Flow Theory of Plasticity

The geometry of a material will change when it is subjected to external loading; this is termed as deformation. In this course we will be mostly concerned with instantaneous or time-independent deformation. The deformation can be elastic or plastic, depending on the material behaviour after the removal of the loading. An elastic deformation is reversible, while a plastic deformation is permanent.

Stress-strain relationship is the major concern in the study of material behaviour. The equations that describe these relationships are known as constitutive equations. The constitutive equations of plasticity consist of:

(1) A yield criterion to determine the stress state when yielding occurs.
(2) A flow rule to describe the increment of plastic strain when yielding occurs.
(3) A strain-hardening rule to describe how the material is strain-hardened as the plastic strain increases., and
(4) Loading-unloading conditions to specify the next move in the loading program.

1. EXPERIMENTAL OBSERVATIONS OF PLASTIC DEFORMATION

The mechanical properties of a material are directly related to the response of the material when it's subjected to mechanical stresses. A wealth of information about a material's mechanical behaviour can be determined from tests like uniaxial tensile test, torsion test, and bulge test.

II. Uniaxial tensile test

In the uniaxial tensile test, the specimen (as shown in Figure 1) is deformed usually until complete rupture or fracture occurs, with a gradually increasing tensile load that is applied uniaxially along the longitudinal axis of the specimen. During testing, deformation is confined to the narrow centre region which has a uniform cross section along its length.

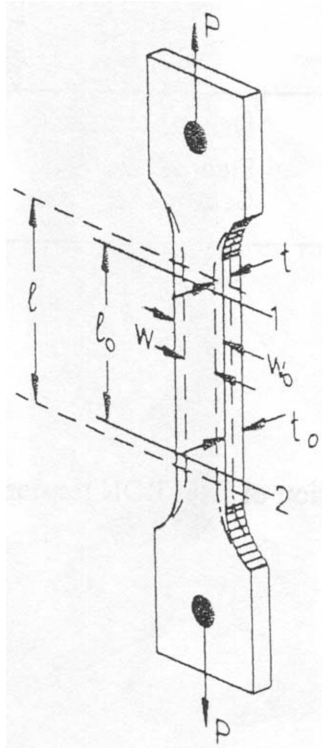

Figure 1. Uniaxial tensile test specimen.

The uniaxial tensile test is a common standard test and is a valuable method of determining important mechanical properties of engineering materials. In the tensile test, the original cross sectional area, A_0, and gage length, l_0, are measured prior to conducting the test. The applied load P and instantaneous gage length l_1 are continuously measured throughout the test using computer-based data acquisition.

For uniaxial loading, the engineering stress, σ_{eng}, is defined by

$$\sigma_{eng} = \frac{P}{A_0}. \qquad (1)$$

The engineering strain is defined according to

$$\varepsilon_{eng} = \frac{l_1 - l_0}{l_0}. \qquad (2)$$

The strain (true strain) is the change in length divided by the instantaneous length,

$$d\varepsilon = \frac{dl}{l}. \qquad (3)$$

The integration gives

$$\varepsilon = \int_0^{\varepsilon} d\varepsilon = \int_{l_0}^{l_1} \frac{dl}{l} = \ln \frac{l_1}{l_0}. \qquad (4)$$

The stress (true stress) is the applied load divided by the instantaneous cross-sectional area, A,

$$\sigma = \frac{P}{A}. \qquad (5)$$

The fundamental distinction between true stress and engineering stress, and true strain and engineering strain, concerns the interrelation between gage length and diameter changes associated with plastic deformation. From Eq. (4), the relation between the strain (true) and engineering strain is

$$\varepsilon = \ln\left(1+\varepsilon_{eng}\right). \tag{6}$$

Since plastic deformation is a const-volume process such that

$$A_0 l_0 = Al = \text{constant}, \tag{7}$$

any extension of the original gage length would produce a corresponding contraction of the gauge cross-section area. The relation between true stress σ and engineering stress σ_{eng} is

$$\sigma = \frac{P}{A} = \frac{P}{A_0}\frac{A_0}{A} = \frac{P}{A_0}\frac{l}{l_0} = \sigma_{eng}\left(1+\varepsilon_{eng}\right) \tag{8}$$

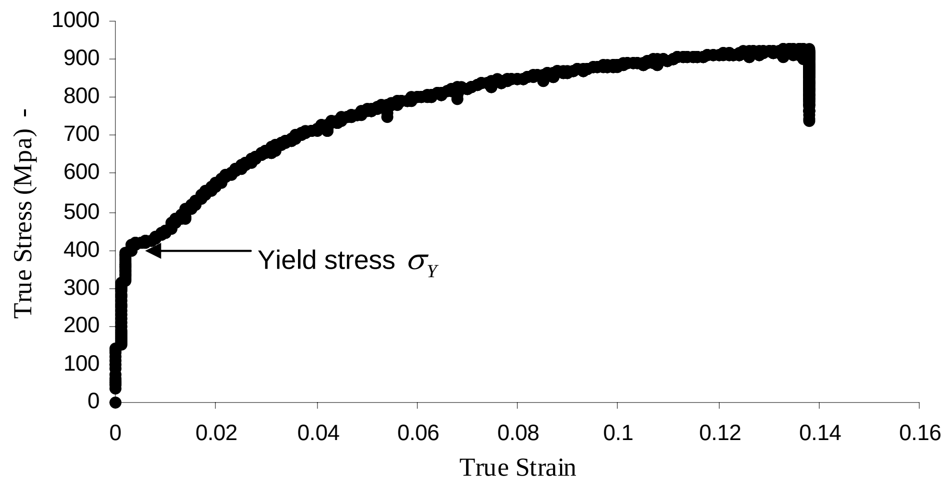

Figure 2. Stress-strain curve of a mild-steel.

Figure 2 shows that the stress (true stress) continues to increase after the yield point, though at a small rate, with increasing strain (true strain). Increasing stress after the yield point shows that the material is becoming stronger as it deforms. This is known as work-hardening or strain-hardening.

As the load is applied to the material, elastic deformation will initially prevail. At some point, the criterion for yield will be reach. After the yield point, plastic deformation in addition to elastic deformation will prevail, and the tangent modulus is taken as the slope of the stress-strain curve at a specified stress level.

III. Idealization of the stress-strain curve

Because of the complex nature of the stress-strain curve, it has become customary to idealize this curve in various ways.

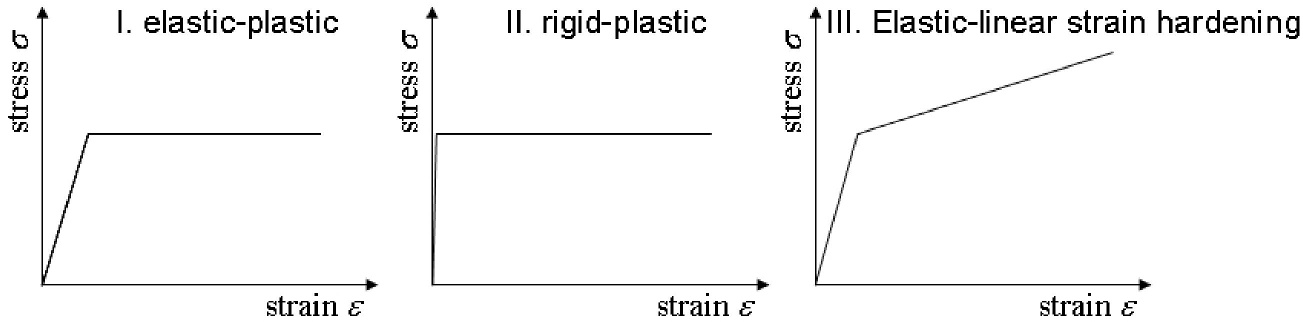

Figure 3. Idealized stress-strain curve.

A perfectly plastic material shows an unlimited amount of deformation or strain, at constant stress, and after the removal of load, the plastic strain or deformation cannot be recovered. The stress-strain curve for an ideal elastic-plastic material is illustrated in Figure 3 curve I. If the elastic deformation is negligibly small, the rigid-plastic idealization is valid (curve II of Figure 3), and for a linear-hardening material, curve III could be a reasonable approximation.

A typical elastic-linear strain hardening curve is shown in Figure 4, where σ_Y is the yield stress, E is the Young's modulus, E_t is the tangential modulus after yielding. If a strain hardened material is unloaded from the elastic-plastic range of deformation and reloaded again, the plastic deformation will not resume at the stress level indicated by the initial yield point σ_Y, but at the stress level reached just before the unloading.

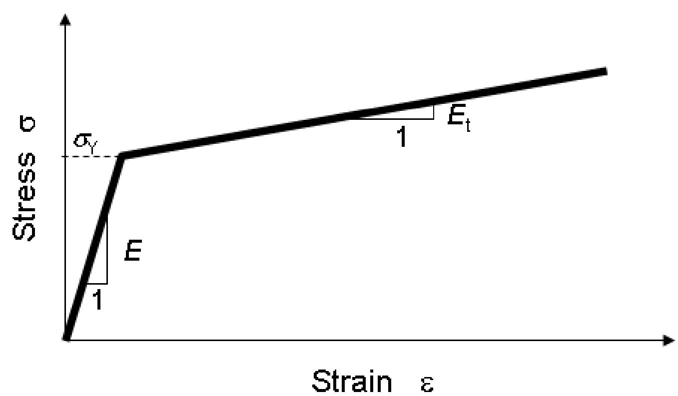

Figure 4. Stress (true) - strain (true) curve from a uniaxial test.

Exercise 1:

The stress-strain curve from a uniaxial test of a metal rod is shown in Figure 4. Assuming $E_t = 0.1E$, calculate the strain for the same material:

(1) Tensile load elastically to stress level of $0.6\sigma_Y$.

(2) Tensile load to stress level of $1.2\sigma_Y$, then unload back to the stress level of $0.6\sigma_Y$.

Solution:

(1) The strain is $0.6\frac{\sigma_Y}{E}$.

(2) The strain is $\frac{\sigma_Y}{E}+0.2\frac{\sigma_Y}{E_t}-0.6\frac{\sigma_Y}{E}=0.4\frac{\sigma_Y}{E}+0.2\frac{\sigma_Y}{E_t}=2.4\frac{\sigma_Y}{E}$.

IV. Energy density

The energy density (energy per unit volume) in deforming a material is given by the area under the stress-strain curve,

$$w=\int_0^{\varepsilon}\sigma \mathrm{d}\varepsilon . \quad (9)$$

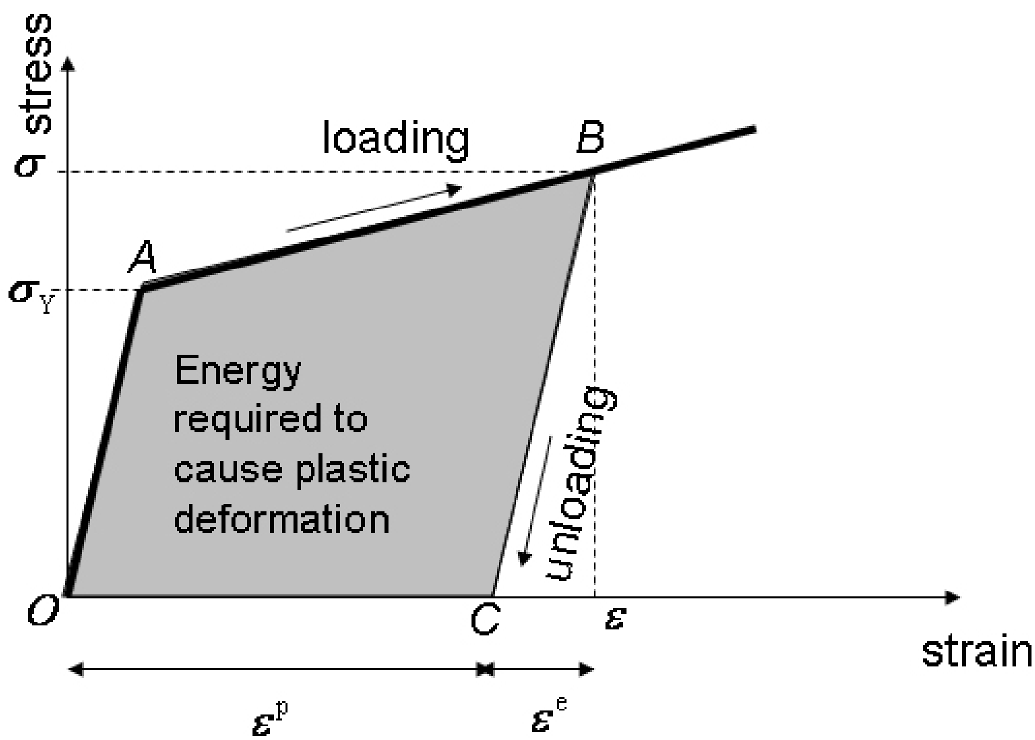

Figure 5. Energy density for a elastic-linear strain hardening material.

For the process $O\rightarrow A\rightarrow B\rightarrow C$ as shown in Figure 5, where $O\rightarrow A$ is elastic deformation, $A\rightarrow B$ is plastic loading, $B\rightarrow C$ is elastic unloading.

The energy required during $O\rightarrow A$ is $w_{O\rightarrow A}=\frac{\sigma_{\mathrm{Y}}^2}{2E}$, where E is the Young's modulus.

The energy required during $A\rightarrow B$ is $w_{A\rightarrow B}=\frac{\sigma_{\mathrm{Y}}+\sigma}{2}\left(\varepsilon-\varepsilon_{\mathrm{Y}}\right)$, where $\varepsilon_{\mathrm{Y}}=\frac{\sigma_{\mathrm{Y}}}{E}$, $\sigma=\sigma_{\mathrm{Y}}+E_t\left(\varepsilon-\varepsilon_{\mathrm{Y}}\right)$, and E_t is the tangential modulus after yielding.

The energy released during $B\rightarrow C$ is $w_{B\rightarrow C}=\frac{\sigma^2}{2E}$.

The plastic energy density dissipated during the process $O \to A \to B \to C$ is

$$w_{O\to A} + w_{A\to B} - w_{B\to C}$$

At point B, the plastic strain $\varepsilon^p = \varepsilon - \varepsilon^e$, where the elastic strain $\varepsilon^e = \dfrac{\sigma}{E}$.

V. Torsion test

Torsion tests are conducted on thin-walled cylindrical tubes, as shown in Figure 6. During the torsion test the angle of twist ϕ and the applied torque *T* are measured as the test proceeds. The radius and thickness of the thin-walled tube are *r* and *t*, respectively. The shear stress

$$\tau = \frac{T}{2\pi r^2 t}, \qquad (10)$$

and the shear strain is

$$\gamma = \frac{\phi r}{l}, \qquad (11)$$

where ϕ is the angular twist (in radius) over a fixed length *l*. The plastic component of shear strain is

$$\gamma^{\mathrm{p}} = \gamma - \gamma^{\mathrm{e}} = \gamma - \frac{\tau}{G}. \qquad (12)$$

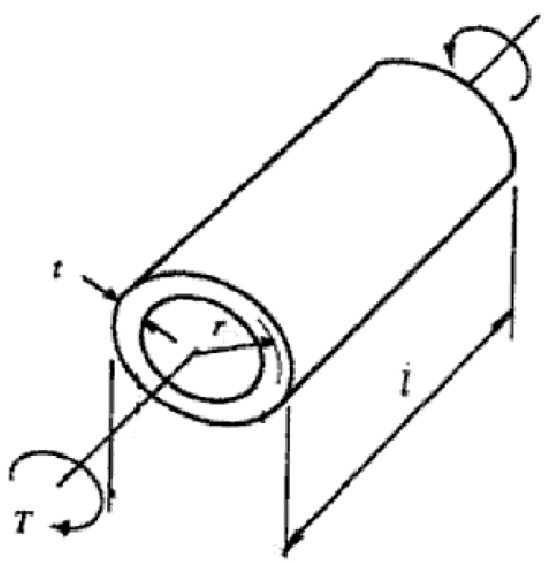

Figure 6. Thin-walled tube under torsion.