Constitutive equations

I. Yielding, condition of loading

Yielding criterion is

$$\bar{\sigma} = Y(\bar{\varepsilon}^p). \tag{1}$$

The subsequent behaviour for a stress state at yielding can be characterized as loading, unloading and neutral.

$$\begin{cases} \mathrm{d}\bar{\sigma} > 0 & \text{loading} \\ \mathrm{d}\bar{\sigma} = 0 & \text{neutral} \\ \mathrm{d}\bar{\sigma} < 0 & \text{unloading} \end{cases}, \tag{2}$$

which can be rewritten as

$$\begin{cases} \sigma'_{ij} d\sigma'_{ij} > 0 & \text{loading} \\ \sigma'_{ij} d\sigma'_{ij} = 0 & \text{neutral} \\ \sigma'_{ij} d\sigma'_{ij} < 0 & \text{unloading} \end{cases}, \tag{3}$$

When an element of material is unloaded from a certain plastic state, it recovers elasticity. The elastic behaviour of the material can be characterized by the same Young's modulus E and Poisson's ν ratio in the previous elastic stage. When the element is reloaded along a certain strain-path, yielding will occur again when the equivalent stress $\bar{\sigma}$ reaches $Y\left(\bar{\varepsilon}^p\right)$.

Exercise 1

For a material point, starting from a free-state increase the stress through uniaxial tension to σ_Y

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} \sigma_Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hold $\sigma_{11} = \sigma_Y$ while increase $\sigma_{22} = \sigma$ monotonically.

(1) What will be the loading condition for the stress state $\begin{pmatrix} \sigma_Y & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}$ when $\sigma = 0$.

(2) Keep increasing σ, when will the stress state $\begin{pmatrix} \sigma_Y & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}$ goes back from elastic deformation to yielding?

Solution:

(1) The deviatoric stress tensor is $\boldsymbol{\sigma}' = \begin{pmatrix} \frac{2\sigma_Y - \sigma}{3} & 0 & 0 \\ 0 & \frac{2\sigma - \sigma_Y}{3} & 0 \\ 0 & 0 & -\frac{\sigma_Y + \sigma}{3} \end{pmatrix}$

Therefore, the equivalent stress is

$$\begin{aligned} \bar{\sigma} &= \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} \\ &= \sqrt{\frac{3}{2}}\sqrt{\left(\frac{2\sigma_Y - \sigma}{3}\right)^2 + \left(\frac{2\sigma - \sigma_Y}{3}\right)^2 + \left(-\frac{\sigma_Y + \sigma}{3}\right)^2} \\ &= \sqrt{\sigma_Y^2 + \sigma^2 - \sigma\sigma_Y} \end{aligned}$$

We have $\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\sigma} = \frac{2\sigma - \sigma_Y}{2\sqrt{\sigma_Y^2 + \sigma^2 - \sigma\sigma_Y}}$.

When $\sigma = 0$, $\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\sigma} < 0$, unloading.

(2) Let $\bar{\sigma} = \sqrt{\sigma_Y^2 + \sigma^2 - \sigma\sigma_Y} = \sigma_Y$, one has $\sigma = \sigma_Y$

II. Deformation

In this section, we consider the elastic-plastic behaviour the solid is characterized by its elastic constants E, ν and by the yield stress $Y(\bar{\varepsilon}^p)$ as a function of accumulated plastic strain $\bar{\varepsilon}^p$ and its slope $h = \dfrac{dY}{d\bar{\varepsilon}^p}$.

A deformation state is described by the stress σ_{ij}, strain ε_{ij}, and the equivalent plastic strain $\bar{\varepsilon}^p$. Under a given stress increment $\mathrm{d}\sigma_{ij}$, the strain increment $\mathrm{d}\varepsilon_{ij}$ can be decomposed into two parts,

$$\mathrm{d}\varepsilon_{ij} = \mathrm{d}\varepsilon_{ij}^e + \mathrm{d}\varepsilon_{ij}^p . \tag{4}$$

The elastic part of the strain is related to stress using the linear elastic equations as

$$\mathrm{d}\varepsilon_{ij} = \frac{1+\nu}{E}\mathrm{d}\sigma_{ij} - \frac{\nu}{E}\mathrm{d}\sigma_{kk}\delta_{ij} . \tag{5}$$

If the deformation is in elastic stage, i.e.,

$$\bar{\sigma} < Y\left(\bar{\varepsilon}^p\right), \tag{6}$$

there is no strain increase, $\mathrm{d}\varepsilon_{ij}^p = 0$.

If the current stage is on the yielding surface, the plastic strain increment $\mathrm{d}\varepsilon_{ij}^p$ depends on the loading or unloading condition.

$$\begin{cases} \mathrm{d}\varepsilon_{ij}^p = \sigma'_{ij}\mathrm{d}\lambda & \text{if } \sigma'_{ij}\mathrm{d}\sigma_{ij} \ge 0 \\ \mathrm{d}\varepsilon_{ij}^p = 0 & \text{if } \sigma'_{ij}\mathrm{d}\sigma_{ij} < 0 \end{cases}, \tag{7}$$

where

$$\mathrm{d}\lambda = \frac{3}{2}\frac{\mathrm{d}\bar{\varepsilon}^p}{\bar{\sigma}} . \tag{8}$$

Therefore, for the loading case

$$\begin{aligned} \mathrm{d}\varepsilon_{ij}^{p} &= \sigma'_{ij}\mathrm{d}\lambda \\ &= \frac{3}{2}\frac{\sigma'_{ij}}{\bar{\sigma}}\mathrm{d}\bar{\varepsilon}^{p} \\ &= \frac{3}{2}\frac{\sigma'_{ij}}{h\bar{\sigma}}\mathrm{d}\bar{\sigma} \\ &= \frac{9}{4}\frac{1}{h}\frac{\sigma'_{ij}}{\bar{\sigma}^{2}}\sigma'_{kl}\mathrm{d}\sigma'_{kl} \end{aligned} . \tag{9}$$

Exercise 2

Following Exercise 1, keep increasing σ, what is the strain tensor when the stress state reaches $\boldsymbol{\sigma} = \begin{pmatrix} \sigma_Y & 0 & 0 \\ 0 & 2\sigma_Y & 0 \\ 0 & 0 & 0 \end{pmatrix}$?