

### Multiaxial loading

- Stress tensor
- Einstein notation
- Principal stresses & directions

#### I. Example of multiaxial stress

Most engineering structures are influenced by complex multiaxial stresses that arise from loading, geometry and/or material inhomogeneity. Below is a thin-walled tube subject to combined tension  $P$  and torsion  $T$ .

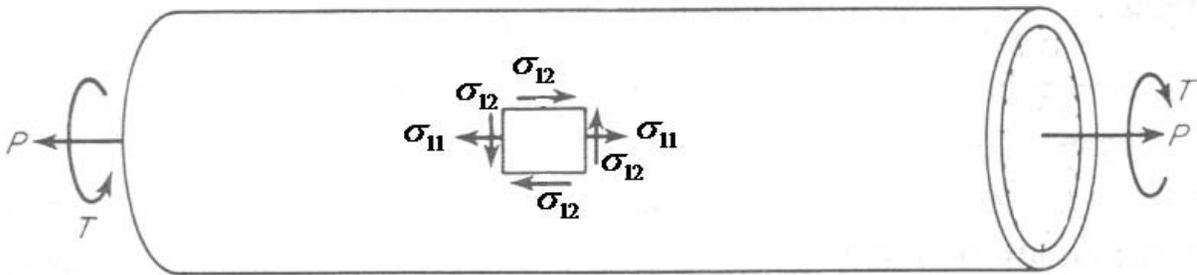


Figure 1. A thin-walled tube subject to combined tension  $P$  and torsion  $T$ .

The shear stress is  $\tau = \frac{T}{2\pi r^2 t}$ , where  $r$  and  $t$  are the radius and thickness of the tube,

respectively. The tensile stress is  $\sigma = \frac{P}{2\pi r t}$ . The stress tensor is  $\sigma = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

#### II. Stress tensor

Matrix representation of a tensor.

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{1}$$

Given the six components of stress at a point with respect to a coordinate system  $(\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{e}^{(3)})$ , we can determine the traction  $\mathbf{t} = (t_1, t_2, t_3)$  acting on any plane, which is described by the unit normal vector  $\mathbf{n} = (n_1, n_2, n_3)$ , through this point.

In matrix expression:

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \tag{2}$$

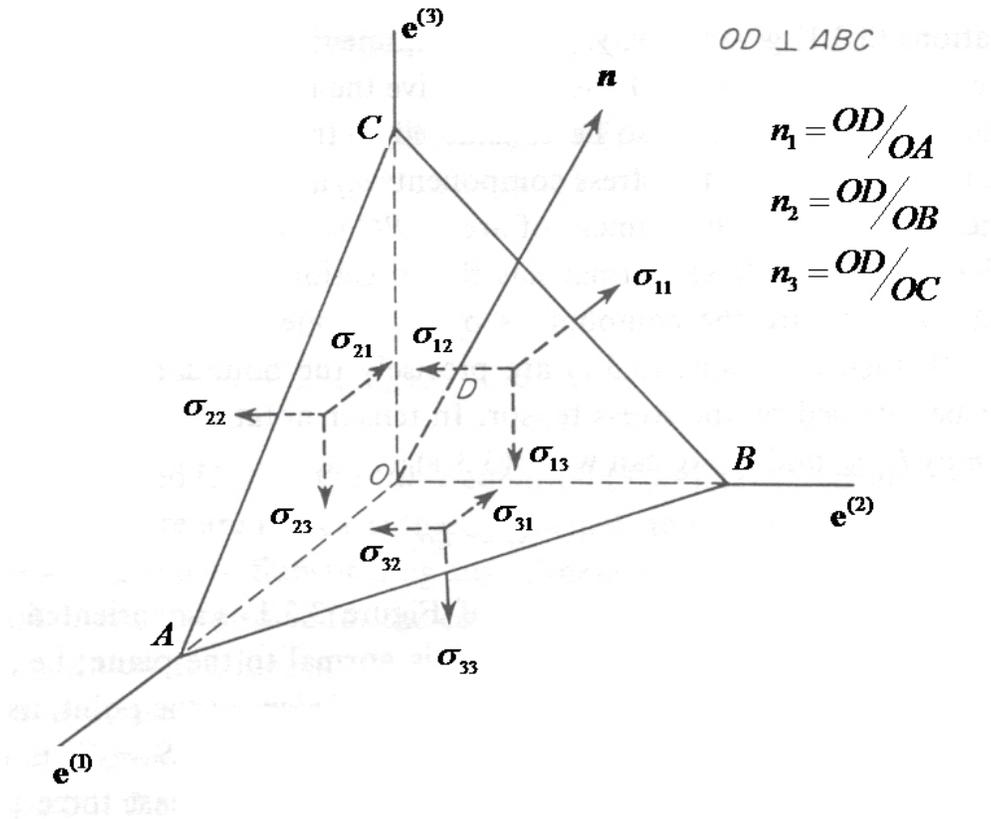


Figure 2. Stress tensor.

Surface traction in tensor expression

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} \tag{3}$$

where  $\mathbf{n}$  is a unit vector normal to a surface,  $\boldsymbol{\sigma}$  is the stress tensor and  $\mathbf{t}$  is the traction vector acting on the surface.

In tensor component expression

$$t_i = \sigma_{ij} n_j \tag{4}$$

### III. Einstein notation

Einstein summation convention is a notational convention useful when dealing with coordinate formulas. It was introduced by Albert Einstein in 1916.

**Summation rule:**

When an index occurs twice in a term summation is to be done over the values 1, 2 and 3 for this index.

$$l = v_i v_i \quad \Leftrightarrow \quad l = \sum_{i=1}^3 v_i v_i \quad (5)$$

**Index rule:**

An index occurring once in a term is valid for each of the values 1, 2 and 3 for this index.

$$t_i = \sigma_{ij} n_j \quad \Leftrightarrow \quad \begin{cases} t_1 = \sum_{j=1}^3 \sigma_{1j} n_j \\ t_2 = \sum_{j=1}^3 \sigma_{2j} n_j \\ t_3 = \sum_{j=1}^3 \sigma_{3j} n_j \end{cases} \quad (6)$$

**Exercise 1**

Write out the expression for the work increment  $dW = \sigma_{ij} d\varepsilon_{ij}$ .

Solution:

$$dW = \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} d\varepsilon_{ij}$$

**IV. Stress tensor transformation**

For a given stress tensor in one coordinate system, we wish to determine the six independent stress components of the same tensor as seen by another coordinate system.

**Exercise 2**

A specimen is subject to uniaxial tension.

A coordinate system  $S$  is set up to describe the stress state in the specimen. The three unit vectors are denoted as  $\mathbf{e}^{(1)}$ ,  $\mathbf{e}^{(2)}$  and  $\mathbf{e}^{(3)} = \mathbf{e}^{(1)} \times \mathbf{e}^{(2)}$ , where  $\mathbf{e}^{(1)}$  is along the tensional direction.

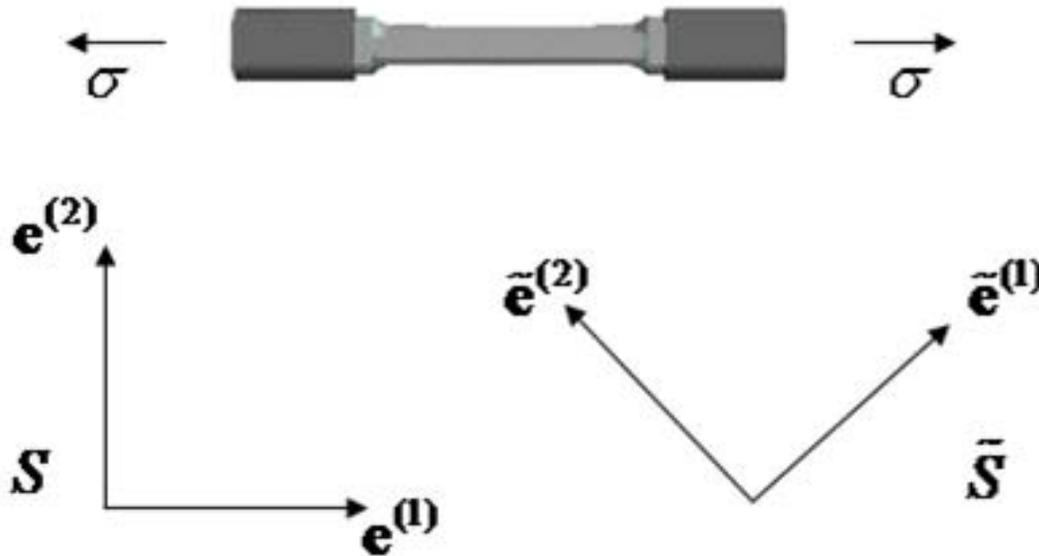


Figure 3. Uniaxial tension viewed from different coordinated system.

There is another coordinate system  $\tilde{S}$  with the three unit vector  $\tilde{\mathbf{e}}^{(1)}$ ,  $\tilde{\mathbf{e}}^{(2)}$  and  $\tilde{\mathbf{e}}^{(3)}$ , where

$$\tilde{\mathbf{e}}^{(1)} = \frac{1}{\sqrt{2}}(\mathbf{e}^{(2)} + \mathbf{e}^{(1)})$$

$$\tilde{\mathbf{e}}^{(2)} = \frac{1}{\sqrt{2}}(\mathbf{e}^{(2)} - \mathbf{e}^{(1)}).$$

$$\tilde{\mathbf{e}}^{(3)} = \mathbf{e}^{(3)}$$

For a point in the specimen,

- (1) determine the tensor components  $\sigma_{ij}$  in the coordinate system  $S$ .
- (2) determine the traction on the face with normal  $\tilde{\mathbf{e}}^{(1)}$ , express it as  $\mathbf{t}^{(1)}$  and  $\tilde{\mathbf{t}}^{(1)}$  in the coordinate system  $S$  and  $\tilde{S}$ , respectively.
- (3) determine the traction on the face with normal  $\tilde{\mathbf{e}}^{(2)}$ , express it as  $\mathbf{t}^{(2)}$  and  $\tilde{\mathbf{t}}^{(2)}$  in the coordinate system  $S$  and  $\tilde{S}$ , respectively.
- (4) determine the traction on the face with normal  $\tilde{\mathbf{e}}^{(3)}$ , express it as  $\mathbf{t}^{(3)}$  and  $\tilde{\mathbf{t}}^{(3)}$  in the coordinate system  $S$  and  $\tilde{S}$ , respectively.
- (5) for the same stress state determine the tensor components  $\tilde{\sigma}_{ij}$  in the coordinate system  $\tilde{S}$ .

Solution:

(1)

In the coordinate system  $S$ , the stress tensor can be written as

$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(2)

In the coordinate system  $S$ , the traction on the face with normal  $\tilde{\mathbf{e}}^{(1)}$  can be expressed as

$$\mathbf{t}^{(1)} = \boldsymbol{\sigma} \cdot \tilde{\mathbf{e}}^{(1)} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{pmatrix} \tilde{e}_1^{(1)} \\ \tilde{e}_2^{(1)} \\ \tilde{e}_3^{(1)} \end{pmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \frac{\sigma}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

In the coordinate system  $\tilde{S}$ , it is

$$\begin{aligned} \tilde{\mathbf{t}}^{(1)} &= \begin{pmatrix} \tilde{\mathbf{e}}^{(1)} \cdot \mathbf{t}^{(1)} \\ \tilde{\mathbf{e}}^{(2)} \cdot \mathbf{t}^{(1)} \\ \tilde{\mathbf{e}}^{(3)} \cdot \mathbf{t}^{(1)} \end{pmatrix} = \begin{pmatrix} \tilde{e}_1^{(1)} & \tilde{e}_2^{(1)} & \tilde{e}_3^{(1)} \\ \tilde{e}_1^{(2)} & \tilde{e}_2^{(2)} & \tilde{e}_3^{(2)} \\ \tilde{e}_1^{(3)} & \tilde{e}_2^{(3)} & \tilde{e}_3^{(3)} \end{pmatrix} \cdot \begin{pmatrix} t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \frac{\sigma}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\sigma}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \end{aligned}$$

(3)

In the coordinate system  $S$ , the traction on the face with normal  $\tilde{\mathbf{e}}^{(2)}$  can be expressed as

$$\mathbf{t}^{(2)} = \boldsymbol{\sigma} \cdot \tilde{\mathbf{e}}^{(2)} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{pmatrix} \tilde{e}_1^{(2)} \\ \tilde{e}_2^{(2)} \\ \tilde{e}_3^{(2)} \end{pmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \frac{\sigma}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

In the coordinate system  $\tilde{S}$ , it is

$$\begin{aligned} \tilde{\mathbf{t}}^{(2)} &= \begin{pmatrix} \tilde{\mathbf{e}}^{(1)} \cdot \mathbf{t}^{(2)} \\ \tilde{\mathbf{e}}^{(2)} \cdot \mathbf{t}^{(2)} \\ \tilde{\mathbf{e}}^{(3)} \cdot \mathbf{t}^{(2)} \end{pmatrix} = \begin{pmatrix} \tilde{e}_1^{(1)} & \tilde{e}_2^{(1)} & \tilde{e}_3^{(1)} \\ \tilde{e}_1^{(2)} & \tilde{e}_2^{(2)} & \tilde{e}_3^{(2)} \\ \tilde{e}_1^{(3)} & \tilde{e}_2^{(3)} & \tilde{e}_3^{(3)} \end{pmatrix} \cdot \begin{pmatrix} t_1^{(2)} \\ t_2^{(2)} \\ t_3^{(2)} \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \frac{\sigma}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \frac{\sigma}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

(4)

In the coordinate system  $S$ , the traction on the face with normal  $\tilde{\mathbf{e}}^{(3)}$  can be expressed as

$$\mathbf{t}^{(3)} = \boldsymbol{\sigma} \cdot \tilde{\mathbf{e}}^{(3)} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{pmatrix} \tilde{e}_1^{(3)} \\ \tilde{e}_2^{(3)} \\ \tilde{e}_3^{(3)} \end{pmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In the coordinate system  $\tilde{S}$ , it is

$$\tilde{\mathbf{t}}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(5)

The stress tensor in the coordinate system  $\tilde{S}$  is

$$\tilde{\boldsymbol{\sigma}} = \begin{bmatrix} \tilde{t}_1^{(1)} & \tilde{t}_2^{(1)} & \tilde{t}_3^{(1)} \\ \tilde{t}_1^{(2)} & \tilde{t}_2^{(2)} & \tilde{t}_3^{(2)} \\ \tilde{t}_1^{(3)} & \tilde{t}_2^{(3)} & \tilde{t}_3^{(3)} \end{bmatrix} = \frac{\sigma}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## V. Principal stresses

Principal stresses and directions are defined as

$$\boldsymbol{\sigma} \cdot \mathbf{n} = t\mathbf{n} . \quad (7)$$

The principal stresses can be obtained from

$$\begin{vmatrix} \sigma_{11} - t & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - t & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - t \end{vmatrix} = 0 \quad (8)$$

The determinant can be rewritten as

$$\begin{vmatrix} \sigma_{11} - t & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - t & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - t \end{vmatrix} = -t^3 + I_1 t^2 + I_2 t + I_3 , \quad (9)$$

where

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$\begin{aligned} I_2 &= \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2 - (\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) \\ &= \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}) \end{aligned} \quad (10)$$

$$I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$$

$I_1$ ,  $I_2$  and  $I_3$  are usually referred to as first, second and third stress invariants, respectively.

### Exercise 3

For the stress state and coordinate system  $S$  and  $\tilde{S}$  as stated in Exercise 1, verify that

- (1)  $I_1$  for  $S$  equals to  $\tilde{I}_1$  for  $\tilde{S}$ .
- (2)  $I_2$  for  $S$  equals to  $\tilde{I}_2$  for  $\tilde{S}$ .
- (3)  $I_3$  for  $S$  equals to  $\tilde{I}_3$  for  $\tilde{S}$ .
- (4)  $K = \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2 + (\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11})$  for  $S$  does not equal to  $\tilde{K} = \tilde{\sigma}_{12}^2 + \tilde{\sigma}_{23}^2 + \tilde{\sigma}_{13}^2 + (\tilde{\sigma}_{11}\tilde{\sigma}_{22} + \tilde{\sigma}_{22}\tilde{\sigma}_{33} + \tilde{\sigma}_{33}\tilde{\sigma}_{11})$  for  $\tilde{S}$ .

### Solution:

(1)

In coordinate system  $S$ ,

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma$$

In coordinate system  $\tilde{S}$ ,

$$I_1 = \tilde{\sigma}_{11} + \tilde{\sigma}_{22} + \tilde{\sigma}_{33} = \sigma$$

(2)

In coordinate system  $S$ ,

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2 = 0$$

In coordinate system  $\tilde{S}$ ,

$$\tilde{I}_2 = \tilde{\sigma}_{11}\tilde{\sigma}_{22} + \tilde{\sigma}_{22}\tilde{\sigma}_{33} + \tilde{\sigma}_{33}\tilde{\sigma}_{11} - \tilde{\sigma}_{12}^2 - \tilde{\sigma}_{23}^2 - \tilde{\sigma}_{13}^2 = 0$$

(3)

In coordinate system  $S$ ,

$$I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} = 0$$

In coordinate system  $\tilde{S}$ ,

$$\tilde{I}_3 = \begin{vmatrix} \tilde{\sigma}_{11} & \tilde{\sigma}_{12} & \tilde{\sigma}_{13} \\ \tilde{\sigma}_{21} & \tilde{\sigma}_{22} & \tilde{\sigma}_{23} \\ \tilde{\sigma}_{31} & \tilde{\sigma}_{32} & \tilde{\sigma}_{33} \end{vmatrix} = 0$$

(4)

In coordinate system  $S$ ,

$$K = \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2 + (\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) = 0$$

In coordinate system  $\tilde{S}$ ,

$$\tilde{K} = \tilde{\sigma}_{12}^2 + \tilde{\sigma}_{23}^2 + \tilde{\sigma}_{13}^2 + (\tilde{\sigma}_{11}\tilde{\sigma}_{22} + \tilde{\sigma}_{22}\tilde{\sigma}_{33} + \tilde{\sigma}_{33}\tilde{\sigma}_{11}) = \frac{\sigma^2}{2}$$

## VI. Yield criterion?

For a stress state  $\boldsymbol{\sigma}$  with matrix expression as  $\boldsymbol{\sigma} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  in the coordinate system  $S$ , the

yield criterion is obvious:  $\sigma = \sigma_Y$ .

Looking at the same stress state in another coordinate system  $\tilde{S}$  as shown in Figure 3, the matrix expression of the same stress tensor is  $\boldsymbol{\sigma} = \frac{\sigma}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . When  $\sigma = \sigma_Y$ , how can we know that this stress state is in yielding, since the matrix expression is so obscure for making judgement?

Next section will resolve this puzzle.