Hardening and flow rule

I. Equivalent stress

The maximum distortion energy criterion, or so called von Mises yield criterion, can be expressed in another form:

$$\bar{\sigma} = \sigma_Y , \tag{1}$$

where $\bar{\sigma}$ is the equivalent yield stress which is defined as

$$\bar{\sigma} = \sqrt{3J_2} . \tag{2}$$

Based on the relation between J_2 and the deviatoric stress tensor, one has

$$\bar{\sigma} = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} . \tag{3}$$

II. Strain hardening

Experiments show that if you plastically deform a solid, then unload it, and then try to re-load it so as to induce further plastic flow, its resistance to plastic flow will have increased. This is known as strain hardening.

The yield criterion will change with the further development of plastic deformation. We assume that the yield criterion accounting for the hardening can be written as

$$\bar{\sigma} = Y(\bar{\varepsilon}^p) , \tag{4}$$

where $\bar{\varepsilon}^p$, called equivalent plastic strain, is a scalar measure of plastic strain tensor defined through time integration of the equivalent plastic strain rate

$$\bar{\varepsilon}^p = \int_{t=0}^{t} \dot{\bar{\varepsilon}}^p dt , \tag{5}$$

where $\varepsilon_{ij}^p = 0$ at the time $t = 0$, and the equivalent plastic strain rate $\dot{\bar{\varepsilon}}^p$ is defined as

$$\dot{\bar{\varepsilon}}^p = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p} . \tag{6}$$

Exercise 1

Proof that $\bar{\varepsilon}^p = \varepsilon^p$ in a uniaxial tensile test in which the specimen is stretched parallel to $\mathbf{e}_1$ direction (ε^p is the plastic strain in $\mathbf{e}_1$ direction).

Solution:

During plastic deformation, $\dot{\varepsilon}_{ii}^p = 0$ due to the incompressibility. Therefore in a uniaxial tension, the plastic strain rate is $\dot{\boldsymbol{\varepsilon}}^p = \begin{pmatrix} \dot{\varepsilon}^p & 0 & 0 \\ 0 & -\frac{1}{2}\dot{\varepsilon}^p & 0 \\ 0 & 0 & -\frac{1}{2}\dot{\varepsilon}^p \end{pmatrix}$, which gives the equivalent plastic strain as

$$
\begin{aligned}
\bar{\varepsilon}^p &= \int \sqrt{\frac{2}{3}\left(\dot{\varepsilon}^p\dot{\varepsilon}_{11}^p + \dot{\varepsilon}_{22}^p\dot{\varepsilon}_{22}^p + \dot{\varepsilon}_{33}^p\dot{\varepsilon}_{33}^p\right)}dt \\
&= \int \sqrt{\frac{2}{3}\left(\dot{\varepsilon}^p\dot{\varepsilon}^p + \frac{1}{4}\dot{\varepsilon}^p\dot{\varepsilon}^p + \frac{1}{4}\dot{\varepsilon}^p\dot{\varepsilon}^p\right)}dt \\
&= \int \dot{\varepsilon}^p dt = \varepsilon^p
\end{aligned}
$$

The equivalent yield stress as a function of equivalent plastic strain, $Y\left(\bar{\varepsilon}^p\right)$, can be obtained from experiment, such as uniaxial test or torsion test, based on the condition that the yield criterion must be satisfied at all times during plastic straining.

In uniaxial stress state, letting axis 1 along the tensile direction, the stress tensor is

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

The deviatoric stress tensor is

$$\boldsymbol{\sigma}' = \begin{pmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{pmatrix}, \tag{8}$$

Therefore the equivalent stress is $\bar{\sigma} = \sigma$. Also Exercise 1 shows that the equivalent strain during uniaxial tensile test $\bar{\varepsilon}^p = \bar{\varepsilon}_{11}^p$. Thus, the uniaxial stress-strain curve can be employed to determine the hardening law for a general stress state, as shown in Exercise 2.

Exercise 2

The stress-strain curve from a uniaxial test is shown in Figure 1. Derive the relation between the equivalent yield stress Y and the equivalent plastic strain $\bar{\varepsilon}^{\mathrm{p}}$.

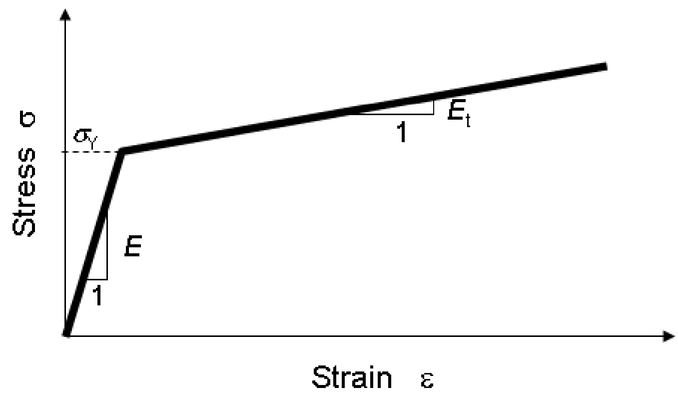

Figure 1, Stress (true) - strain (true) curve from a uniaxial test.

Solution:

During the plastic deformation, from Figure 1

$$\sigma = \sigma_{\mathrm{Y}} + E_{\mathrm{t}}\left(\varepsilon - \varepsilon_{\mathrm{Y}}\right),$$

where σ_{Y} is the initial yield stress, and

$$\varepsilon_{\mathrm{Y}} = \frac{\sigma_{\mathrm{Y}}}{E}$$

is the strain at the yield point.

Since

$$\varepsilon = \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{p}},$$

where

$$\varepsilon^{\mathrm{e}} = \frac{\sigma}{E}$$

is the elastic part of the strain and ε^{p} is the plastic part, one has

$$\sigma = \sigma_{\mathrm{Y}} + \frac{EE_{\mathrm{t}}}{E - E_{\mathrm{t}}}\varepsilon^{\mathrm{p}}.$$

Since for uniaxial tension, the equivalent stress $\bar{\sigma} = \sigma$, and the equivalent plastic strain $\bar{\varepsilon}^{\mathrm{p}} = \varepsilon^{\mathrm{p}}$, the relation between the equivalent yield stress and the equivalent plastic strain is

$$Y = \sigma_{\mathrm{Y}} + \frac{EE_{\mathrm{t}}}{E - E_{\mathrm{t}}}\bar{\varepsilon}^{\mathrm{p}}.$$

III. Flow rule

To complete the plastic stress-strain relations, we need a way to predict the plastic strains induced by stressing the material beyond the yield point. Specifically, given:

1. The current stress applied to the material
2. The current yield stress (characterized by $Y\left(\bar{\varepsilon}^p\right)$ for isotropic hardening)
3. A small increase in stress $d\sigma_{ij}$ applied to the solid

we wish to determine the small change in plastic strain $d\varepsilon_{ij}^p$.

The plastic strains are usually derived from the yield criterion *f*. A material that has its plastic flow law derived from *f* is said to have an "associated"' flow law, i.e., the flow law is associated with *f*.

The flow rule specifies the increment of plastic strain once the material has yielded. The early work was known as Levy-Mises equation, which specifies the incremental of total strain as

$$\dot{\varepsilon}_{ij} = \sigma'_{ij}\dot{\lambda} \tag{9}$$

where $\dot{\lambda}$ is a scalar factor of proportionality. The equation was later extended to allow for the elastic strain and takes the form

$$\begin{aligned} \dot{\varepsilon}_{ij} &= \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p \\ \dot{\varepsilon}_{ij}^p &= \sigma'_{ij}\dot{\lambda} \end{aligned} \tag{10}$$

which is known as Prandtl-Reuss equation. The total strain rate $\dot{\varepsilon}_{ij}$ is the sum of the elastic strain rate $\dot{\varepsilon}_{ij}^e$ and elastic strain rate $\dot{\varepsilon}_{ij}^p$. When elastic strain can be ignored, the total strain rate is assumed to be equal to the plastic strain rate, thus the Levy-Mises equation and the Prandtl-Reuss equation are identical. In the following, we use the Prandtl-Reuss equation.

According to the definition of the equivalent strain rate

$$\dot{\bar{\varepsilon}}^p = \sqrt{\frac{2}{3}\sigma'_{ij}\sigma'_{ij}}\,\dot{\lambda}\,.$$

Therefore, the relation between $\dot{\lambda}$ and the equivalent plastic strain rate $\dot{\bar{\varepsilon}}^p$ is

$$\dot{\lambda} = \frac{3}{2}\frac{\dot{\bar{\varepsilon}}^p}{\bar{\sigma}}\,. \tag{11}$$

The information of $\dot{\bar{\varepsilon}}^p$ with the stress rate can be determined based on the hardening rule which will be discussed in the next chapter.