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Miscellanea

The distribution of the difference between two *t*-variates

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Summary

In this paper, the difference between two correlated t variables is divided by a function of their sample correlation and the distribution of the resulting quantity is examined. Functions of the sample correlation are found for which this quantity is approximately pivotal and has a t distribution, asymptotically. Simulations show that the asymptotic results hold well for small sample sizes. The results yield a useful test for comparing the difference in standardised scores of an individual with those of a group of controls. The test assumes that sampling is from a bivariate normal distribution and robustness of the test to departure from normality is examined.

Some key words: Asymptotic t distribution; Bivariate t distribution; Dissociation; Standardised difference.

1. Introduction

We consider the distribution of the difference between two correlated Student variates, t_1 and t_2 say, where t_1 corresponds to the x observations and t_2 to the y observations from a sample of size n from a bivariate normal distribution. We consider the asymptotic distribution of a quantity $(t_1 - t_2)/h$, where h is a function of the sample correlation coefficient. We find an h for which $(t_1 - t_2)/h$ is independent of the population correlation and has approximately a t distribution with error of order $O(n^{-2})$. We also give a function h that enables tail areas of the distribution of $(t_1 - t_2)/h$ to be approximated by tail areas of a t distribution with error of order $O(n^{-3})$. This can be used for hypothesis testing and yields a reasonably straightforward function of h and $t_1 - t_2$ that has approximately a t distribution with error of order $O(n^{-3})$. Simulation results show that the method works very well for values of n as low as 10 and that it gives useful results for n as low as 5. Application of the method to a problem from psychology is also considered.

2. Asymptotic results

Define

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad \frac{M}{n-1} = \begin{pmatrix} s_1^2 & rs_1s_2 \\ rs_1s_2 & s_2^2 \end{pmatrix} \tag{1}$$

and assume that M has a Wishart distribution with scale parameter Σ and $n-1$ degrees of freedom, so that $E(M) = (n-1)\Sigma$. Also, suppose that (u, v) is independent of M and

$$(u, v)' \sim N\{(\mu_1, \mu_2)', \Sigma/n\}. \tag{2}$$

The most common situation where these distributions arise is where (u, v) is the sample mean and $M/(n-1)$ is the sample variance-covariance matrix for a random sample of n observations from a bivariate normal distribution with mean $(\mu_1, \mu_2)'$ and variance-covariance matrix Σ.

Let $t_1 = (u-\mu_1)s_1^{-1}n^{1/2}$ and $t_2 = (v-\mu_2)s_2^{-1}n^{1/2}$. Interest centres on $t_1 - t_2$ and the aim is to derive a variable from it whose asymptotic distribution is tractable and does not depend on the unknown parameters ρ, σ_1^2, σ_2^2, μ_1 and μ_2. To this end, we consider variables of the form

$$w = (t_1 - t_2) \Big/ \left\{ 2 - 2r + \frac{b_0 + b_1 r + b_2 r^2}{n-1} + \frac{c_0 + c_1 r + c_2 r^2 + c_3 r^3}{(n-1)^2} \right\}^{1/2}, \tag{3}$$

where $b_0, b_1, b_2, c_0, \ldots, c_3$ are values to be specified. Siddiqui (1967) shows that the joint probability density function of t_1, t_2 and r is

$$\begin{aligned} f_1(t_1, t_2, r) = {} & [(n-2)\Gamma(n)(1-\rho^2)^{n/2}/\{(2\pi)^{3/2}\Gamma(n+\tfrac{1}{2})\}] \\ & \times [\{1 + t_1^2/(n-1)\}\{1 + t_2^2/(n-1)\}]^{-n/2}(1-r^2)^{(n-4)/2} \\ & \times (1 - a_0 - a_1 r)^{-n+1/2} F\{\tfrac{1}{2}, \tfrac{1}{2}, n+\tfrac{1}{2}, \tfrac{1}{2}(1 + a_0 + a_1 r)\}, \end{aligned} \tag{4}$$

where

$$a_0 = \{\rho t_1 t_2/(n-1)\}[\{1 + t_1^2/(n-1)\}\{1 + t_2^2/(n-1)\}]^{-1/2},$$
$$a_1 = \rho[\{1 + t_1^2/(n-1)\}\{1 + t_2^2/(n-1)\}]^{-1/2}$$

and $F(a, b, c, x) = 1 + (ab/c)(x/1!) + [a(a+1)b(b+1)/\{c(c+1)\}](x^2/2!) + \ldots$ is the Gauss hypergeometric series.

Starting from equation (4), we used the computer algebra system Maple to obtain the asymptotic distribution of w; details are given in the Appendix. Its density function is

$$\begin{aligned} f_2(w) = {} & (2\pi)^{-1/2} e^{-w^2/2} \\ & \times \Bigg(1 + \Bigg[-5 - 2b_2 + 2w^2 + 2b_2 w^2 + w^4 + b_1(w^2 - 1) \\ & \qquad + (1 - w^2)\left\{ \frac{(2 + b_2)(1-\rho)^2 + b_0 + b_1 + b_2}{1-\rho} \right\} \Bigg] \Big/ \{4(n-1)\} \\ & \quad + \Bigg[(b_0 - 2)\phi_0 + b_1\phi_1 + (b_2 + 2)\phi_2 + (c_0 + c_1\rho + c_2\rho^2 + c_3\rho^3)\frac{1 - w^2}{4(1-\rho)} \\ & \qquad - \frac{19 + 24\rho + 4\rho^2}{32} + \frac{w^2(3 - 2\rho^2)}{8} + \frac{w^4(7 + 4\rho + 2\rho^2)}{16} + \frac{3w^8 - 28w^6}{96} \Bigg] \Big/ (n-1)^2 \Bigg) \\ & + O(n^{-3}), \end{aligned} \tag{5}$$

where the functions ϕ_0, ϕ_1 and ϕ_2 do not involve n, their formulae being

$$\begin{aligned} \phi_0 = {} & \{(w^4 - 2w^2 - 1)(b_0 + 2b_2\rho^2) + 2(-2 + 5w^2 + 4w^4 - w^6 + 3\rho + 3w^2\rho - 7w^4\rho + w^6\rho - 12w^2\rho^2 \\ & + 4w^4\rho^2 - 2\rho^3 + 2w^2\rho^3)\}\{32(1-\rho)^2\}^{-1}, \\ \phi_1 = {} & \{(w^4 - 2w^2 - 1)(2b_0\rho + b_1\rho^2 + 2b_2\rho^3 - 4) + 2(-\rho + w^2\rho + 5w^4\rho - w^6\rho - \rho^2 + w^2\rho^2 - 5w^4\rho^2 \\ & + w^6\rho^2 - 6w^2\rho^3 + 2w^4\rho^3)\}\{32(1-\rho)^2\}^{-1}, \\ \phi_2 = {} & \{b_2\rho^4(w^4 - 2w^2 - 1) + 2(4 - 4w^2 + 12w^2\rho - 4w^4\rho - 11\rho^2 - w^2\rho^2 + 9w^4\rho^2 - w^6\rho^2 + 3\rho^3 \\ & - 9w^2\rho^3 - 3w^4\rho^3 + w^6\rho^3 + 5\rho^4 - 2w^2\rho^4 - w^4\rho^4 - 2\rho^5 + 2w^2\rho^5)\}\{32(1-\rho)^2\}^{-1}. \end{aligned}$$

From equation (5) the distribution of w is approximately normal with error of order $O(n^{-1})$. To obtain a higher-order approximation, let $g(.)$ be a $t(n-1)$ distribution. Then

$$g(w)=(2\pi)^{-1/2}e^{-w^2/2}[1+(w^4-2w^2-1)/\{4(n-1)\} + (3w^8-28w^6+30w^4+12w^2+3)/\{96(n-1)^2\}]+O(n^{-3}). \quad (6)$$

If we set $b_0=2$, $b_1=0$ and $b_2=-2$, comparison of (5) and (6) shows that $f_2(.)$ is approximately a t density and is independent of ρ. If we make these substitutions in (3) and set $c_0=c_1=c_2=c_3=0$,

$$w_1:=(t_1-t_2)\bigg/\left\{2-2r+\frac{2(1-r^2)}{n-1}\right\}^{1/2}\sim t(n-1), \quad (7)$$

with error of order $O(n^{-2})$. Also, with the same values for b_0, b_1 and b_2, straightforward integration gives

$$\int_k^\infty \{g(w)-f_2(w)\}dw = ke^{-k^2/2}\{8(2\pi)^{1/2}(1-\rho)\}^{-1} \times \{2c_0-5-k^2+(2c_1-1-k^2)\rho+(2c_2+5+k^2)\rho^2 + (2c_3+1+k^2)\rho^3\}/(n-1)^2+O(n^{-3}).$$

Consequently, if we set $c_0=(5+k^2)/2$, $c_1=(1+k^2)/2$, $c_2=-(5+k^2)/2$ and $c_3=-(1+k^2)/2$, then the tail area $\int_k^\infty f_2(w)dw$ is independent of ρ and approximately equals the corresponding tail area of a $t(n-1)$ distribution, with error of order $O(n^{-3})$.

The result may be used to test the hypothesis that $E(u,v)=(\mu_1,\mu_2)$. For a test at a specified significance level, let k be the corresponding critical value of a $t(n-1)$ distribution. Put

$$w_2(k)=(t_1-t_2)\bigg/\left[2-2r+\frac{2(1-r^2)}{n-1}+\frac{\{5+r+k^2(1+r)\}(1-r^2)}{2(n-1)^2}\right]^{1/2}. \quad (8)$$

Then the hypothesis is rejected at the specified significance level if and only if $|w_2(k)|>k$. To obtain a p-value, solve $w_2(k)=k$, which is a quadratic equation in k^2. Choosing the positive root gives

$$k=[\{-b+(b^2-4ac)^{1/2}\}/2a]^{1/2}, \quad (9)$$

where

$$a=(1+r)(1-r^2),\quad b=(1-r)\{4(n-1)^2+4(1+r)(n-1)+(1+r)(5+r)\},$$
$$c=-2(t_1-t_2)^2(n-1)^2.$$

Then the p-value equals $2\,\mathrm{pr}(t>k)$, where $t\sim t(n-1)$.

Equation (9) also enables the approximation given by $w_2(k)$ to be expressed in a form that does not involve k. Let $w_2^*=\mathrm{sgn}(t_1-t_2)[\{-b+(b^2-4ac)^{1/2}\}/2a]^{1/2}$. Then $w_2^*\sim t(n-1)$, approximately, with error of order $O(n^{-3})$.

To examine the accuracy of approximations, simulations were conducted to compare $\mathrm{pr}(w_1>k)$ and $\mathrm{pr}\{w_2(k)>k\}$ with the tail area of a $t(n-1)$ distribution. Accuracy varies with the sample size, n, and the correlation, ρ, and we took values 5, 10, 20, 50 and 100 for n and $-0{\cdot}95$, $-0{\cdot}8$, $-0{\cdot}5$, $-0{\cdot}2$, $0{\cdot}0$, $0{\cdot}2$, $0{\cdot}5$, $0{\cdot}8$ and $0{\cdot}95$ for ρ. For each combination of n and ρ, 10^6 simulations were conducted in each of which a sample of n observations was generated from a bivariate normal distribution with mean $(0,0)'$, unit variances and correlation ρ. The sample means, variances and correlation were determined and the corresponding sample value of w_1 was calculated. The sample values of $w_2(k)$ were also calculated for k set equal to the 40th, 30th, 20th, 10th, 5th, 2·5, 1·0 and 0·5 percentiles of a $t(n-1)$ distribution. These values of w_1 and $w_2(k)$ were compared with each value of k and the proportions of the simulations for which they exceeded each k were recorded. A selection of these results are given in Tables 1 and 2, together with their standard errors. Table 1 shows that the distribution of w_1 is well approximated by the t distribution for $n\geqslant 20$ but the

Table 1. *Percentages of the simulations for which w_1 exceeded specified percentiles, k, of a $t(n-1)$ distribution, for various sample sizes, n, and correlations, ρ. Standard errors* (SD) *for the percentages in each column are also given*

		Correct percentage: $100 \times \mathrm{pr}(t > k)$							
n	ρ	40	30	20	10	5	2·5	1·0	0·5
5	0·95	40·79	31·55	22·23	12·57	7·32	4·35	2·22	1·35
5	0·50	40·57	31·11	21·62	11·85	6·66	3·83	1·88	1·11
5	0·00	40·36	30·72	21·03	11·16	6·07	3·36	1·59	0·91
5	−0·50	40·13	30·34	20·48	10·58	5·53	2·92	1·30	0·72
5	−0·95	40·05	30·02	20·06	10·07	5·05	2·54	1·02	0·53
10	0·95	40·15	30·34	20·46	10·57	5·51	2·91	1·27	0·69
10	0·50	40·14	30·24	20·33	10·39	5·35	2·80	1·19	0·62
10	0·00	40·11	30·19	20·23	10·23	5·20	2·65	1·11	0·58
10	−0·50	40·03	30·04	20·10	10·10	5·09	2·58	1·05	0·53
10	−0·95	40·03	30·04	20·03	10·02	5·01	2·50	1·01	0·50
20	0·95	40·05	30·09	20·16	10·14	5·12	2·59	1·06	0·54
20	0·50	40·05	30·04	20·07	10·10	5·09	2·58	1·06	0·54
20	0·00	40·04	30·02	20·02	10·05	5·05	2·53	1·02	0·51
20	−0·50	40·02	30·02	20·00	10·00	5·01	2·50	1·00	0·51
20	−0·95	39·97	29·95	19·92	9·95	4·99	2·48	1·00	0·50
	SD	0·05	0·05	0·04	0·03	0·02	0·02	0·01	0·01

Table 2. *Percentages of the simulations for which $w_2(k)$ exceeded specified percentiles, k, of a $t(n-1)$ distribution, for various sample sizes, n, and correlations, ρ. Standard errors* (SD) *for the percentages in each column are also given*

		Correct percentage: $100 \times \mathrm{pr}(t > k)$							
n	ρ	40	30	20	10	5	2·5	1·0	0·5
5	0·95	40·27	30·53	20·77	10·92	5·75	2·98	1·20	0·57
5	0·50	40·19	30·37	20·53	10·55	5·42	2·72	1·05	0·49
5	0·00	40·11	30·23	20·28	10·25	5·17	2·56	0·97	0·44
5	−0·50	40·05	30·08	20·15	10·12	5·06	2·50	0·93	0·41
5	−0·95	40·00	30·01	20·02	10·00	4·99	2·49	0·99	0·48
10	0·95	40·03	30·09	20·12	10·11	5·10	2·57	1·03	0·51
10	0·50	40·02	30·02	20·02	10·06	5·04	2·51	0·99	0·49
10	0·00	40·00	29·99	19·97	9·99	5·00	2·51	0·99	0·49
10	−0·50	40·01	30·04	20·05	10·02	5·01	2·50	0·99	0·49
10	−0·95	40·00	30·02	20·03	9·99	4·99	2·49	0·98	0·50
20	0·95	40·00	30·00	20·01	10·01	5·01	2·51	1·00	0·50
20	0·50	40·02	29·99	19·99	10·01	5·01	2·52	1·01	0·51
20	0·00	40·03	30·02	20·01	9·99	4·99	2·50	1·00	0·50
20	−0·50	40·01	30·04	20·02	10·02	5·01	2·50	1·00	0·50
20	−0·95	39·99	29·96	19·97	9·99	4·99	2·50	1·00	0·50
	SD	0·05	0·05	0·04	0·03	0·02	0·02	0·01	0·01

approximation is poor, particularly in the extreme tails of the distribution, for $n = 5$. For $n = 10$, the approximation would be usable for many purposes. Table 2 shows that $\mathrm{pr}\{w_2(k) > k\}$ is approximated reasonably well by the t distribution for $n \geqslant 10$ and the approximation would be adequate for many purposes for $n \geqslant 5$.

To try to improve on the approximations given by w_1 and $w_2(k)$, we also considered variables of the form

$$w^* = (t_1 - t_2) \Big/ \left\{ 2 - 2r + \frac{b_0 + b_1 r + b_2 r^2}{n-1} + \frac{c_0 + c_1 r + c_2 r^2 + c_3 r^3}{(n-1)^2} + \frac{d_0 + d_1 r + d_2 r^2 + d_3 r^3 + d_4 r^4}{(n-1)^3} \right\},$$

which is a natural extension of equation (3). The same approach that gave $w_2(k)$ was followed, except that sufficient terms were retained in asymptotic expansions to give an accuracy of order $O(n^{-4})$. However, simulations revealed that for small n the resulting approximation is much poorer than $w_2(k)$. In summary, the results show that inferences about $t_1 - t_2$ should be based on $w_2(k)$ $(\equiv w_2^*)$ unless tractability is important and n is of a reasonable size, $n \geqslant 20$ say, when the greater simplicity of w_1 may make it preferable.

3. Tests for dissociations

Motivation for this paper stemmed from the use of dissociation tests for single case studies in psychology and academic neurology. Suppose for example that a patient with neurological damage is given two tests of memory, one that measures short-term memory, such as digit span, and the other long-term memory, such as learning a list of digits that exceeds the individual's immediate digit span. If the person's performance on one of the tests differs markedly from their performance on the other, then the person is said to exhibit a dissociation. This form of evidence is important in attempts to fractionate the human cognitive system into its constituent parts, and to determine whether different parts of the brain perform the different tests. Usually, scores on the two tests will not be comparable until they have been standardised, which is achieved by administering the tests to a group of controls. The size of the group is typically small, particularly if controls must match the patient for specific covariates, such as age, sex and number of years in education, say (Crawford & Howell, 1998).

Let x and y denote the scores of a control on two tests and assume that

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \Sigma \right), \tag{10}$$

where Σ is defined in equation (1). If x^* and y^* denote the scores of a case, then the case is said to have a dissociation with respect to the two tests if

$$\left| \frac{x^* - \mu_x}{\sigma_1} - \frac{y^* - \mu_y}{\sigma_2} \right| \tag{11}$$

is so large that x^* and y^* are unlikely to be the scores of a control. We assume that μ_x, μ_y, σ_1 and σ_2 are unknown and must be estimated from a control sample. To relate a test of dissociation to results in § 2, suppose that a sample of n controls are asked to perform the two tasks. The sample covariance matrix of their scores may be denoted by $M/(n-1)$, as then $M \sim \text{Wi}\,(n-1, \Sigma)$, consistent with the definition of M in equation (1). Let $\bar{x}$ and $\bar{y}$ be the mean scores of the controls and define s_1^2, s_2^2 and r by equation (1). Also, let

$$u = (x^* - \bar{x})/(n+1)^{1/2}, \quad v = (y^* - \bar{y})/(n+1)^{1/2},$$

where x^* and y^* are the scores of the patient. Clearly (u, v) is independent of M. Moreover, under the null hypothesis that (x^*, y^*) is an observation from the distribution in equation (10), (u, v) has the bivariate normal distribution given in equation (2) with $(\mu_1, \mu_2)' = (0, 0)'$. Hence, under the null hypothesis, the approximations given in § 2 apply to the distribution of $t_1 - t_2$, where

$$t_1 = u s_1^{-1} n^{1/2}, \quad t_2 = v s_2^{-1} n^{1/2}.$$

From equation (11), to test for a dissociation it is natural to base a test statistic on the function

$$\theta = \frac{x^* - \bar{x}}{s_1} - \frac{y^* - \bar{y}}{s_2}.$$

As $t_1 - t_2 = \theta n^{1/2}/(n+1)^{1/2}$, the observed values of w_1 and $w_2(k)$ are both suitable statistics for testing for a dissociation.

Other tests for a dissociation have been proposed by Payne & Jones (1957) and Crawford et al. (1998). The test of Payne & Jones approximates the distribution of $\theta/(2-2r)^{1/2}$ by a normal distribution while Crawford et al. approximate the distribution of $\theta\{n/(n+1)\}^{1/2}/(2-2r)^{1/2}$ by a $t(n-1)$-distribution. Simulation results comparing these methods with the methods proposed here are reported in Crawford & Garthwaite (2004). They determined the sizes of Type 1 errors for a significance level nominally of 5% and consider values of 5, 10, 20, 50 and 100 for n and 0·0, 0·2, 0·5 and 0·8 for ρ. The new test based on $w_2(k)$ performed much better than the older tests. Its biases for $n = 5$ were smaller than those for $n = 20$ with the test of Crawford et al. (1998) or for $n = 100$ with the test of Payne & Jones.

A concern with dissociation tests is that the sizes of their Type 1 errors might be sensitive to departures from normality. As a referee noted, one might suppose that approximate normality would apply to an average of controls even for small n, but a key assumption appears to be that the observations from the case are exactly normal. To examine robustness of the test based on $w_2(k)$ to skewness, simulations were conducted in which samples were drawn from Azzalini's bivariate skew-normal distribution (Azzalini & Dalla Valle, 1996), rather than a bivariate normal distribution. Parameters were varied so that the marginal distributions of x and y had classical coefficients of skewness that covered all combinations of 0·0, 0·3 and 0·7. There are restrictions on the parameter space for which the skew-normal distribution is defined, but correlations of $\pm 0{\cdot}8$, $\pm 0{\cdot}5$, $\pm 0{\cdot}2$ and 0 were examined where possible, and the size of the control group was set at $n = 5$, $n = 10$ or $n = 20$. For each combination, 10^6 simulations were conducted. The size of the Type 1 error varied between 4·99% and 6·03% when the nominal significance level was set at 5%, and between 0·98% and 1·26% when it was set at 1%. Hence the test seems reasonably robust to skewness.

To examine its characteristics with heavy-tailed distributions, further simulations were run with samples from a bivariate t distribution, using the same correlations and sample sizes as in the simulations for skewness. When (x, y) values were drawn from a bivariate t distribution on three degrees of freedom, the size of the Type 1 error ranged from 6·5% to 8·7% for a nominal 5% significance level and from 2·2% to 3·5% for the 1% level. With a bivariate t distribution on six degrees of freedom, discrepancies between nominal and actual significance levels were smaller but again quite large: Type 1 error ranged from 5·9% to 7·1% for the 5% level and from 1·3% to 2·2% for the 1% level. Hence, results of the test should be interpreted with caution if the sampling distribution might be heavy-tailed.

To facilitate use of the test based on $w_2(k)$, a program implementing it has been written and may be downloaded from http://www.abdn.ac.uk/psy086/dept/tvardiff.htm.

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Appendix

Asymptotic distribution of w

The asymptotic expansion given in equation (5) was obtained using Maple. The main steps were as follows.

Step 1. Define $h_1(\alpha, \beta) = 1 + \beta/(4\alpha) + 9\beta^2/\{16\alpha(\alpha+1)2!\}$, so that $h_1(\alpha, \beta) = F(\frac{1}{2}, \frac{1}{2}, \alpha, \beta) + O(\alpha^{-3})$. First $n-1$ was replaced by m^2 to facilitate asymptotic expansions in powers of m and then the probability density function of (t_1, t_2, r) was written as

$$f_1(t_1, t_2, r) = \psi_1\psi_2\psi_3 h_1\{m^2 + \tfrac{3}{2}, (1 + a_0 + a_1 r)/2\},$$

where

$$a_1 = \frac{\rho}{\{(1 + t_1^2/m^2)(1 + t_2^2/m^2)\}^{-1/2}}, \quad a_0 = \frac{a_1 t_1 t_2}{m^2},$$

$$\psi_1 = \frac{(m^2-1)\Gamma(m^2+1)(1-\rho^2)^{1/2}}{(2\pi)^{3/2}\Gamma(m^2 + \frac{3}{2})},$$

$$\psi_2 = \{(1 + t_1^2/m^2)(1 + t_2^2/m^2)(1 - a_0 - a_1 r)\}^{-1/2}(1 - r^2)^{-3/2},$$

$$\psi_3 = \left\{\frac{(1 + t_1^2/m^2)(1 + t_2^2/m^2)(1 - a_0 - a_1 r)^2}{(1 - r^2)(1 - \rho^2)}\right\}^{-m^2/2}.$$

Step 2. The transformation from r to $v = m(r - \rho)$ was made by substituting for r in ψ_1, ψ_2, ψ_3 and h_1. Multiplication by the Jacobian of the transformation, m^{-1}, was deferred until Step 8.

Step 3. The term ψ_3 was expanded in powers of m^{-1} to $O(m^{-6})$ and multiplied by ψ_2 and h_1, giving the kernel of the density function of (t_1, t_2, v).

Step 4. The transformation $z_1 = t_1 + t_2$, $z_2 = t_1 - t_2$, $v = v$ was applied to the kernel of the density function of (t_1, t_2, v).

Step 5. The result from Step 4 is the kernel of the density function of (z_1, z_2, v). This was expanded in powers of m^{-1} to $O(m^{-6})$, and z_1 was integrated out, giving the kernel of the density function of (z_2, v).

Step 6. The expression $v - \rho/m$ was substituted for r in

$$\{2 - 2r + (b_0 + b_1 r + b_2 r^2)/m^2 + (c_0 + c_1 r + c_2 r^2 + c_3 r^3)/m^4\}.$$

Denote the result by ξ.

Step 7. The next aim was to apply the transformation $w = z_2\xi^{-1/2}$. To this end, $\xi^{1/2}$ was expanded in powers of m^{-1} to $O(m^{-6})$. Also, $w\xi^{1/2}$ was substituted for z_2 in the kernel of the density function of (z_2, v) and was expanded in powers of m^{-1} to $O(m^{-6})$. Functions are so long that the memory capacity allocated to Maple is exceeded unless functions are partitioned into sections. For this reason, the latter expansion was partitioned according to the power of m^{-1}. Each element of the partition was multiplied by the relevant part of the expansion of the Jacobian so that terms of order $O(m^{-6})$ or above were omitted, and then v was integrated out.

Step 8. The results from these integrations were added together and multiplied by ψ_1/m. Note that ψ_1/m is of order $O(1)$. This gave the probability density function of w, which was expanded in powers of m^{-1} to obtain the form given in equation (5).

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