ME 7953: Simulations in Materials

Fall 2002

Problem Set 6 (Friday, 11/8/2002)

Problems are due at the beginning of the class on Friday, 11/15/2002.

Finite Element Method (FEM) is the focus of this week's problems. The problems will help us to understand the basics of FEM.

1) Inside a triangle element or not

Write a MATLAB function to determine if a point (x,y) is inside a triangle element or not.

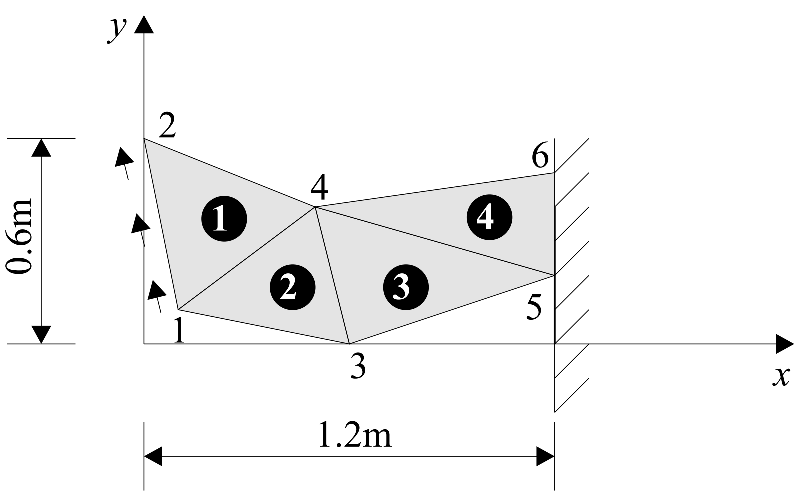
```
function [in] = inside_elem(x1,y1,x2,y2,x3,y3,x,y)
```

A triangle element has three nodes, their coordinates are (x_1,y_1), (x_2,y_2), (x_3,y_3). The function returns 1 if the point (x,y) is inside (or on the edge of) the element, or returns 0 if not. Test the function to see the results of

```
inside_elem(0,0,1,0,0,1,0.5,0.2)
inside_elem(0,0,1,0,0,1,-0.5,0)
inside_elem(0,0,1,0,-0.5,1,0.1,0.1)
inside_elem(0,0,1,0,-0.5,1,0.6,0.55)
inside_elem(0,0,0,1,1,0,0.5,0.2)
```

2) External body force

Consider a two-dimensional beam problem.



We have 6 nodes, their coordinates are

```
coor=(0.1,0.1; 0,0.6; 0.6,0; 0.5,0.4; 1.2,0.2; 1.2,0.5);
```

The coordinates matrix coor is a 6×3 matrix. The first row gives the *x* and *y* coordinates of node 1, the second row gives the *x* and *y* coordinates of node 2, ….

We have 4 elements. Each element contains 3 nodes. We thus construct a 4×3 matrix, elmt, as below to record the element-node relationship.

```
elmt=(2,1,4; 1,3,4; 4,3,5; 4,5,6);
```

The first row gives the three nodes of element 1, the second row gives the three nodes of element 2, …. The node-order should be in anticlockwise. Mathematically it means that

$$\det\begin{bmatrix} 1 & \text{coor(elmt(e,1),1)} & \text{coor(elmt(e,1),2)} \\ 1 & \text{coor(elmt(e,2),1)} & \text{coor(elmt(e,2),2)} \\ 1 & \text{coor(elmt(e,3),1)} & \text{coor(elmt(e,3),2)} \end{bmatrix} > 0 \text{, for element } e = 1,2,3,4$$

Assume the body force density is

$$b_x = 0$$

$$b_y = 10\text{m/s}^2,$$

where the unit is force per mass. The mass density of the material is $\rho = 10^3 \text{kg/m}^3$. Compute the external body force on node *n*:

$$F_x^{(n)} = \frac{1}{3}\sum_{e \supset n} A^e \rho b_x$$

$$F_y^{(n)} = \frac{1}{3}\sum_{e \supset n} A^e \rho b_y \quad \text{, } n \text{ from 1 to 6,}$$

where the summation is over all the elements surrounding node *n*. For example, node 1 is only surrounded by element 1 and 2; node 2 is surrounded by elements 1 only; and node 3 is surrounded by elements 2 and 4, *etc*. A^e is the area of the element, which can be calculate from

$$A^e = \frac{1}{2}\det\begin{bmatrix} 1 & x_a & y_b \\ 1 & x_b & y_b \\ 1 & x_c & y_c \end{bmatrix}.$$

where *a*, *b* and *c* are three nodes of a element. In MATLAB with the above definite of elmt and coor, the area can be calculated with

```
A=0.5*det([1, coor(elmt(e,1),1), coor(elmt(e,1),2); …
           1, coor(elmt(e,2),1), coor(elmt(e,2),2); …
           1, coor(elmt(e,3),1), coor(elmt(e,3),2)])
```

where `elmt(e,1), elmt(e,2)` and `elmt(e,3)` are local nodes *a*, *b* and *c* of element `e`, respectively.

<u>Question:</u> Please give the external body force in two vector-forms, each with 6 components, as

```
F_x
F_y
```

3) Traction force

Computer the traction force on nodes. Traction force on node *n* can be calculated from

$$T_x^{(n)} = \sum_e \int_{\partial\Omega^e} t_x N^{(n)} ds$$

$$T_y^{(n)} = \sum_e \int_{\partial\Omega^e} t_y N^{(n)} ds$$

One triangular element has three edges. The integral along $\partial\Omega^e$ can be estimated on each edge of the element *e* using mid-point rule. In the equations, t_x, t_y are the *x* and *y* components of the boundary tractions. For our problem here, the traction 5000N/m^2 is along edge 1-2, with directions shown in the figure.

The traction forces on node 5 and 6 are unknowns.

<u>Question:</u> Please give the traction forces for node 1, 2, 3 and 4 as:

```
T_x(1), T_x(2), T_x(3), T_x(4)
T_y(1), T_y(2), T_y(3), T_y(4)
```

3) Stiffness

The material parameters for the problem are

$$E = 10^9\text{Pa}$$
$$\nu = 0.3$$

In MATLAB, use

```
E=10e9;
nu=0.3;
```

<u>Question:</u> Calculate the stiffness $K_{xx}^{(3,4)}$, $K_{xy}^{(3,4)}$, $K_{yx}^{(3,4)}$, $K_{yy}^{(3,4)}$.

The process can be:

```
Kxx=0;
Kxy=0;
Kyx=0;
Kyy=0;
for e=1:4
  %Find if element e contains both node 3 and 4.
  %Also, the corresponding local order i for global node 3,
  $and j for global node 4.
  %
```

```
    %calculate element-stiffness kxx between i and j
    %calculate element-stiffness kxy between i and j
    %calculate element-stiffness kyx between i and j
    %calculate element-stiffness kyy between i and j
    Kxx = Kxx + kxx;
    Kxy = Kxy + kxy;
    Kyx = Kyx + kyx;
    Kyy = Kyy + kyy;
  end
```

The element-stiffness can be calculated from:

$$k_{xx}^{(e,i,j)} = A^e E\left(\frac{1}{1-\nu^2}\frac{\partial N^{(i)}}{\partial x}\frac{\partial N^{(j)}}{\partial x} + \frac{1}{2(1+\nu)}\frac{\partial N^{(i)}}{\partial y}\frac{\partial N^{(j)}}{\partial y}\right)$$

$$k_{xy}^{(e,i,j)} = A^e E\left(\frac{\nu}{1-\nu^2}\frac{\partial N^{(i)}}{\partial y}\frac{\partial N^{(j)}}{\partial x} + \frac{1}{2(1+\nu)}\frac{\partial N^{(i)}}{\partial x}\frac{\partial N^{(j)}}{\partial y}\right)$$

$$k_{yx}^{(e,i,j)} = A^e E\left(\frac{1}{2(1+\nu)}\frac{\partial N^{(i)}}{\partial y}\frac{\partial N^{(j)}}{\partial x} + \frac{\nu}{1-\nu^2}\frac{\partial N^{(i)}}{\partial x}\frac{\partial N^{(j)}}{\partial y}\right),$$

$$k_{yy}^{(e,i,j)} = A^e E\left(\frac{1}{2(1+\nu)}\frac{\partial N^{(i)}}{\partial x}\frac{\partial N^{(j)}}{\partial x} + \frac{1}{1-\nu^2}\frac{\partial N^{(i)}}{\partial y}\frac{\partial N^{(j)}}{\partial y}\right)$$

where *i* and *j* can be 1 (local node *a*), 2 (local node *b*), or 3 (local node *c*), and A^e is the area of the element *e*. The derivatives are:

$$\frac{\partial N^{(a)}}{\partial x} = \frac{y_b - y_c}{2A^e}, \quad \frac{\partial N^{(a)}}{\partial y} = \frac{x_c - x_b}{2A^e}$$

$$\frac{\partial N^{(b)}}{\partial x} = \frac{y_c - y_a}{2A^e}, \quad \frac{\partial N^{(b)}}{\partial y} = \frac{x_a - x_c}{2A^e}$$

$$\frac{\partial N^{(c)}}{\partial x} = \frac{y_a - y_b}{2A^e}, \quad \frac{\partial N^{(c)}}{\partial y} = \frac{x_b - x_a}{2A^e}$$

4) Stiffness

<u>Question:</u> Calculate the stiffness $K_{xx}^{(4,4)}$, $K_{xy}^{(4,4)}$, $K_{yx}^{(4,4)}$, $K_{yy}^{(4,4)}$.

The process can be:

```
  Kxx=0;
  Kxy=0;
  Kyx=0;
  Kyy=0;
  for e=1:4
    %Find if element e contains node 4.
    %Also, the corresponding local order i for global node 4.
    %
    %calculate element-stiffness kxx(e,i,i)
    %calculate element-stiffness kxy(e,i,i)
```

```
    %calculate element-stiffness kyx(e,i,i)
    %calculate element-stiffness kyy(e,i,i)
    Kxx = Kxx + kxx;
    Kxy = Kxy + kxy;
    Kyx = Kyx + kyx;
    Kyy = Kyy + kyy;
  end
```

The element stiffness can be calculated from

$$k_{xx}^{(e,i,i)} = A^e E\left(\frac{1}{1-\nu^2}\left(\frac{\partial N^{(i)}}{\partial x}\right)^2 + \frac{1}{2(1+\nu)}\left(\frac{\partial N^{(i)}}{\partial y}\right)^2\right)$$

$$k_{xy}^{(e,i,i)} = \frac{A^e E}{2(1-\nu)}\frac{\partial N^{(i)}}{\partial y}\frac{\partial N^{(i)}}{\partial x}$$

$$k_{yx}^{(e,i,i)} = \frac{A^e E}{2(1-\nu)}\frac{\partial N^{(i)}}{\partial y}\frac{\partial N^{(i)}}{\partial x},$$

$$k_{yy}^{(e,i,i)} = A^e E\left(\frac{1}{2(1+\nu)}\left(\frac{\partial N^{(i)}}{\partial x}\right)^2 + \frac{1}{1-\nu^2}\left(\frac{\partial N^{(i)}}{\partial y}\right)^2\right)$$

where i can be 1 (local node a), 2 (local node b), or 3 (local node c).