Distortion energy and von Mises yield criterion

The strain energy density at a material point consists of two parts. One is from the change of volume caused by the hydrostatic stress. The other is from the change of shape (**distortion energy** U_d) cause by the **deviatoric stress**.

Experiments show that for most metals **hydrostatic stress** does not affect yielding. Based on this observation, the **von Mises yield criterion** states that a material point with stress state $\boldsymbol{\sigma}$ starts yielding when the distortion energy U_d exceeds certain value.

In this chapter, we study the concept of hydrostatic stress, deviatoric stress, distortion energy and von Mises yield criterion

I. Elastic strain energy

For a linear elastic material, the strain elastic energy density U (energy per volume) for a stress state σ_{ij} is

$$U = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\frac{\partial u_j}{\partial x_i},$$

where u_j is the displacement. With the Einstein notation, the strain energy density can be written as

$$U = \frac{1}{2}\sigma_{ij}\frac{\partial u_j}{\partial x_i}.$$

Define the strain tensor as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right), \text{ for } i, j = 1,2,3, \tag{1}$$

one has

$$U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}. \tag{2}$$

II. Isotropic linear-elastic constitutive law

To calculate the distortion energy (the energy related to the shape changing) for an isotropic linear elastic material, we need to know the stress-strain relationship which is

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}. \tag{3}$$

III. Hydrostatic and deviatoric stresses

For stress σ_{ij}, define the hydrostatic stress σ_H as

$$\sigma_H = \frac{1}{3}\sigma_{kk}, \tag{4}$$

and deviatoric stress tensor as

$$\sigma'_{ij} = \sigma_{ij} - \sigma_H \delta_{ij}, \tag{5}$$

where δ_{ij} is called Kronecker delta, and is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}. \tag{6}$$

For Kronecker delta,

$$\delta_{ii} = \sum_{i=1}^{3} \delta_{ii} = 3. \tag{7}$$

It is obvious that

$$\sigma'_{ii} = \sigma_{ii} - \sigma_H \delta_{ii} = \sigma_{ii} - 3\sigma_H = 0. \tag{8}$$

Exercise 1

Proof that the volume change related to the deviatoric stress tensor σ'_{ij} for an isotropic linear-elastic material is zero.

Solution:

The volume strain is

$$\frac{\Delta V}{V} = \varepsilon_{ii}.$$

According to the isotropic linear-elastic constitutive law described by Eq. (3)

$$\varepsilon_{ii} = \frac{1+\nu}{E}\sigma'_{ii} - \frac{\nu}{E}\sigma'_{kk}\delta_{ii} = 0.$$

Therefore, $\Delta V = 0$.

Exercise 2

Proof that the shape change related to the hydrostatic stress tensor $\sigma_H \delta_{ij}$ for an isotropic linear-elastic material is zero.

Solution:

According to Eq. (3), the strain tensor correspond to the hydrostatic stress tensor $\sigma_H \delta_{ij}$ is

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_H \delta_{ij} - \frac{\nu}{E}\sigma_H \delta_{kk}\delta_{ij} = \frac{1-2\nu}{E}\sigma_H \delta_{ij} .$$

The shear components of the above strain are all zero, which means there is no shape change.

IV. J_2

The principal stresses for the deviatoric stress tensor $\boldsymbol{\sigma}'$ can be obtained from

$$\begin{vmatrix} \sigma'_{11} - t' & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} - t' & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} - t' \end{vmatrix} = 0$$

The above equation can be written in the form

$$t'^3 - J_1 t'^2 - J_2 t' - J_3 = 0$$

where

$$J_1 = \sigma'_{11} + \sigma'_{22} + \sigma'_{33} = 0$$

$$\begin{aligned} J_2 &= \sigma'^2_{12} + \sigma'^2_{23} + \sigma'^2_{13} - \left(\sigma'_{11}\sigma'_{22} + \sigma'_{22}\sigma'_{33} + \sigma'_{33}\sigma'_{11}\right) \\ &= \frac{1}{2}\left(\sigma'_{ij}\sigma'_{ij} - \sigma'_{ii}\sigma'_{jj}\right) \end{aligned} \quad (9)$$

$$J_3 = \begin{vmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{vmatrix}$$

It can be proved that J_2 can also be written as

$$J_2 = \frac{1}{2}\sigma'_{ij}\sigma'_{ij} \tag{10}$$

Using the elements of the original stress tensor $\boldsymbol{\sigma}$, J_2 can be expressed as

$$J_2 = \frac{1}{6}\left[\left(\sigma_{11}-\sigma_{22}\right)^2+\left(\sigma_{22}-\sigma_{33}\right)^2+\left(\sigma_{33}-\sigma_{11}\right)^2+6\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right)\right] \tag{11}$$

V. Distortion energy

According to Eq. (3), the strain tensor ε'_{ij} caused by the deviatoric stress tensor σ' can be derived from

$$\varepsilon'_{ij} = \frac{1+\nu}{E}\sigma'_{ij}\,. \tag{12}$$

According to Eq. (2), the distortion energy is

$$U_d = \frac{1}{2}\sigma'_{ij}\varepsilon'_{ij}\,,$$

which can be further written as

$$U_d = \frac{1+\nu}{2E}\sigma'_{ij}\sigma'_{ij}$$

Since Eq. (10)

$$U_d = \frac{1+\nu}{E}J_2 \tag{13}$$

VI. von Mises yield criterion

The maximum distortion energy criterion (also called von Mises yield criterion) assumes that yielding begins when the distortion energy equals the distortion energy at yield in uniaxial tension.

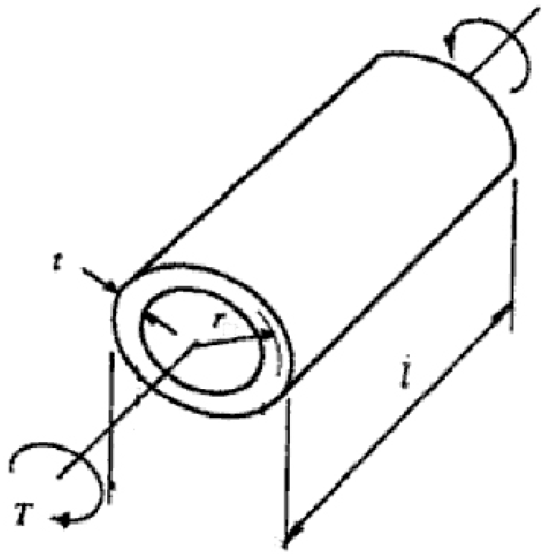
J_2 at the yield point in uniaxial tension is $\sigma_Y^2/3$. According to Eq. (13), the yield criterion is

$$J_2 = \frac{1}{3}\sigma_Y^2\,. \tag{14}$$

Exercise 3

The yield shear stress in a torsion test is τ_Y. Prove that

$$\tau_Y = \frac{1}{\sqrt{3}}\sigma_Y .$$



Solution:

At the yielding point, the stress tensor in a tube subject to torsion is

$$\begin{bmatrix} 0 & \tau_Y & 0 \\ \tau_Y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For this state,

$$J_2 = \tau_Y^2 .$$

Therefore,

$$\tau_Y^2 = \frac{1}{3}\sigma_Y^2 .$$