# Tutorial 2

1. Find the equivalent stress $\bar{\sigma}$ of a plane stress tensor $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution:

The deviatoric stress is $\boldsymbol{\sigma}' = \begin{bmatrix} \frac{2\sigma_1 - \sigma_2}{3} & 0 & 0 \\ 0 & \frac{2\sigma_2 - \sigma_1}{3} & 0 \\ 0 & 0 & -\frac{\sigma_1 + \sigma_2}{3} \end{bmatrix}$.

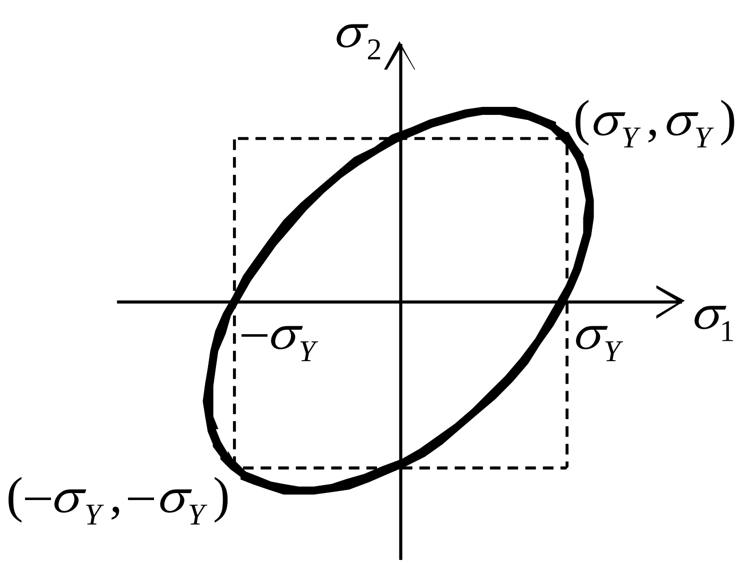
The equivalent stress is

$$\bar{\sigma} = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \sqrt{\frac{3}{2}\left[\left(\frac{2\sigma_1 - \sigma_2}{3}\right)^2 + \left(\frac{2\sigma_2 - \sigma_1}{3}\right)^2 + \left(-\frac{\sigma_1 + \sigma_2}{3}\right)^2\right]} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}.$$

2. Initially material will yield at $\bar{\sigma} = \sigma_Y$. For the plane stress problem studied above, sketch the yield locus in $\sigma_1$ and $\sigma_2$ axes.

Solution:

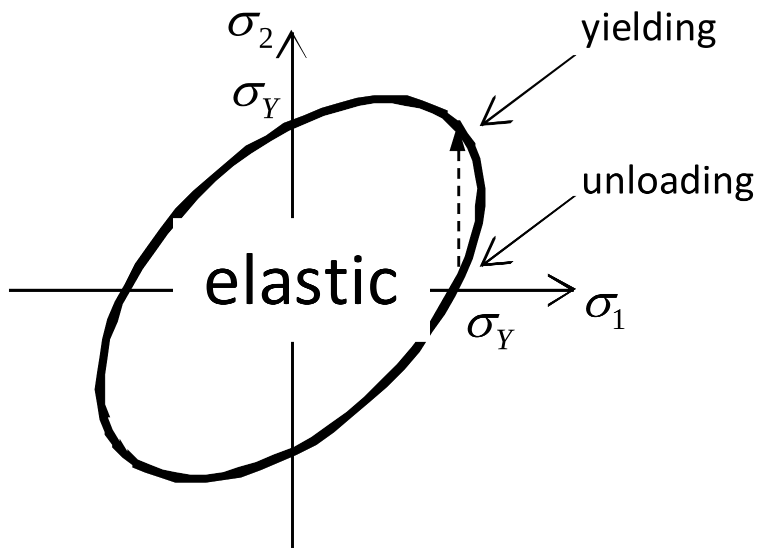
The equation of the yield locus is $\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_Y^2$.



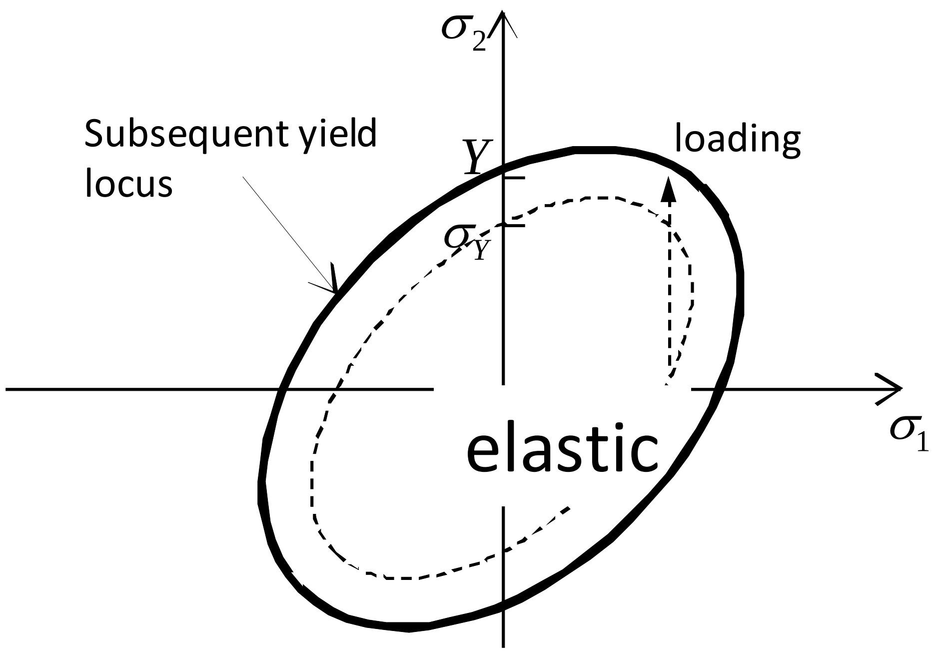
3. Assuming isotropic hardening. Holding $\sigma_1 = \sigma_Y$ ($\sigma_Y$ is the initial yield stress) and increase $\sigma_2$ from 0, describe the behaviour.

Solution:

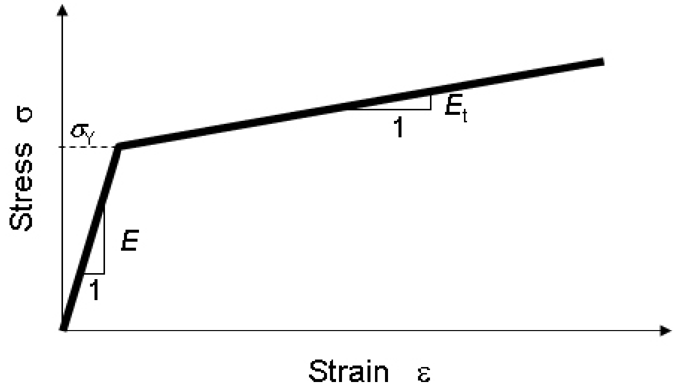
Stage I, $\sigma_2 \le \sigma_Y$:




Stage II, $\sigma_2 > \sigma_Y$:



4. For a linear-elastic and linear plastic model, the following figure shows the stress-strain relation during uniaxial tension.




From the above model, derive the slop, $h = \dfrac{dY}{d\bar{\varepsilon}^p}$, of the equivalent yield stress function $Y(\bar{\varepsilon}^p)$.

Solution:

For uniaxial tension, $Y = \sigma$, $d\bar{\varepsilon}^p = d\varepsilon$.

Therefore, $h = \dfrac{dY}{d\bar{\varepsilon}^p} = \dfrac{d\sigma}{d\varepsilon^p} = \dfrac{d\sigma}{d\varepsilon - d\varepsilon^e} = \dfrac{1}{\dfrac{d\bar{\varepsilon}}{d\sigma} - \dfrac{d\bar{\varepsilon}^e}{d\sigma}} = \dfrac{1}{\dfrac{1}{E_t} - \dfrac{1}{E}}$.

5. What is the plastic strain tensor when $\sigma_2 = 2\sigma_Y$ ?

Solution:

The plastic strain $\varepsilon_{ij}^p = 0$ when $\sigma_2 = \sigma_Y$ . The integration for $\sigma_2 : \sigma_Y \to 2\sigma_Y$ will give the final plastic strain.

We use the formula $\mathrm{d}\varepsilon_{ij}^p = \frac{9}{4}\frac{1}{h}\frac{\sigma'_{ij}}{\bar{\sigma}^2}\sigma'_{kl}\mathrm{d}\sigma'_{kl}$ to calculate the plastic strain increment.

The deviatoric stress tensor is $\boldsymbol{\sigma}' = \frac{1}{3}\begin{bmatrix} 2\sigma_Y - \sigma_2 & 0 & 0 \\ 0 & 2\sigma_2 - \sigma_Y & 0 \\ 0 & 0 & -\sigma_Y - \sigma_2 \end{bmatrix}$.

$$\Rightarrow \mathrm{d}\boldsymbol{\sigma}' = \frac{\mathrm{d}\sigma_2}{3}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\Rightarrow \sigma'_{kl}\mathrm{d}\sigma'_{kl} = \frac{(2\sigma_2 - \sigma_Y)}{3}\mathrm{d}\sigma_2 .$$

Using the formula $\mathrm{d}\varepsilon_{ij}^p = \frac{9}{4}\frac{1}{h}\frac{\sigma'_{ij}}{\bar{\sigma}^2}\sigma'_{kl}\mathrm{d}\sigma'_{kl}$

$$\mathrm{d}\boldsymbol{\varepsilon}^p = \frac{3(2\sigma_2 - \sigma_Y)}{4h}\frac{\mathrm{d}\sigma_2}{\sigma_Y^2 + \sigma_2^2 - \sigma_Y\sigma_2}\boldsymbol{\sigma}'$$

$$\Rightarrow \mathrm{d}\boldsymbol{\varepsilon}^p = \frac{2\sigma_2 - \sigma_Y}{4h\left(\sigma_Y^2 + \sigma_2^2 - \sigma_Y\sigma_2\right)}\begin{bmatrix} 2\sigma_Y - \sigma_2 & 0 & 0 \\ 0 & 2\sigma_2 - \sigma_Y & 0 \\ 0 & 0 & -\sigma_Y - \sigma_2 \end{bmatrix}\mathrm{d}\sigma_2 .$$

$$\Rightarrow \quad \begin{cases} \mathrm{d}\varepsilon_{11}^{\mathrm{p}} = \dfrac{(2\sigma_2 - \sigma_Y)(2\sigma_Y - \sigma_2)}{4h\left(\sigma_Y^2 + \sigma_2^2 - \sigma_Y\sigma_2\right)}\mathrm{d}\sigma_2 \\ \mathrm{d}\varepsilon_{22}^{\mathrm{p}} = \dfrac{(2\sigma_2 - \sigma_Y)^2}{4h\left(\sigma_Y^2 + \sigma_2^2 - \sigma_Y\sigma_2\right)}\mathrm{d}\sigma_2 \\ \mathrm{d}\varepsilon_{33}^{\mathrm{p}} = -\dfrac{(2\sigma_2 - \sigma_Y)(\sigma_Y + \sigma_2)}{4h\left(\sigma_Y^2 + \sigma_2^2 - \sigma_Y\sigma_2\right)}\mathrm{d}\sigma_2 \\ \mathrm{d}\varepsilon_{12}^{\mathrm{p}} = \mathrm{d}\varepsilon_{23}^{\mathrm{p}} = \mathrm{d}\varepsilon_{31}^{\mathrm{p}} = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} \varepsilon_{11}^{\mathrm{p}} = \dfrac{1}{4h}\displaystyle\int_{\sigma_Y}^{2\sigma_Y} \dfrac{(2\sigma_2 - \sigma_Y)(2\sigma_Y - \sigma_2)}{\sigma_Y^2 + \sigma_2^2 - \sigma_Y\sigma_2}\mathrm{d}\sigma_2 = \left(\dfrac{\pi}{2\sqrt{3}} + 2 + \dfrac{3}{2}\ln(3)\right)\dfrac{\sigma_Y}{4h} = 0.14\dfrac{\sigma_Y}{h} \\ \varepsilon_{22}^{\mathrm{p}} = \dfrac{1}{4h}\displaystyle\int_{\sigma_Y}^{2\sigma_Y} \dfrac{(2\sigma_2 - \sigma_Y)^2}{\sigma_Y^2 + \sigma_2^2 - \sigma_Y\sigma_2}\mathrm{d}\sigma_2 = \left(4 - \dfrac{\pi}{\sqrt{3}}\right)\dfrac{\sigma_Y}{4h} = 0.55\dfrac{\sigma_Y}{h} \\ \varepsilon_{33}^{\mathrm{p}} = -\dfrac{1}{4h}\displaystyle\int_{\sigma_Y}^{2\sigma_Y} \dfrac{(2\sigma_2 - \sigma_Y)(\sigma_Y + \sigma_2)}{\sigma_Y^2 + \sigma_2^2 - \sigma_Y\sigma_2}\mathrm{d}\sigma_2 = \left(\dfrac{\pi}{2\sqrt{3}} - 2 - \dfrac{3}{2}\ln(3)\right)\dfrac{\sigma_Y}{4h} = -0.69\dfrac{\sigma_Y}{h} \\ \varepsilon_{12}^{\mathrm{p}} = \varepsilon_{23}^{\mathrm{p}} = \varepsilon_{31}^{\mathrm{p}} = 0 \end{cases}$$

6. What is the elastic strain tensor when $\sigma_2 = 2\sigma_Y$ ?

<u>Solution:</u>

We use the formula $\varepsilon_{ij}^{e} = \dfrac{1+\nu}{E}\sigma_{ij} - \dfrac{\nu}{E}\sigma_{kk}\delta_{ij}$ to calculate the elastic strain.

From $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \sigma_{kk} = \sigma_1 + \sigma_2$.

Therefore, the elastic strain tensor is

$$\boldsymbol{\varepsilon}^{e} = \frac{1+\nu}{E}\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{\nu}{E}(\sigma_1 + \sigma_2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{E}\begin{bmatrix} \sigma_1 - \nu\sigma_2 & 0 & 0 \\ 0 & \sigma_2 - \nu\sigma_1 & 0 \\ 0 & 0 & -\nu(\sigma_1 + \sigma_2) \end{bmatrix}.$$

7. What is the equivalent yield stress $Y$ when $\sigma_2 = 2\sigma_Y$ ?

Solution:

$$Y^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_Y^2 + \left(2\sigma_Y\right)^2 - \sigma_Y\left(2\sigma_Y\right) = 3\sigma_Y^2 .$$

$\Rightarrow$ Equivalent yield stress is $Y = \sqrt{3}\sigma_Y$ .

8. What is the equivalent yield strain $\bar{\varepsilon}^p$ when $\sigma_2 = 2\sigma_Y$ ?

Solution:

$$\bar{\varepsilon}^p = \int_{\sigma_Y}^{\sqrt{3}\sigma_Y} \frac{1}{\frac{d\bar{\sigma}}{d\bar{\varepsilon}^p}} d\bar{\sigma} = \left(\sqrt{3}-1\right)\frac{\sigma_Y}{h} .$$

9. Verify that $\bar{\varepsilon}^p \neq \sqrt{\frac{2}{3}\left(\varepsilon_{ij}^p \varepsilon_{ij}^p\right)}$ when $\sigma_2 = 2\sigma_Y$ .