Tutorial 1

1. The strain energy density W during the deformation can be calculated from
    $$W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \,.$$ (1)
    Assume that the elastic part of deformation can be ignored, prove that
    $$W = \int_0^{\varepsilon_{ij}} \sigma'_{ij} d\varepsilon_{ij} \,.$$ (2)
    where σ'_{ij} is the deviatoric stress.

Solution:

>   Elastic deformation is ignored $\Rightarrow d\varepsilon_{ij} = d\varepsilon_{ij}^p$.
>
>   According to plastic flow rule, $d\varepsilon_{ij}^p = \sigma'_{ij} d\lambda$.
>
>   Therefore, $W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} = \int_0^{\varepsilon_{ij}} \sigma_{ij} \sigma'_{ij} d\lambda$.
>
>   Since $\sigma_{ij}\sigma'_{ij} = \left(\sigma'_{ij} + \sigma_H \delta_{ij}\right)\sigma'_{ij} = \sigma'_{ij}\sigma'_{ij} + \sigma_H \delta_{ij}\sigma'_{ij} = \sigma'_{ij}\sigma'_{ij}$, one has
>
> $$W = \int_0^{\varepsilon_{ij}} \sigma'_{ij}\sigma'_{ij} d\lambda = \int_0^{\varepsilon_{ij}} \sigma'_{ij} d\varepsilon_{ij}^p = \int_0^{\varepsilon_{ij}} \sigma'_{ij} d\varepsilon_{ij} \,.$$

2. Show that the equivalent stress of a stress tensor σ_{ij} can be expressed as
    $$\bar{\sigma} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{11}-\sigma_{22}\right)^2 + \left(\sigma_{22}-\sigma_{33}\right)^2 + \left(\sigma_{33}-\sigma_{11}\right)^2 + 6\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2\right)} \,.$$ (3)

Solution:

The deviatoric stress tensor is

$$\boldsymbol{\sigma}' = \begin{bmatrix} \dfrac{2\sigma_{11}-\sigma_{22}-\sigma_{33}}{3} & \sigma_{12} & \sigma_{13} \\[2mm] \sigma_{12} & \dfrac{2\sigma_{22}-\sigma_{11}-\sigma_{33}}{3} & \sigma_{23} \\[2mm] \sigma_{13} & \sigma_{23} & \dfrac{2\sigma_{33}-\sigma_{11}-\sigma_{22}}{3} \end{bmatrix}.$$

$\Rightarrow$

$$\bar{\sigma} = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}}$$
$$= \sqrt{\frac{3}{2}}\sqrt{\left(\frac{2\sigma_{11}-\sigma_{22}-\sigma_{33}}{3}\right)^2 + \left(\frac{2\sigma_{22}-\sigma_{11}-\sigma_{33}}{3}\right)^2 + \left(\frac{2\sigma_{33}-\sigma_{11}-\sigma_{22}}{3}\right)^2 + 2\sigma_{12}^2 + 2\sigma_{13}^2 + 2\sigma_{23}^2} \,.$$

3. Show that the von Mises yield criterion can be expressed as

$$\sigma_1'^2 + \sigma_2'^2 + \sigma_3'^2 = 2\tau_Y^2,$$ (4)

where σ_1', σ_2' and σ_3' are principal stresses of the deviatoric stresses tensor σ_{ij}', and τ_Y is the yield shear stress.

Solution:

In the coordinate system with axes along principal directions, the deviatoric tensor can be

written as $\begin{bmatrix} \sigma_1' & 0 & 0 \\ 0 & \sigma_2' & 0 \\ 0 & 0 & \sigma_3' \end{bmatrix}$. Therefore equivalent stress is $\bar{\sigma} = \sqrt{\frac{3}{2}\sigma_{ij}'\sigma_{ij}'} = \sqrt{\frac{3}{2}\left(\sigma_1'^2 + \sigma_2'^2 + \sigma_3'^2\right)}$.

In the coordinate system with axis 1 along the torsion axis, the deviatoric tensor at the yielding

state can be written as $\begin{bmatrix} 0 & \tau_Y & 0 \\ \tau_Y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore the equivalent stress is $\bar{\sigma} = \sqrt{\frac{3}{2}\sigma_{ij}'\sigma_{ij}'} = \sqrt{3}\tau_Y$.

Equivalent stress is a variable independent of the coordinate system chosen, therefore the two

equivalent stress should be the same, i.e., $\sqrt{\frac{3}{2}\left(\sigma_1'^2 + \sigma_2'^2 + \sigma_3'^2\right)} = \sqrt{3}\tau_Y$.

4. Determine the final yield stress of a billet, made from a material with an initial stress-strain curve $\sigma = 18 + 10\varepsilon^{1/2}$, that has been compressed uniaxially by 25% and then extended uniaxially by 25% of its intermediate length.

Solution:

The uniaxial tensile test gives the initial (i.e., the equivalent plastic strain is zero at the beginning) true stress-strain relation as $\sigma = \left(18 + 10\varepsilon^{1/2}\right)$ MPa. In this curve, the elastic stage is assumed as rigid.

This curve gives the relation between the equivalent stress $\bar{\sigma}$ and the equivalent plastic strain $\bar{\varepsilon}^p$ as $\bar{\sigma} = 18 + 10\sqrt{\bar{\varepsilon}^p}$.

The final equivalent strain consists of two parts, $\bar{\varepsilon}^p = \bar{\varepsilon}_1^p + \bar{\varepsilon}_2^p$. Part 1 (25% compression) gives

$\bar{\varepsilon}_1^p = \left|\ln\frac{3}{4}\right| = -\ln\frac{3}{4}$. Part 2 (25% tension) gives $\bar{\varepsilon}_2^p = \ln\frac{5}{4}$. Therefore, $\bar{\varepsilon}^p = -\ln\frac{3}{4} + \ln\frac{5}{4} = \ln\frac{5}{3}$.

The equivalent stress $\bar{\sigma} = \left(18 + 10\sqrt{\ln\dfrac{5}{3}}\right)$MPa. In uniaxial tension the equivalent stress and the

uniaxial stress are the same, therefore, the final yield stress is $\left(18 + 10\sqrt{\ln\dfrac{5}{3}}\right)$MPa

5.   The deformation of a metal element is described by the strain rate tensor

$\dot{\varepsilon} = \begin{bmatrix} 2 & 0 & \sqrt{2} \\ 0 & -3 & 0 \\ \sqrt{2} & 0 & 1 \end{bmatrix} \times 10^{-3}\text{s}^{-1}$. Assuming that elastic strains are negligible (Levy-Mises flow

rule can be applied), determine:
(i) The principal strain rate;
(ii) The deviatoric stress tensor σ', where the maximum principal deviatoric stress for the
       element is 100MPa ;
(iii) The associated stress tensor for the element, where its mean stress is 100MPa;
(iv) The unit normals for the surfaces on which the principal stresses act.

Solution:

(i) Denote the principal strain rate as $\lambda \times 10^{-3}\text{s}^{-1}$. Solving the equation

$\begin{vmatrix} 2-\lambda & 0 & \sqrt{2} \\ 0 & -3-\lambda & 0 \\ \sqrt{2} & 0 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow \begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= 0 \\ \lambda_3 &= -3 \end{aligned}$, so the principal strain rates are $\begin{cases} \dot{\varepsilon}_1 = 3 \times 10^{-3}\text{s}^{-1} \\ \dot{\varepsilon}_2 = 0 \\ \dot{\varepsilon}_3 = -3 \times 10^{-3}\text{s}^{-1} \end{cases}$.

(ii) In the principal coordinate system, the strain rate tensor is $\begin{bmatrix} \dot{\varepsilon}_1 & 0 & 0 \\ 0 & \dot{\varepsilon}_2 & 0 \\ 0 & 0 & \dot{\varepsilon}_3 \end{bmatrix}$. According to

the flow rule The deviatoric stress tensor is $\begin{bmatrix} \dot{\varepsilon}_1 & 0 & 0 \\ 0 & \dot{\varepsilon}_2 & 0 \\ 0 & 0 & \dot{\varepsilon}_3 \end{bmatrix}\dfrac{1}{\dot{\lambda}}$. Since $\dfrac{\dot{\varepsilon}_1}{\dot{\lambda}} = 100\text{MPa}$,

$\dot{\lambda} = 3 \times 10^{-5}\dfrac{1}{\text{MPa} \cdot \text{s}}$.

Therefore in the principal coordinate system, the deviatoric stress tensor is

$$100 \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{MPa}.$$

In the original coordinate system (for strain rate tensor $\dot{\varepsilon} = \begin{bmatrix} 2 & 0 & \sqrt{2} \\ 0 & -3 & 0 \\ \sqrt{2} & 0 & 1 \end{bmatrix} \times 10^{-3}\text{s}^{-1}$), the

deviatoric stress tensor can be expressed as $\dfrac{100}{3} \times \begin{bmatrix} 2 & 0 & \sqrt{2} \\ 0 & -3 & 0 \\ \sqrt{2} & 0 & 1 \end{bmatrix} \text{MPa}.$

(iii) The hydrostatic stress (mean stress) is 100MPa.

In the principal coordinate system, the stress tensor is

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \times 100\text{MPa} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 100\text{MPa}.$$

In the original coordinate system, the deviatoric stress tensor can be expressed as

$$\left(\begin{bmatrix} \dfrac{2}{3} & 0 & \dfrac{\sqrt{2}}{3} \\ 0 & -1 & 0 \\ \dfrac{\sqrt{2}}{3} & 0 & \dfrac{1}{3} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \times 100\text{MPa} = \begin{bmatrix} \dfrac{5}{3} & 0 & \dfrac{\sqrt{2}}{3} \\ 0 & 0 & 0 \\ \dfrac{\sqrt{2}}{3} & 0 & \dfrac{4}{3} \end{bmatrix} \times 100\text{MPa}$$

(v) Fining the unit vector of the principal directions: $\underline{n}_1, \underline{n}_2$ and $\underline{n}_3$.

1st principal direction

$$\begin{bmatrix} -1 & 0 & \sqrt{2} \\ 0 & -6 & 0 \\ \sqrt{2} & 0 & -2 \end{bmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow n_2 = 0, \ n_1 = \sqrt{2}n_3 \text{ and } n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow n_3 = \dfrac{1}{\sqrt{3}}$$

$$\underline{n}_1 = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}_1 = \dfrac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix}$$

2nd principal direction

$$\begin{bmatrix} 2 & 0 & \sqrt{2} \\ 0 & -3 & 0 \\ \sqrt{2} & 0 & 1 \end{bmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow n_2 = 0, \ n_3 = -\sqrt{2}n_1 \text{ and } n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow n_1 = \dfrac{1}{\sqrt{3}}$$

$$\underline{n}_2 = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ -\sqrt{2} \end{pmatrix},$$

3rd principal direction

$$\begin{bmatrix} 5 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 4 \end{bmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{n}_3 = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$