Strongly bounded groups

(in progress)

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One of my personal motivations

(M, ω) – a closed symplectic manifold.
G – a group.

Is there an inclusion?

$$G \longrightarrow \mathrm{Ham}(M, \omega)$$

Investigate
conj. inv. norms,
find restrictions,
find invariants...

Hofer's norm:
conjugation invariant,
nondiscrete,
unbounded,
separable...

Definitions and notation

- ▷ G – a group.
- ▷ $C(g)$ – the conjugacy class of g.
- ▷ S – a symmetric conjugation invariant subset.
- ▷ The number of conjugacy classes in S:

$$\#S = \min\left\{n \in \mathbb{N} \,|\, S = \cup_{i=1}^{n} C\left(g_i^{\pm 1}\right)\right\}$$

Standing assumption:

G is generated by S such that $\#S < \infty$.

Examples: finitely generated, simple, semisimple Lie, infinite braid, $\mathrm{Diff}_0(M)$, $\mathrm{Ham}(M, \omega)$...

Definitions and notation

- The word norm:

$$\|g\|_S = \min\left\{n \in \mathbb{N} \,|\, g = s_1 \cdots s_n\,, s \in S\right\}$$

- The diameter: $\delta(S) = \sup_g \|g\|_S$.
- The sup-diameter:

$$\Delta_k(G) = \sup\left\{\delta(S) \,|\, \#S < k+1\right\}.$$

- Inequalities:

$$0 \leq \Delta_1(G) \leq \Delta_2(G) \leq \cdots \leq \Delta_\infty(G) \leq \infty.$$

Boundedness

$$0 \leq \Delta_1(G) \leq \Delta_2(G) \leq \cdots \leq \Delta_\infty(G) \leq \infty.$$

- ▷ G is bounded if .... $\delta(S) < \infty$.
- ▷ G is *strongly* bounded if .... $\exists k$ $\Delta_k(G) < \infty$.
- ▷ G is *uniformly* bounded if ..... $\Delta_\infty(G) < \infty$.

Examples.

- ▷ $\mathrm{SL}(3, \mathbb{Z})$ is bounded. [Elementary bounded generation]
- ▷ $\mathrm{Diff}_0^c(\mathbb{R}^n)$ is bounded. [Burago-Ivanov-Polterovich]
- ▷ G – simple $\Longrightarrow$ $\Delta_1(G) = \Delta_\infty(G)$.

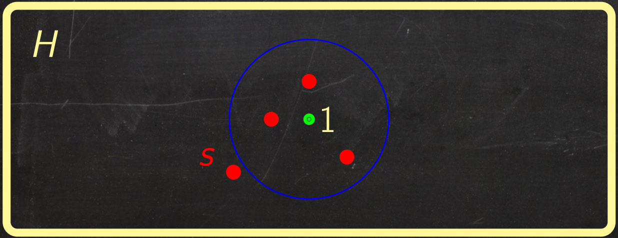
Nonsqueezing Lemma

Assume:

- (H, ν) – a group with a conj. invariant norm
- $G \to (H, \nu)$ – an injective homomorphism
- $S \subset G$ and $\#S < k + 1$.

Then:

$$\max_{s \in S} \nu(s) \geq \frac{\text{diameter}(G, \nu)}{\Delta_k(G)}.$$



At least one generator has to be away from the identity.

Application: Hamiltonian actions

- $G \to (H, \nu)$ – as above.

Corollary. If G is uniformly simple then the ν-topology is discrete on G.

Corollary. If $G \subset \mathrm{Ham}(M, \omega)$ is uniformly simple then it is countable.
Proof: The Hofer topology is separable.

Examples

- $\Delta_\infty(\mathrm{SL}(n,\mathbb{R})) = O(n^2)$ . . . . . uniformly bounded.
- $\Delta_1(\mathrm{SO}(3)) = \infty$ . . . . not strongly bounded.
- $\Delta_\infty(\mathrm{SL}(n,\mathbb{Z})) = \infty$ . . . . . not uniformly bounded.
- $\Delta_k(\mathrm{SL}(n,\mathbb{Z})) = O(n^2 k)$ . . . . . . strongly bounded.
- $\Delta_k(\mathrm{SO}(3,\mathbb{Z}[1/5])) = \infty$ . . not strongly bounded.
- Assume:
  - $\mathfrak{R}$ – ring with m maximal ideals;
  - $\mathrm{SL}(n,\mathfrak{R})$ – elementary boundedly generated

  Then:

  $\Delta_\infty(\mathrm{SL}(n,\mathfrak{R})) \le \Delta_m(\mathrm{SL}(n,\mathfrak{R}))$.

  $\Delta_\infty(\mathrm{SL}(n,\mathfrak{R})) < \infty$ . . . . uniformly bounded.

Theorem

Assume:

- ▷ $\mathfrak{R}$ – a principal ideal domain,
- ▷ $SL(n, \mathfrak{R})$ – elementary bounded generation,

Then:

$$\Delta_k(SL(n, \mathfrak{R})) \leq (8n + 4)kb.$$

Example.

$$\Delta_\infty(PSL(n, p^m)) \leq (8n + 4)(4n - 1).$$

A special case of [Liebeck-Shalev]

Application: Hamiltonian actions

- $(M, \omega) =$ a closed symplectic manifold.

Recall: A uniformly simple group $G \subset \mathrm{Ham}(M, \omega)$ is countable.

The following groups cannot be subgroups of $\mathrm{Ham}(M, \omega)$:

- simple noncompact Lie, [Delzant for smooth actions]
- some groups acting on linear orders,

[uniform simplicity due to Gal-Gismatullin-Lazarovich]

- $\mathrm{SL}(n, \mathfrak{R})$ for a suitable ring $\mathfrak{R}$,
- $\mathrm{Diff}_0(\mathbb{S}^1)$. [uniform simplicity due to Tsuboi]

Open problem

Is there a closed symplectic manifold (M, ω) such that

$$\mathrm{SL}(3, \mathbb{Z}) \subset \mathrm{Ham}(M, \omega)?$$

Remarks:

- No, if (M, ω) is a closed surface. [Polterovich]
- $\mathrm{SL}(3, \mathbb{Z}) \subset \mathrm{Ham}(M, \omega)$ for some noncompact M.
- $\mathrm{SL}(3, \mathbb{Z}) \subset \mathrm{Diff}_0(M, \mathrm{vol})$ for some closed M.

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