots, x_9$.

Consider a family of polynomials $a \cdot p_1 + b \cdot p_2$, where $[a : b] \in \mathbf{CP}^1$. The corresponding (possibly singular) manifolds all intersect at $x_i$. Blowing up $\mathbf{CP}^2$ at these nine points makes them all disjoint and hence we can define a map

$$M := \mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2 \to \mathbf{CP}^1.$$

This map is a singular fibration with finite number of singular fibres. This shows that $M$ admits a symplectic submanifold diffeomorphic to a torus with trivial normal bundle. $\diamondsuit$

**Exercise 3.6.** Show that every finitely presented group can be the fundamental group of a closed symplectic 4-manifold. Since it is impossible to classify finitely presented groups it is also impossible to classify symplectic 4-manifolds.

$\diamondsuit$

3.D. **Symplectic bundles.** Let $(M,\omega)$ be a symplectic manifold. A bundle $M \to E \to B$ is called symplectic (Hamiltonian) if the structure group is a subgroup of $\mathrm{Symp}(M,\omega)$ ($\mathrm{Ham}(M,\omega)$).

We know that the cartesian product of two symplectic manifolds is symplectic. The following theorem by Thurston gives a condition under which the total space of a symplectic bundle admits a symplectic form.

**Theorem 3.7.** *Let $(M,\omega) \xrightarrow{i} E \xrightarrow{\pi} (B,\omega_B)$ be a compact symplectic bundle over a symplectic base. If there exists a cohomology class $a \in H^2(E;\mathbf{R})$ such that $i^* a = [\omega]$ then there exist a representative $\Omega \in a$ and a positive real number $C > 0$ such that $\Omega + t \cdot \pi^* \omega_B$ is a symplectic form for every $|t| > C$.*

**Example 3.8.** Consider the bundle $T^2 \to S^1 \times S^3 \to S^2$ which is the product of the Hopf bundle with the circle. It is a symplectic bundle over a symplectic base but the total space does not admit a symplectic form. $\diamondsuit$

**Exercise 3.9.** Let $\Sigma$ be a closed oriented surface different from the torus. Show that any oriented bundle $\Sigma \to E \to (B, \omega_B)$ satisfies the hypothesis of the Thurston theorem.

More generally, let $(M, \omega) \to E \to (B, \omega_B)$ be a symplectic bundle. If $[\omega] = t \cdot c_1(M, \omega)$ for a nonzero constant $t \in \mathbf{R}$ then the bundle satisfies the hypothesis of the Thurston theorem. $\diamondsuit$

It is not difficult to show that a Hamiltonian bundle over a symplectic base satisfies the hypothesis of the Thurston theorem. The following is an open problem posed by Lalonde and McDuff.

**Exercise 3.10.** Let $(M, \omega) \to E \to B$ be a Hamiltonian bundle. Is it true that the inclusion of the fibre $i\colon M \to E$ induces a surjection on the real cohomology $i^*\colon H^*(E; \mathbf{R}) \to H^*(M, \mathbf{R})$? $\diamondsuit$

**Exercise 3.11.** Give an example of a symplectic but not Hamiltonian bundle such that the inclusion of the fibre induces a surjection on cohomology. $\diamondsuit$

**Example 3.12.** The total space of the bundle $\mathbf{CP}^1 \to \mathbf{CP}^3 \to \mathbf{HP}^1 = \mathbf{S}^4$ is symplectic and the inclusion of the fibre is a symplectic submanifold but the base is not sympectic. $\diamondsuit$

## 4. Symplectic group actions

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. If $G$ acts on a manifold $M$ then every $X \in \mathfrak{g}$ defines a vector field $\underline{X}$ on $M$ tangent to an orbit. Conversely, every vector tangent to an orbit arises this way.

Consider the dual Lie algebra $\mathfrak{g}^\vee$ and define a two form $\omega \in \Omega^2(\mathfrak{g}^\vee)$ by

$$\omega_\xi(\underline{X}, \underline{Y}) := \xi[X, Y]$$

where $\xi \in \mathfrak{g}^\vee$ and $X, Y \in \mathfrak{g}$.

**Exercise 4.1.** Show that this two form is a symplectic form on every orbit of the coadjoint action of $G$ on $\mathfrak{g}^\vee$. Moreover, the action of $G$ preserves the symplectic form.

Such a symplectic manifold is shortly called a coadjoint orbit. This is a rich source of symplectic actions of Lie groups. $\diamondsuit$

If $G$ is a semisimple Lie group then the Killing form defines an equivariant isomorphism between the Lie algebra and its dual. Hence in this case the adjoint orbits are also symplectic.

**Exercise 4.2.** Let $T \subset G$ be a maximal torus in a compact Lie group. Let $\mathfrak{t} \subset \mathfrak{g}$ be the corresponding inclusion of the Lie algebras. Show that every adjoint orbit $M$ has an element in $\mathfrak{t}$. It follows that $M$ is diffeomorphic to $G/H$ where $H$ is a subgroup containing the maximal torus.

The converse is virtually true. That is, if $H \subset G$ is a closed subgroup containing a maximal torus then the homogeneous space $G/H$ admits an invariant symplectic form and its (finite) universal cover is a coadjoint orbit. $\diamondsuit$

**Example 4.3.** Let $M$ be the set of complex structures compatible with the standard scalar product on the oriented vector space $\mathbf{R}^{2n}$. Observe that $\mathrm{SO}(2n)$ acts transitively on $M$ and the isotropy subgroup of the standard complex structure is $\mathrm{U}(n)$ embedded in $\mathrm{SO}(2n)$ in the standard way (how?). Thus the space of complex structures admits a natural symplectic form. $\diamondsuit$

**Exercise 4.4.** Let $(M,g)$ be a Riemannian manifold and let $E \to M$ be a natural bundle where the fibre over a point $x \in M$ consists of complex structures in $T_xM$ compatible with the Riemannian metric. This bundle is clearly Hamiltonian with respect to the symplectic form from the previous example. It is called the *twistor bundle*.

Show that if $g$ has sufficiently pinched sectional curvature then the total space admits a symplectic form such that the fibres are symplectic submanifolds. $\diamondsuit$

**Example 4.5.** The total space of the twistor bundle

$$\mathrm{SO}(2n)/\,\mathrm{U}(n) \to \mathrm{SO}(2n+1)/\,\mathrm{U}(n) \to \mathbf{S}^{2n}$$

is symplectic and the fibres are symplectic submanifolds. This example we can see directly because the total space is a coadjoint orbit. Notice that if $n=2$ then we obtain the bundle $\mathbf{CP}^1 \to \mathbf{CP}^3 \to \mathbf{S}^4$.

Let $M = \Gamma\backslash\,\mathrm{SO}^+(2n,1)/\,\mathrm{SO}(2n)$ be a hyperbolic manifold. Then the total space of the twistor bundle

$$\mathrm{SO}(2n)/\,\mathrm{U}(n) \to \Gamma\backslash\,\mathrm{SO}^+(2n,1)/\,\mathrm{U}(n) \to M$$

admits a symplectic form with symplectic fibres.

It is known that the fundamental group of a closed hyperbolic manifold is never Kähler. Hence the above example is not Kähler. In a recent paper Jonny Evans computed the quantum cohomology of these twistor spaces. $\diamondsuit$

**Exercise 4.6.** Is there an example of a closed hyperbolic four-manifold admitting a symplectic form? $\diamondsuit$

Having a family of examples of symplectic manifolds admitting Hamiltonian actions of Lie groups we can construct many examples of nontrivial Hamiltonian bundles.

**Exercise 4.7.** Let $M = G/H$ be a coadjoint orbit of a semisimple compact and connected Lie group. Construct a nontrivial Hamiltonian bundle $M \to E \to \mathbf{S}^4$. ♢

**Exercise 4.8.** Show that the action of $\mathrm{N}_3(\mathbf{R}) \times \mathbf{R}$ on the Kodaira-Thurston manifold is symplectic but not Hamiltonian.

Construct more general examples of this sort. More precisely, let $\psi\colon \mathbf{R}^2 \to \mathrm{GL}(2, \mathbf{R})$ be a homomorphism. The semidirect product $G := \mathbf{R}^2 \times_\psi \mathbf{R}^2$ is diffeomorphic to $\mathbf{R}^4$ and let's equip it with the standard symplectic form. Give examples of $\psi$ such that the action of $G$ on itself preserves the standard symplectic form.

Give examples of $\psi$ such that it restricts to $\psi\colon \mathbf{Z}^2 \to \mathrm{SL}(2, \mathbf{Z})$. Let $\Gamma := \mathbf{Z}^2 \times_\psi \mathbf{Z}^2$ and give examples of $\psi$ such that the quotient $M := \Gamma\backslash G$ is a closed symplectic manifold.

Of course, $G$ acts on $M$ symplectically. For $\psi(x, y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ we obtain the Kodaira-Thurston example. ♢

## 5. A theorem of Delzant

**Theorem 5.1.** *Let $(M, \omega)$ be a closed symplectic manifold and let $G \subset \mathrm{Symp}(M, \omega)$ be a connected Lie group. Let $H := G \cap \mathrm{Ham}(M, \omega)$. Then the following statements are true.*

(1) *If $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$ is the Levi decomposition of the Lie algebra then $[\mathfrak{r}, \mathfrak{s}] = 0$.*

(2) *If $G$ is semisimple then $G$ is compact.*

(3) *If $G$ is nilpotent then $H$ is central.*

(4) *If $G$ is solvable then $H$ is abelian and $G$ is metabelian.*

*Proof.* The Lie algebra $\mathfrak{symp}(M, \omega)$ of the group of symplectic diffeomorphisms of $(M, \omega)$ consists of the vector fields $X$ such that $L_X\omega = 0$. It can be identified with the space of closed one-forms on $M$. The Lie algebra of the group of Hamiltonian diffeomorphisms consists of the vector fields corresponding to exact one-forms under this identification.

Let $\mathfrak{g} \subset \mathfrak{symp}(M, \omega)$ be the Lie algebra of $G$. Let $X \in \mathfrak{g}$ and consider the complexified endomorphism $\mathrm{ad}^c_X \in \mathrm{End}(\mathfrak{g} \otimes \mathbf{C})$. Let $\lambda$ be an eigenvalue of $\mathrm{ad}^c_X$ and let $Y$ be the corresponding eigenvector. That is, we have

$$[X, Y] = \lambda Y.$$

In the first step we shall show that $\lambda$ is a purely imaginary number.

Consider a function $\omega(X, Y)\colon M \to \mathbf{C}$.

**Exercise 5.2.** Show that $L_X(\omega(X,Y)) = \lambda\omega(X,Y)$. ♢

Let $\psi\colon \mathbf{R} \to \mathrm{Symp}(M,\omega)$ be the flow of the vector field $X$. For every point $x \in M$ we obtain a function $F_x\colon \mathbf{R} \to \mathbf{C}$ defined by

$$F_x(t) := \omega(X,Y)(\psi_t(x)).$$

**Exercise 5.3.** Show that for every $x \in M$.

$$\frac{d}{dt}F_x = \lambda F_x$$

and observe that the claim **(2)** of Theorem 5.1 follows.

Indeed, since $M$ is closed, $F_x$ is bounded and it follows from the above equation that $\lambda$ is purely imaginary. This implies that the Killing form of $G$ is negatively defined and hence the group is compact. ♢

Now, we analyse the restriction of $\mathrm{ad}_X^c$ to its eigenspaces.

**Lemma 5.4.** *If $E_0 \subset \mathfrak{g}$ is the Jordan subspace associated to the eigenvalue zero then $\mathrm{ad}_X^2\colon E_0 \to E_0$ and $\mathrm{ad}_X^2 = 0$.*

*Proof of Lemma 5.4.* Let $Y_0 \in E_0$ and let $Y_1 := [X,Y_0]$, $Y_2 := [X,Y_1]$. Observe that there exists a nonzero $Y_0 \in E_0$ such that $[X,Y_2] = 0$. Our aim is to show that $Y_2 = 0$.

Let $H_1 := \omega(X,Y_0)$ and $H_2 := \omega(X,Y_1)$. These are Hamiltonian functions for $Y_1$ and $Y_2$ respectively.

If $\psi_t$ is the flow of $X$ then

$$\begin{aligned} \frac{d}{dt}(H_1 \circ \psi_t) &= (L_X H_1) \circ \psi_t \\ &= \omega(X,Y_1) \circ \psi_t \\ &= H_2 \circ \psi_t. \end{aligned}$$

Since $[X,Y_2] = 0$ the function $H_2$ is constant along the trajectories of the vector field $X$. It the follows from the compactness of $M$ that $H_2$ is identically zero. Otherwise the function $H_1$ could not be bounded. This implies that $Y_2 = \mathrm{ad}_X^2(Y_0) = 0$. □

Now we prove the claim **(3)** of the theorem. We need to show that $\mathfrak{h} := \mathfrak{g} \cap \mathfrak{ham}(M,\omega)$ is central.

Since $G$ is nilpotent, the eigenvalues of $\mathrm{ad}_X$ are equal to zero for all $X \in \mathfrak{g}$. Let $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$ be any two elements and let $Z := [X,Y]$. We need to prove that $Z = 0$. Observe that we can assume that $[X,Z] = 0$.

Let $H_1$ be a Hamiltonian function of $Y$ and observe that $H_2 := \omega(X,Y)$ is a Hamiltonian of $Z$. Moreover $L_X(H_2) = \omega(X,Z)$ is constant and, since $H_2$ has a critical point according to the compactnes of $M$, this constant is equal to zero.

Notice that $L_X H_1$ is a Hamiltonian for $Z$ and we get that $L_X H_1 = H_2 + C$, where $C \in \mathbf{R}$ is a constant. Since $H_2$ is constant along the flowlines of $X$ we obtain that $H_2 - C = 0$ and hence $Z = 0$ as claimed.

**Exercise 5.5.** Prove the remaining statements of the theorem. $\diamondsuit$

$\square$

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