Lecture 2. Morse homology and Floer homology.

Additional bibliography: F. Laudenbach, Symplectic geometry and Floer homology, Ensaios Matematicos 7 (2004), 1-50

§1. Compactness and gluing in Morse theory.

N finite dim. mfd, $f : N \longrightarrow \mathbb{R}$ proper Morse function, g Riemannian metric

s.t. (f, g) satisfies the Morse - Smale condition:

$$\forall\, p, q \in \mathrm{Crit}(f), \quad W^u(p) \pitchfork W^s(q)$$ (generic condition in the space of Riemannian metrics)

Notation: $\hat{\mathcal{M}}(p,q) = W^u(p) \pitchfork W^s(q)$

$\mathcal{M}(p,q) = W^u(p) \pitchfork W^s(q) \big/ \mathbb{R}$

$|p| = \mathrm{ind}(p)$

$\begin{cases} \text{empty if } |p| < |q| \\ \text{point if } |p| = |q|, \text{ i.e. } p = q. \\ \dim = |p| - |q| \quad \text{if } |p| > |q| \text{ and if nonempty} \end{cases}$

Theorem (compactness): Given $p, q \in \mathrm{Crit}(f)$ and a sequence $[\gamma_n] \in \mathcal{M}(p,q)$, $n \geqslant 1$ there exist points $x^0, x^1, \ldots, x^\ell \in \mathrm{Crit}(f)$, $x^0 = p$, $x^\ell = q$

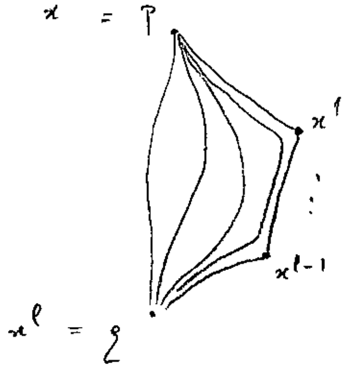
trajectories $[\gamma_i] \in \mathcal{M}(x^{i-1}, x^i)$, $i = 1, \ldots, \ell$

subsequence $[\gamma_{n_k}] \subset [\gamma_n]$

such that $[\gamma_{n_k}]$ converges to the "broken trajectory" $([\gamma_1], \ldots, [\gamma_\ell])$ in the following sense:

$\forall\, i = 1, \ldots, \ell$, $\exists$ "shifts" $s^i_{n_k}$ such that

$$\gamma_{n_k}(\cdot + s^i_{n_k}) \longrightarrow \gamma_i$$ uniformly on compact sets, together with all derivatives.



Crucial ingredient in the proof is the following:

<u>Lemma</u>: Any sequence γ_n of gradient trajectories running from p to q has a subsequence which converges uniformly on compact sets with all derivatives to a gradient trajectory $\gamma : \mathbb{R} \longrightarrow M$.

<u>Proof</u>: Since $\dot{\gamma}_n = -\nabla f(\gamma_n)$ and $\operatorname{im}(\gamma_n) \subset f^{-1}([f(q), f(p)])$ compact.

<u>Step 1</u>. $\quad \Rightarrow \dot{\gamma}_n$ uniformly bounded

Thus γ_n is an equicontinuous & bounded family

By Arzelà - Ascoli: $\exists$ subsequence, still denoted γ_n, converging unif. cpt sets to $\gamma : \mathbb{R} \longrightarrow M$.

<u>Step 2</u>. We now show that γ is a gradient trajectory.

Since $\dot{\gamma}_n = -\nabla f(\gamma_n) \rightsquigarrow \dot{\gamma}_n$ converges unif. cpt sets to vector field v along γ that satisfies $v = -\nabla f(\gamma)$.

But $\dot{\gamma}_n \longrightarrow \dot{\gamma}$ in distributional sense, hence $\dot{\gamma} = -\nabla f(\gamma)$ ▨

<u>Idea of proof of cptness theorem</u>: once we have obtained a limit γ, if its endpoint $\gamma(-\infty)$ is different from p, we shift so that $f(\gamma_n(\cdot + s_n)|_{s=0}) > f(\gamma(-\infty))$ & apply lemma again. Conclude inductively. ▨

<u>Highlight #4</u>: Proof of compactness theorem in Morse theory.

<u>Corollary</u>: If $|p| - |q| = 1$, $\mathcal{M}(p,q)$ is a compact 0-dim. mfd, hence a finite set.

Question: What about $\mathcal{M}(p,q)$ if $|p|-|q|=2$?

We know it is a 1-dimensional manifold, hence each component is either a circle or an open interval.

Theorem (gluing) There is a 1:1 bijective correspondence between the noncompact ends of $\mathcal{M}(p,q)$ and

$$\bigsqcup_{\substack{|r|=|p|-1\\ =|q|+1}} \mathcal{M}(p,r)\times\mathcal{M}(r,q)$$

Rephrasing: There exists "gluing map" defined for $R_0>0$ large enough

$$\# : \bigsqcup_{r} \mathcal{M}(p,r)\times\mathcal{M}(r,q)\times[R_0,\infty[\longrightarrow \mathcal{M}(p,q)$$

such that:
i) $\#$ is diffeomorphism onto its image
ii) the complement of the image has compact closure
iii) $\#([\gamma],[\gamma'],R) \longrightarrow ([\gamma],[\gamma'])$ as $R\to\infty$

convergence to broken trajectory in previous sense.

Remark: We can write the gluing theorem in a more appealing way as

$$\partial\mathcal{M}(p,q) = \bigsqcup_{\substack{|r|=|p|-1\\ =|q|+1}} \mathcal{M}(p,r)\times\mathcal{M}(r,q)$$

In the case $|p|-|q|>2$, the analogous statement is

$$\partial\mathcal{M}(p,q) = \bigsqcup_{x^1,\dots,x^{l-1}} \mathcal{M}(p,x^1)\times\mathcal{M}(x^1,x^2)\times\dots\times\mathcal{M}(x^{l-1},q)$$

Exercise: State a mathematically rigorous gluing theorem for the case $|p|-|q|>2$.

Exercise: Test your understanding of the gluing theorem for Floer trajectories (cf. Highlight # 8) by proving the gluing theorem for Morse gradient trajectories.

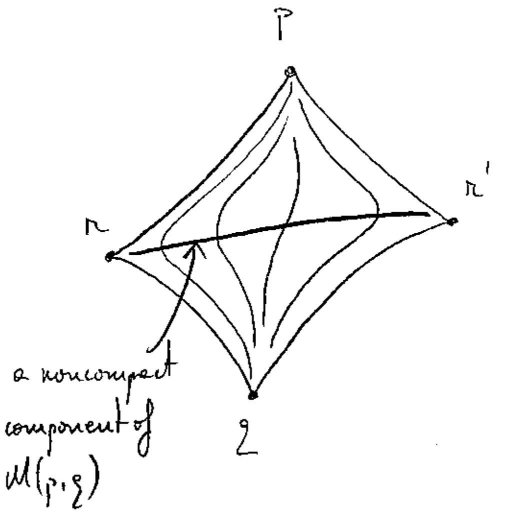
ok (finiteness of $\mathcal{M}(p,q)$ if $|p|-|q|=$

<u>Consequence of compactness and gluing</u>: ∂ is well-defined and $\partial^2=0$.

$$\partial^2(p) = \partial\Big(\sum_{|r|=|p|-1} \#_{alg}\,\mathcal{M}(p,r)\, r\Big) = \sum_{|r|=|p|-1} \#_{alg}\,\mathcal{M}(p,r) \sum_{|q|=|r|-1} \#_{alg}\,\mathcal{M}(r,q)\, q$$

$$= \sum_{|q|=|p|-2} \Bigg[\sum_{\substack{|r|=|p|-1\\ =|q|+1}} \#_{alg}\mathcal{M}(p,r)\cdot \#_{alg}\mathcal{M}(r,q) \Bigg] q$$

$$= \sum_{|q|=|p|-2} \Bigg(\sum_{\substack{|r|=|p|-1\\ =|q|+1}} \ \sum_{\substack{([\gamma],[\gamma'])\\ \in \mathcal{M}(p,r)\times\mathcal{M}(r,q)}} \varepsilon([\gamma])\cdot\varepsilon([\gamma']) \Bigg) \cdot q$$



Now each pair $([\gamma],[\gamma'])$ corresponds to a noncompact end of $\mathcal{M}(p,q)$.

The latter is oriented, and each of its two boundary components inherits an orientation, i.e. a sign (in dimension 0).

<u>Exercise</u>: This sign equals $\varepsilon([\gamma])\cdot\varepsilon([\gamma'])$.

Consequence: in the sum $\sum \varepsilon([\gamma])\cdot\varepsilon([\gamma'])$ the terms cancel two-by-two, so that $\partial^2=0$ ∎

§2. Floer theory

(M, ω) symplectically aspherical manifold.

$\langle \omega, \pi_2 \rangle = 0$ · excludes holomorphic spheres
· allows to define action functional on $\mathcal{L}M$, without going to a cover

$\langle c_1, \pi_2 \rangle = 0$ · allows for $\mathbb{Z}$-grading

More generally: Floer homology (Hamiltonian version) can be defined

· over $\mathbb{Z}$ for monotone symplectic mfds

$$c_1 = \lambda \omega \quad , \quad \lambda > 0$$

· over $\mathbb{Q}$ for arbitrary symplectic mfds.

· with coeff. in a fancy coefficient ring (Novikov ring) to account for the presence of holomorphic spheres

$H : S^1 \times M \longrightarrow \mathbb{R}$ "Hamiltonian"

X_H^t time dependent Hamiltonian vector field, defined by $\omega(X_H^t, \cdot) = dH_t$

$\mathcal{A}_H : \mathcal{L}_0 M \longrightarrow \mathbb{R}$, $\mathcal{A}_H(\gamma) = -\int_{D^2} \bar{\gamma}^* \omega - \int_{S^1} H(t, \gamma(t))\, dt$

(contractible loops in M

with $\bar{\gamma} : D^2 \longrightarrow M$ smooth extension of γ to the disc.

<u>Note</u>: $\langle \omega, \pi_2 \rangle = 0$ ensures that $\mathcal{A}_H$ is well-defined, indep. of choice of $\bar{\gamma}$.

$\mathrm{Crit}(\mathcal{A}_H) \overset{\text{not.}}{=} \mathcal{P}(H) = $ 1-periodic orbits of H

Standing assumption : All 1-periodic orbits of H are nondegenerate, meaning that the linearized time 1 flow has no eigenvalues $= 1$.

$$\forall \gamma \in \mathcal{P}(H), \quad \det\left(\left(\varphi^1_H\right)_*(\gamma(0)) - \mathrm{Id}\right) \neq 0 .$$

Exercise : (i) this is a generic condition in the space of all Hamiltonians.

(ii) it is equivalent to $\mathcal{A}_H$ being Morse (see lecture 3)

(iii) if H is time-independent, the condition is never satisfied along a non-constant 1-periodic orbit.

(iv) if H is time-independent and C^2-small, then it has only constant 1-per. orbits and the condition is equivalent to H being Morse

Almost complex structure J_t, $t \in S^1$ on M, compatible with ω

$\rightsquigarrow$ induces L^2-metric on $\mathcal{L}M$:

$$\xi, \eta \in \Gamma(\gamma^* TM) = T_\gamma \mathcal{L}M$$

$$\langle \xi, \eta \rangle := \int_{S^1} \omega\left(\xi(t), J_t(\gamma(t)) \cdot \eta(t)\right) dt .$$

Differential of $\mathcal{A}_H$ is

$$d\mathcal{A}_H(\gamma) \cdot \xi = \int_{S^1} \omega\left(\dot\gamma - X^t_H(\gamma(t)), \xi(t)\right) dt$$

L^2-gradient vector field of $\mathcal{A}_H$ is

$$\nabla \mathcal{A}_H(\gamma) = J_t(\gamma(t)) \cdot \left(\dot\gamma(t) - X^t_H(\gamma(t))\right)$$

Equation of negative gradient lines $u : \mathbb{R} \longrightarrow \mathcal{L}_0 M \quad \equiv u : \mathbb{R}\times S^1 \to M$

$$\boxed{\partial_s u + J_t(u)\left(\partial_t u - X_H^t(u)\right) = 0}$$

Floer's equation — marvelous mixture between
- the Cauchy-Riemann equation ($H \equiv 0$)
- the usual gradient line equation (H, J, u independent of t).

Note: With my conventions $J X_H = \nabla H$, so that in the case when H, J, u are independent of t the equation is $\partial_s u = \nabla H(u)$, positive gradient lines

Remark: Consider the case $M = \mathbb{C}^n$, $J = i$, $H = 0$.

i) Floer's equation becomes $\partial_s u + i\partial_t u = 0$, i.e. u is holomorphic. In particular $u(s, \cdot)$ is analytic.

Thus the L^2-flow of $\mathcal{A}_H$ is ill-defined on $\mathcal{L}_0 M$!

ii) Given loop $\gamma = \sum_{k\in\mathbb{Z}} z_k e^{ikt}$, its action is $\int x\,dy = \pi \sum_{k\in\mathbb{Z}} k|z_k|^2$.

This quadratic form has infinite-dimensional positive and negative eigenspaces $\longrightarrow$ even if the flow were defined, the index of a critical point would be infinite, hence Morse theory would not apply.

Floer's idea: L^2-gradient trajectories that connect critical points of $\mathcal{A}_H$ come in finite-dimensional families and capture topology.

<u>Definition</u> : Given $x, y \in \mathcal{P}(H)$, denote

$$\widehat{\mathcal{M}}(x,y) = \left\{ u : \mathbb{R}\times S^1 \longrightarrow M \ : \ \bar{\partial}_{J,H} u = 0 \ , \ \begin{array}{l} u(-\infty,\cdot) = x \\ u(+\infty,\cdot) = y \end{array} \right\}$$

$\mathcal{M}(x,y) = \widehat{\mathcal{M}}(x,y)/_{\mathbb{R}}$ if $x \neq y$. "moduli space of Floer trajectories"

<u>Definition</u> : $FC = FC(H,J) = \bigoplus_{x\in\mathcal{P}(H)} \mathbb{Z}\langle x\rangle$

$$\partial : FC \longrightarrow FC$$

$$\partial x = \sum_{\substack{y \\ \dim \mathcal{M}(x,y) = 0}} \#_{alg}\mathcal{M}(x,y)\cdot y \qquad \text{"Floer complex".}$$

this is made possible by a "choice of coherent orientations".

<u>Theorem</u> (Floer) Assume M is closed and $\langle [\omega], \pi_2(M)\rangle = 0$. Then:
[Definition]

- ∂ is well-defined for a generic choice of $J = (J_t)$
- $\partial^2 = 0$
- $FH(H,J)$ is independent of the choice of H and J.

More precisely : a generic homotopy $(H_0, J_0) \longrightarrow (H_1, J_1)$ induces a "continuation isomorphism" $\sigma : FH(H_0,J_0) \longrightarrow FH(H_1,J_1)$ and the latter is independent of the choice of homotopy.

<u>Theorem</u> (Floer) : Same assumptions. We have
[Computation]

$$FH(H,J) \simeq H^{\cdot}(M)$$

singular cohomology

We explain Theorem - Definition and Theorem - Computation in lectures 3 & 4.

Highlight #4. Compactness theorem for gradient trajectories in Morse theory

Setup: $f : M \longrightarrow \mathbb{R}$ proper Morse function, g Riem. metric, M finite dim. mfd

$$\mathcal{M}(p,q) = W^u(p) \cap W^s(q) / \mathbb{R}$$

Theorem: Let $p, q \in \mathrm{Crit}(f)$ and $[\gamma_n] \in \mathcal{M}(p,q)$. There exist

- points $x^0 = p, x^1, \ldots, x^\ell = q \in \mathrm{Crit}(f)$
- trajectories $[\gamma_i] \in \mathcal{M}(x^{i-1}, x^i)$, $i = 1, \ldots, \ell$
- subsequence $[\gamma_{n_k}] \subset [\gamma_n]$

s.t. $\forall i = 1, \ldots, \ell$, $\exists$ shifts $s^i_{n_k}$ s.t. $\gamma_{n_k}(\cdot + s^i_{n_k}) \longrightarrow \gamma_i$ uniformly on cpct sets together with all derivatives (C^∞_{loc})

Lemma: The sequence γ_n has a subsequence that converges in C^∞_{loc} to a gradient traj. γ'

Proof (already seen in lecture 2)

- f proper $\Rightarrow \mathrm{im}(\gamma_n) \subset f^{-1}([f(q), f(p)])$ compact
- $\dot\gamma_n = -\nabla f(\gamma_n)$ and ∇f bded on cpct sets $\Rightarrow \dot\gamma_n$ unif. bded

The conclusion then follows from the Arzelà - Ascoli theorem.

(which is the basis of all compactness theorems for function spaces).

Note: convergence of derivatives is formal consequence of the equation $\dot\gamma = -\nabla f(\gamma)$. ▨

Proof of the theorem

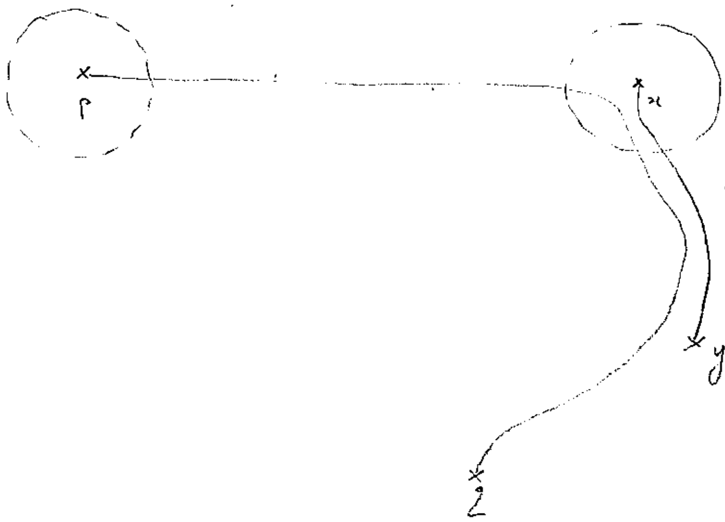
Step 1 (exercise). Let $\gamma_n \xrightarrow{C^\infty_{loc}} \gamma'$ with $\gamma_n \in \widehat{\mathcal{M}}(p,q)$ and $\gamma' \in \widehat{\mathcal{M}}(x,y)$. Then

$$f(p) \geq f(x) \geq f(y) \geq f(q)$$

Step 2: Assume $f(p) = f(x)$. Then $p = x$. Similarly, if $f(y) = f(q)$ then $y = q$.

It is of course enough to prove the first assertion. Assume by contradiction $p \neq x$.

Fix disjoint balls B_p, B_x around p and x which do not contain other critical points than p and x. By possibly shrinking B_p we can assume w.l.o.g. that $\mathrm{im}\,\gamma' \cap B_p = \emptyset$.



Let $s_n = \inf_s \gamma_n(s) \in \partial B_p$ (finite, since $\gamma_n(s) \in \operatorname{int} B_p$ for $s \ll 0$).

The sequence $\gamma_n(\cdot + s_n)$ has convergent subsequence in C^∞_{loc} to $\tilde{\gamma} \in \mathcal{M}(\tilde{x}, \tilde{y})$.

this is automatic, since $\gamma_n(-\infty) = p$.

<u>Claim</u>: $\forall \varepsilon, \exists n(\varepsilon), \forall n \geq n(\varepsilon), \; f(p) \geq f(\gamma_n(s_n)) \geq f(x) - \varepsilon$.

<u>Fix $\varepsilon > 0$</u>. Choose σ s.t. $f(x) - f(\gamma(\sigma)) < \frac{\varepsilon}{2}$ and $\gamma(\sigma) \in B_x$

Since $\gamma_n(\sigma) \longrightarrow \gamma(\sigma)$, $\exists n(\varepsilon)$, $\forall n \geq n(\varepsilon)$, $\gamma_n(\sigma) \in B_x$ and

$$f(\gamma(\sigma)) - f(\gamma_n(\sigma)) < \frac{\varepsilon}{2}.$$

Thus $f(x) - f(\gamma_n(\sigma)) < \varepsilon$, i.e. $f(\gamma_n(\sigma)) > f(x) - \varepsilon$.

But our defi. of s_n ensures $f(\gamma_n(s_n)) > f(\gamma_n(\sigma))$

This proves the claim.

Passing to the limit $n \to \infty$ we obtain

$$\forall \varepsilon, \; f(p) \geq f(\tilde{\gamma}(0)) \geq f(x) - \varepsilon \overset{\text{assumption}}{=} f(p) - \varepsilon.$$

Thus

$$f(p) \geq f(\tilde{x}) \geq f(\tilde{\gamma}(0)) = f(p),$$ which implies that $\tilde{\gamma}$ is a constant trajectory — a contradiction with $\tilde{\gamma}(0) \in \partial B_p$, which is disjoint from $\mathrm{Crit}(f)$. $\boxtimes$

Step 3 : Let us fix $\varepsilon > 0$ s.t. $[f(p)-\varepsilon, f(p)[$ is regular interval and choose s_n^1 s.t. $f(\gamma_n(s_n^1)) = f(p) - \varepsilon$.

By Lemma, Step 1, and Step 2 we get subsequence γ_{n_k} s.t.

$$\gamma_{n_k}(\cdot + s_{n_k}^1) \xrightarrow{C^\infty_{loc}} \gamma^1 \in \hat{\mathcal{M}}(p, x^1).$$

If $x^1 = q$, we are done

If not, we have $f(x^1) > f(q)$ by Step 2. We repeat this same procedure with $\varepsilon > 0$ chosen such that $[f(x_1)-\varepsilon, f(x_1)[$ is regular interval, and s_n^2 s.t. $f(\gamma_n(s_n^2)) = f(x_1) - \varepsilon$.

We continue inductively. The process is finite since the interval $[f(q), f(p)]$ contains a finite number of critical values.

$\square$