Impact of Cellular Materials

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1 Introduction

Cellular materials, such as metal foams, are used as impact energy absorbers in crash and blast protection due to their unique constitutive behaviour. Three stages can be identified in the stress-strain curve of the uniaxial compression of cellular materials, as shown in Figure 1.

> Stage I: Deformation is in the form of bending of the cell walls and edges and is, in general, reversible. At the end of this stage some cells suffer collapse. This may be due to elastic buckling, plastic deformation or fracture.
>
> Stage II: The almost constant compressive stress appears in a wide range of strain. Collapse occurs successively until all cells are collapsed. The deformation in this stage is unrecoverable.
>
> Stage III: Cell walls and edges contact each other and are crushed, giving rise to a steeply rising stress.

Energy absorbers for crush and blast protection are chosen so that the plateau stress is just below that which will cause damage to the packaged object; the best choice is then the one which has the longest plateau, and therefore absorbs the most energy. Cellular materials fit the idealization.

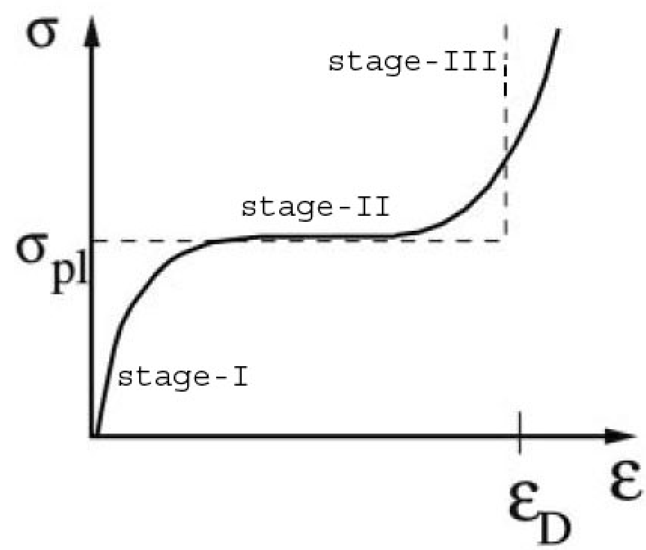


Figure 1. Idealization of stress-strain curve in the rigid, perfect-plastic, locking (RPPL) model for cellular materials. In stage-I, the material is treated as rigid. In stage-II, the material is in perfect plasticity with yield stress at $\sigma_{\rm pl}$. In stage-III, the material is again treated as rigid, with the densification strain at $\varepsilon_{\rm D}$.

For impact analysis, Reid and Peng (1997) firstly treated cellular materials subject to uniaxial compression using a simplified rigid, perfectly-plastic, locking (RPPL) model as shown in Figure 1, where stage-I of the constitutive behaviour is treated as rigid, the second stage is treated as perfect plastic with the yielding plateau stress $\sigma_{\rm pl}$. The second stage ended with the locking (densification) strain $\varepsilon_{\rm D}$, and the third stage is again idealized as rigid. Tan et al. (2002) described a RPPL idealization of foam materials.

For the crush and blast protection, strain densification and the associated shock propagation are important. For simplicity, one dimensional behavior of RPPL cellular materials are considered using shock-wave analysis.

2 Crash protection

In crash protection the absorber must absorb the kinetic energy of the moving object without reaching its densification strain $\varepsilon_{\rm D}$ – then the stress it transmits never exceeds the plateau stress.

The focus of the study is to investigate the dynamic energy losses of a plate with areal density, *m*; during a high-velocity impact on a metal foam layer.

Consider the impact of one end of a stationary rod by a massive striker at a velocity $v_{\rm im}$, as denoted in Figure 2(a). The rod is homogeneous and made from a RPPL material. After impact a shock wave moves from the impacted end to the opposite fixed end of the rod, as shown in Figure 2(b). Assuming rigid-plastic behavior, the stress ahead of the shock wave is compressive with magnitude equals $\sigma_{\rm pl}$. The material behind the shock front has attained a strain $\varepsilon_{\rm D}$, its particle velocity is $v_{\rm d}$, its density has been raised from the initial value ρ_0 to the densification value $\rho_{\rm D}$, and the compressive stress has been raised to $\sigma_{\rm d}$.

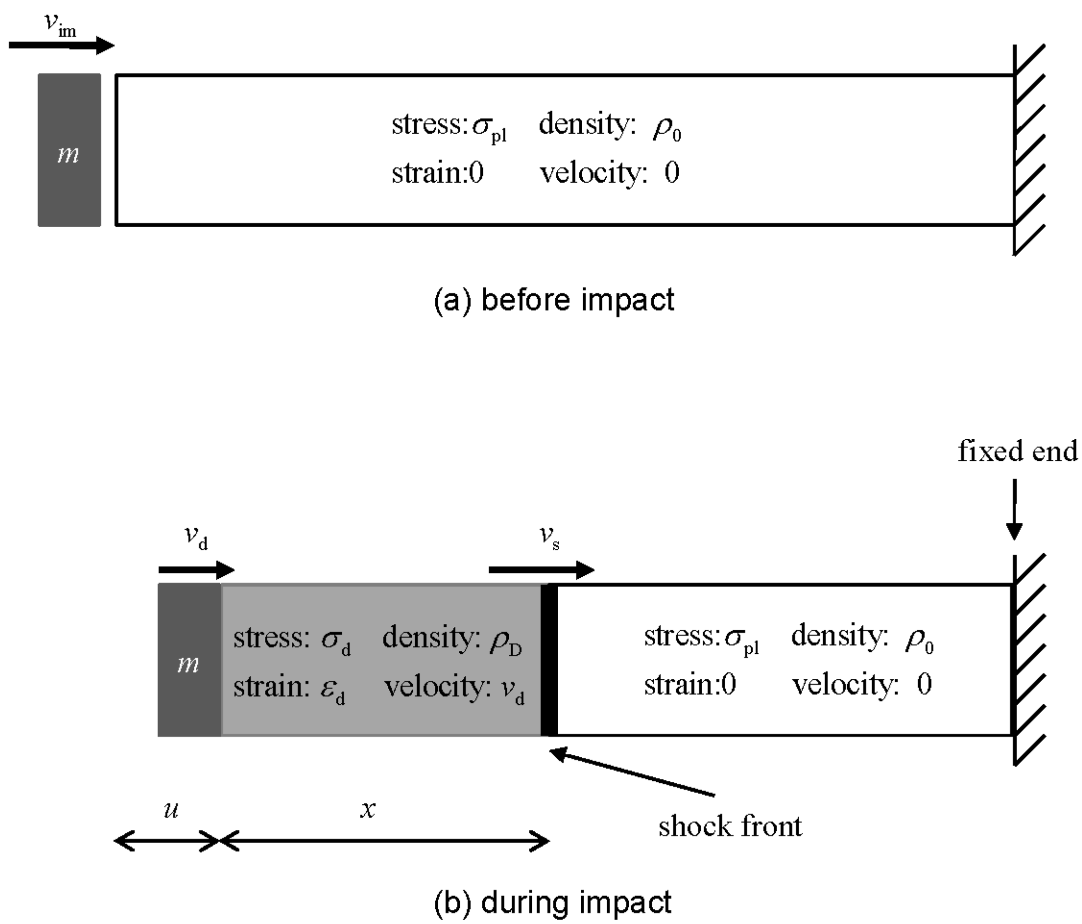


Figure 2. Plastic shock wave propagation through a cellular material. The impact plate has an areal mass *m* that shares the same velocity of the cellular materials behind the shock front. For cellular material, the distal portion is in white color, and the incident portion is in gray color. For the distal portion, material velocity is 0, density is ρ_0, and the compressive stress is σ_{pl}. For the incident portion, material velocity is v_d, density is ρ_D, and the compressive stress is σ_d. The velocity of shock front is v_s.

2.1 Basic assumptions

Assumptions are made for analytically investigate the shock behavior in a one-dimensional rod:

1. There is a sharp shock front separating the compressed and undeformed regions of the foam. The micrograph in Figure 3 clearly shows a sharp shock front separating the compressed and undeformed regions of the foam.

2. After the densification, the particle velocities of the densified materials are equal to each other and equal to the velocity of the moving rigid plate.

3. The densified layer has a density which is spatially constant.

4. The deformation rate that affects the deformation and failure modes of cellular materials is ignored.

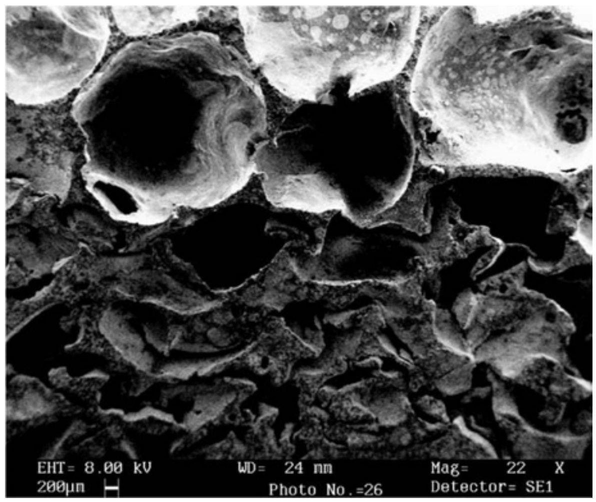


Figure 3. A scanning electron micrograph of the metal foam specimen shown in Figure 2 and sectioned along its perpendicular plane. (from Radford et al., 2005).

2.2 Shock wave analysis

Here we consider the case where the impact speed is lower than the elastic wave speed Therefore, in the undeformed region ahead of the shock wave, the stress is equal to the plateau stress $\sigma_{\rm pl}$, while the strain and particle velocity are zero.

Conservation of mass across the shock front gives

$$\frac{\rho_0}{\rho_{\rm D}} = 1 - \varepsilon_{\rm D} . \qquad (1)$$

The compacted region travels with the same (reducing) velocity at the mass and has been compressed to the densification strain $\varepsilon_{\rm D}$,

$$\varepsilon_{\mathrm{D}} = \frac{u}{x+u}, \tag{2}$$

where u is the displacement of the rigid mass at time t, and x is the deformed length of the crushed cellular material. The cellular material behind the shock front is idealized as rigid. The instantaneous velocity of the rigid mass with respect to a stationary frame is the same as that of the condensed part, and is denoted as

$$v_{\mathrm{d}} = \frac{\mathrm{d}u}{\mathrm{d}t}. \tag{3}$$

The shock velocity v_{s} is faster than particles velocity v_{d} after the shock, and this gives the increase of x with respect of time,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_{\mathrm{s}} - v_{\mathrm{d}}. \tag{4}$$

Therefore, the relation between the shock velocity v_{s} and the condensed part velocity v_{d} is

$$v_{\mathrm{s}} = \frac{v_{\mathrm{d}}}{\varepsilon_{\mathrm{D}}}, \tag{5}$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1-\varepsilon_{\mathrm{D}}}{\varepsilon_{\mathrm{D}}} v_{\mathrm{d}}. \tag{6}$$

Considering the momentum change of the body consists of the impact plate and the cellular material behind the shock front, one has

$$\mathrm{d}\left[\left(m+\rho_{\mathrm{D}}x\right)v_{\mathrm{d}}\right] = -\sigma_{\mathrm{pl}}\mathrm{d}t, \tag{7}$$

where m is the areal density of the impact plate. The above equation gives

$$v_{\mathrm{d}}\mathrm{d}\left[\left(m+\rho_{\mathrm{D}}x\right)v_{\mathrm{d}}\right] + \sigma_{\mathrm{pl}}\frac{\varepsilon_{\mathrm{D}}}{1-\varepsilon_{\mathrm{D}}}\mathrm{d}x = 0,$$

which can be rewritten as

$$\frac{1}{\sigma_{\mathrm{pl}}\frac{\varepsilon_{\mathrm{D}}}{1-\varepsilon_{\mathrm{D}}} + \rho_{D}v_{\mathrm{d}}^{2}}\mathrm{d}v_{\mathrm{d}}^{2} + \frac{2}{m+\rho_{\mathrm{D}}x}\mathrm{d}x = 0.$$

Solving the above equation, one has

$$\left(v_{\mathrm{d}}^2\rho_{\mathrm{D}}+\frac{\varepsilon_{\mathrm{D}}}{1-\varepsilon_{\mathrm{D}}}\sigma_{\mathrm{pl}}\right)\left(m+\rho_{\mathrm{D}}x\right)^2=\mathrm{const}\,.$$

With the initial condition at the impact that: $v_{\mathrm{d}}=v_{\mathrm{im}}$ and $x=0$, one has

$$\left(v_{\mathrm{d}}^2\rho_{\mathrm{D}}+\frac{\varepsilon_{\mathrm{D}}}{1-\varepsilon_{\mathrm{D}}}\sigma_{\mathrm{pl}}\right)\left(m+\rho_{\mathrm{D}}x\right)^2=\left(v_{\mathrm{im}}^2\rho_{\mathrm{D}}+\frac{\varepsilon_{\mathrm{D}}}{1-\varepsilon_{\mathrm{D}}}\sigma_{\mathrm{pl}}\right)m^2\,,$$

which gives the particle velocity behind the shock as

$$v_{\mathrm{d}}=v_{\mathrm{C}}\sqrt{\left[1+\left(\frac{v_{\mathrm{im}}}{v_{\mathrm{C}}}\right)^2\right]\left(\frac{m}{m+\rho_{\mathrm{D}}x}\right)^2-1}\,, \tag{8}$$

where the characteristic velocity v_{C} is defined as

$$v_{\mathrm{C}}=\sqrt{\frac{\varepsilon_{\mathrm{D}}\sigma_{\mathrm{pl}}}{\rho_0}}\,. \tag{9}$$

The evolution of x can be solved from the following equation

$$\frac{dx}{dt}=\frac{1-\varepsilon_{\mathrm{D}}}{\varepsilon_{\mathrm{D}}}v_{\mathrm{C}}\sqrt{\frac{1+\tilde{v}^2}{\left(1+\frac{\rho_{\mathrm{D}}x}{m}\right)^2}-1}\,,$$

where

$$\tilde{v}=\frac{v_{\mathrm{im}}}{v_{\mathrm{C}}}\,, \tag{10}$$

The integration gives

$$\int_0^x\frac{dx}{\sqrt{\frac{1+\tilde{v}^2}{\left(1+\rho_{\mathrm{D}}x/m\right)^2}-1}}=\frac{1-\varepsilon_{\mathrm{D}}}{\varepsilon_{\mathrm{D}}}v_{\mathrm{C}}t+\mathrm{const}\,.$$

which is

$$\frac{1-\varepsilon_{\mathrm{D}}}{\varepsilon_{\mathrm{D}}}v_{\mathrm{C}}t+\sqrt{\frac{m^2\tilde{v}^2}{\rho_{\mathrm{D}}^2}-\frac{2m}{\rho_{\mathrm{D}}}x-x^2}=\mathrm{const}\,.$$

With the initial condition $x=0$ at $t=0$, one has

$$\frac{1-\varepsilon_{\mathrm{D}}}{\varepsilon_{\mathrm{D}}} v_{\mathrm{C}} t + \sqrt{\frac{m^2 \tilde{v}^2}{\rho_{\mathrm{D}}^2} - \frac{2m}{\rho_{\mathrm{D}}} x - x^2} = \frac{m\tilde{v}}{\rho_{\mathrm{D}}} .$$

Solve the equation for *x*, one has the time evolution of the length of the crushed foam as

$$x = -\frac{m}{\rho_{\mathrm{D}}} + \sqrt{\left(\frac{m}{\rho_{\mathrm{D}}}\right)^2 - \left(\frac{1-\varepsilon_{\mathrm{D}}}{\varepsilon_{\mathrm{D}}} v_{\mathrm{C}} t\right)^2 + 2\frac{m}{\rho_{\mathrm{D}}} \frac{1-\varepsilon_{\mathrm{D}}}{\varepsilon_{\mathrm{D}}} v_{\mathrm{im}} t} , \tag{11}$$

which can be written in a nondimensionalized form as

$$\frac{x}{m / \rho_{\mathrm{D}}} = \sqrt{1 - \tilde{t}^2 + 2\tilde{v}\tilde{t}} - 1 , \tag{12}$$

where the nondimensionalized time $\tilde{t}$ is defined as

$$\tilde{t} = \frac{t}{m\varepsilon_{\mathrm{D}} / \rho_0 v_{\mathrm{C}}} . \tag{13}$$

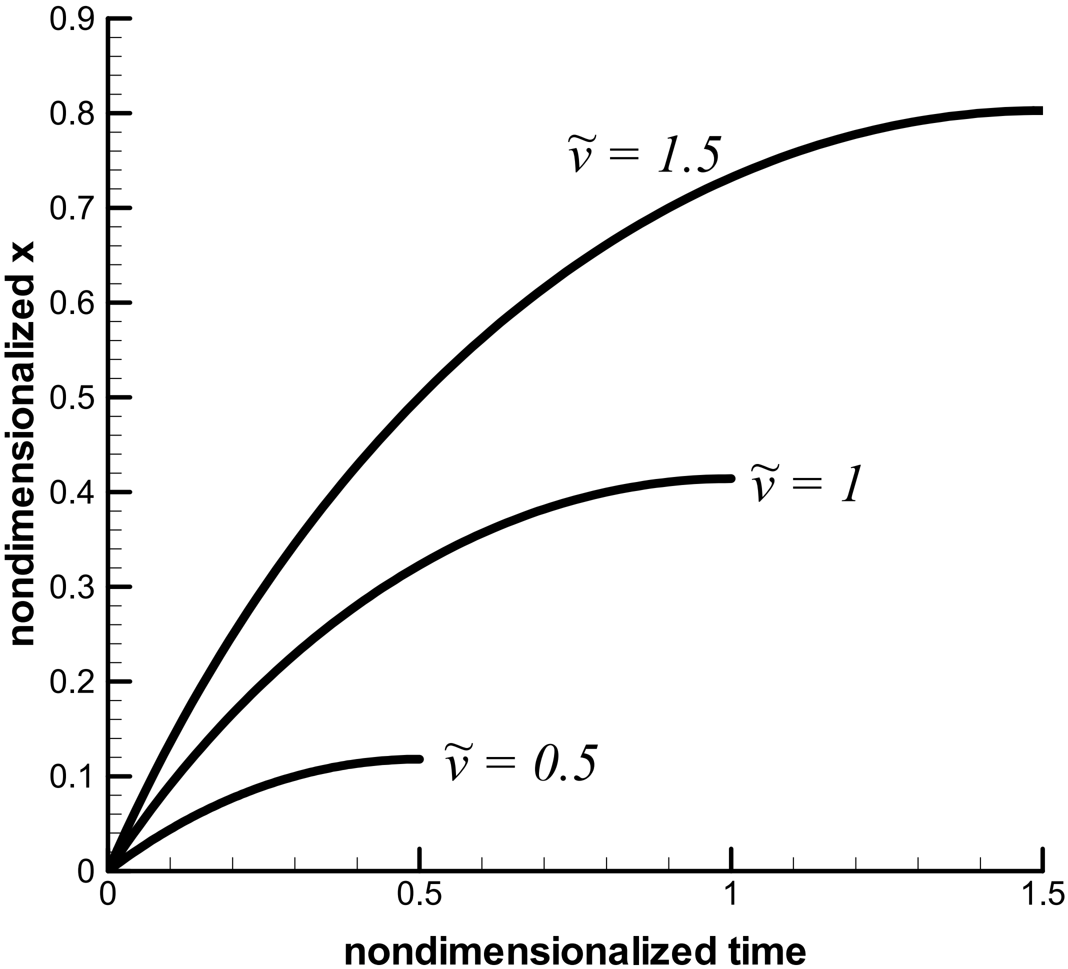


Figure 4. Nondimensionalized length of the crushed cellular material, $\frac{x}{m/\rho_D}$, as a function of nondimensionalized time, $\frac{t}{\varepsilon_D m / \rho_0 v_C}$, for different impact velocities at $v_{im} = \tilde{v}\sqrt{\frac{\varepsilon_D \sigma_{pl}}{\rho_0}}$.

For fully crush, which means no shock wave or shock wave speed is zero,

$$\frac{dx}{dt} = 0, \quad (14)$$

which gives the maximum length of the crushed cellular material as

$$\frac{x_{\max}}{m/\rho_D} = \sqrt{1+\tilde{v}^2} - 1, \tag{15}$$

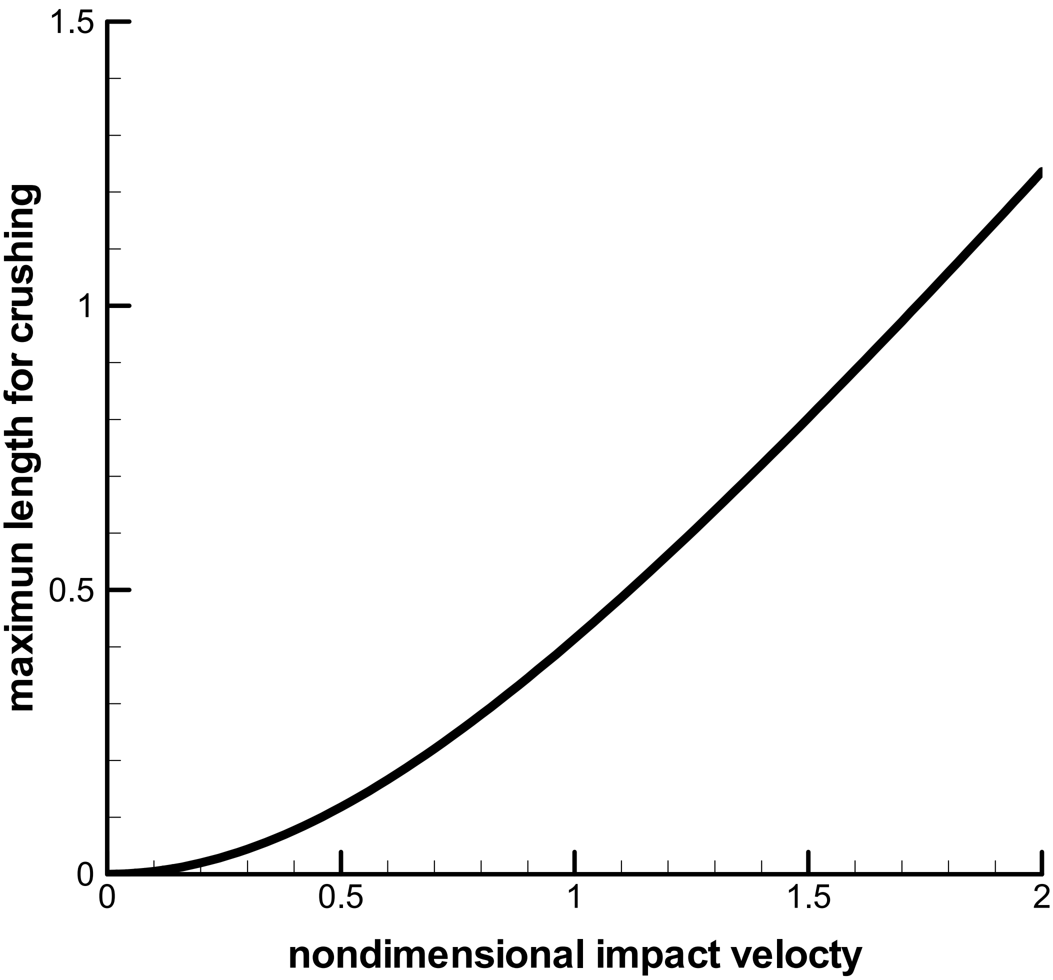


Figure 5. Nondimensionalized maximum length of the crushed cellular material, $\frac{x_{\max}}{m/\rho_D}$, as a function of the nondimensionalized impact velocities $v_{\mathrm{im}} = \tilde{v}\sqrt{\frac{\varepsilon_D \sigma_{\mathrm{pl}}}{\rho_0}}$.

According to Eq. (12), the time to reach the maximum crushing length $x_{\max}$ is

$$\tilde{t} = \tilde{v}, \tag{16}$$

which means

$$t = \frac{m v_{\mathrm{im}}}{\sigma_{\mathrm{pl}}}. \tag{17}$$

According to Eq. (15), to crush a impact velocity of v_{im}, the mass of the cellular material for energy absorbing M_{cellular} should be at least

$$\frac{M_{\mathrm{cellular}}}{M_{\mathrm{impact}}} = \sqrt{1 + \left(\frac{v_{\mathrm{im}}}{v_{\mathrm{C}}}\right)^2} - 1. \tag{18}$$

where M_{impact} is the mass of the impact object. For the case of $v_{\mathrm{im}} = v_{\mathrm{C}}$, $\frac{M_{\mathrm{cellular}}}{M_{\mathrm{impact}}} = \sqrt{2} - 1$, i.e, crush a impact velocity of v_{C}, the ratio of the mass of the cellular material for energy absorbing and that of the impact object should be at least 1.4.

For car crash, the impact velocity is about $v_{\mathrm{im}} = 30\mathrm{m/s}$. The following table gives the material properties for several typical cellular materials for energy absorbing.

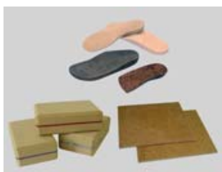
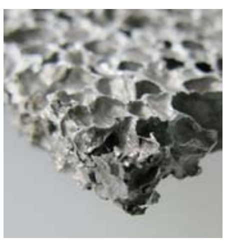
	Image	Density ρ_0 ($\mathrm{kg/m^3}$)	Plateau stress σ_{pl} (MPa)	Densification strain ε_{D}	Characteristic velocity v_{C} (m/s)
polyurethane foam		34	0.25	0.55	45
Cork foam		164	1	0.55	41
Alporas metal foam		200-250	1.6-1.8	0.7-0.82	54

Table 1. Material properties for some typical cellular materials.

A cellular rod can fully crush the impact up to velocity of

$$v_{\mathrm{im}} = v_{\mathrm{C}} \sqrt{\left(1 + \frac{M_{\mathrm{cellular}}}{M_{\mathrm{impact}}}\right)^2 - 1}\,. \tag{19}$$

2.2.1 Stress behind the shock front

Conservation of momentum across the shock front gives the compressive stress (compressive stress is in positive sign) behind the shock σ_{d} as

$$\sigma_{\mathrm{d}} = \sigma_{\mathrm{pl}} + \frac{\rho_0}{\varepsilon_{\mathrm{D}}} v_{\mathrm{d}}^2 . \tag{20}$$

Using Eq. (8), the stress behind the shock can be written as a function of the length of crushed foam x,

$$\sigma_{\mathrm{d}} = \sigma_{\mathrm{pl}} \left[1 + \left(\frac{v_{\mathrm{im}}}{v_{\mathrm{C}}} \right)^2 \right] \left(\frac{m}{m + \rho_{\mathrm{D}} x} \right)^2 . \tag{21}$$

Using Eq. (11) for x as a function of time t, and the nondimensionalized notation for impact velocity $\tilde{v} = \frac{v_{\mathrm{im}}}{v_{\mathrm{C}}}$ and time $\tilde{t} = \frac{t}{m\varepsilon_{\mathrm{D}} / \rho_0 v_{\mathrm{C}}}$, Eq. (21) provides

$$\frac{\sigma_{\mathrm{d}}}{\sigma_{\mathrm{pl}}} = \frac{1 + \tilde{v}^2}{1 + 2\tilde{v}\tilde{t} - \tilde{t}^2} . \tag{22}$$

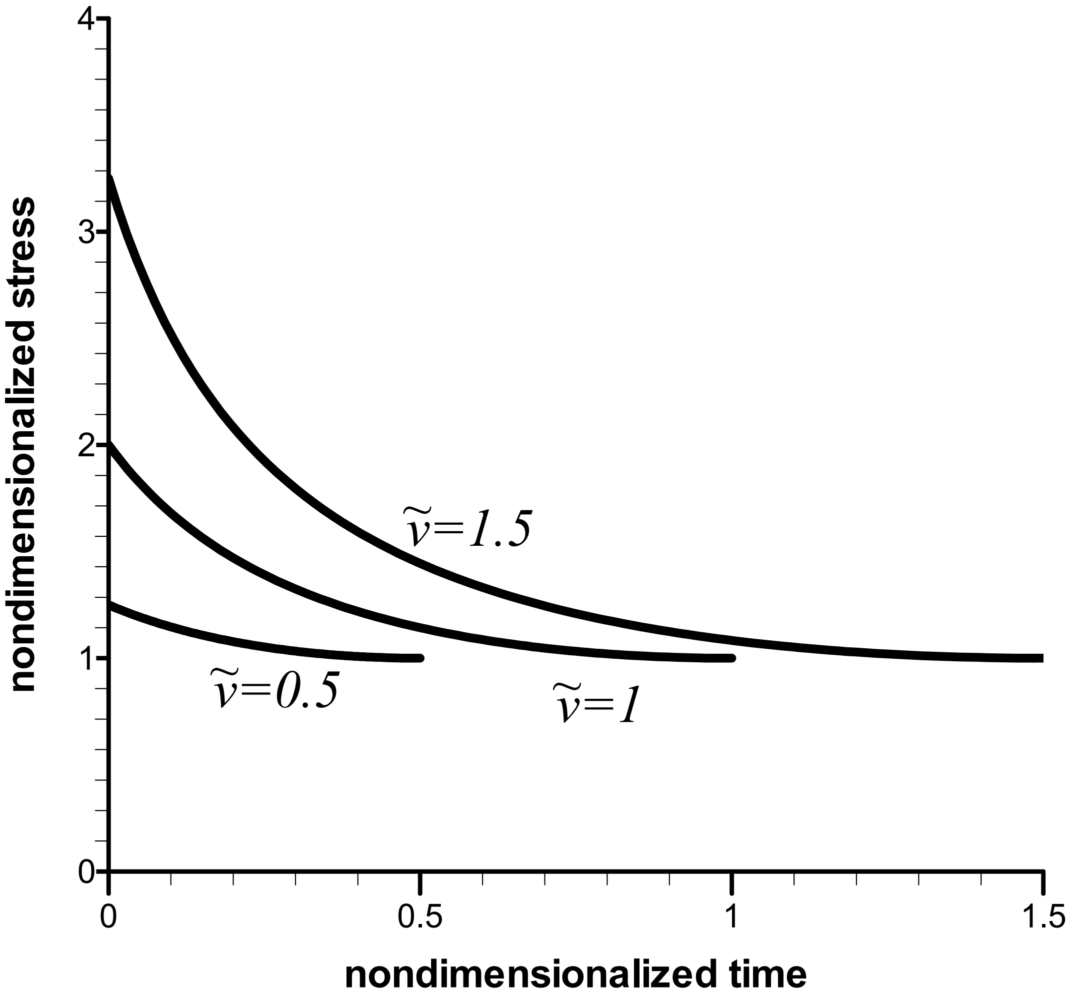


Figure 6. Nondimensionalized stress behind the shock front, $\dfrac{\sigma_{\mathrm{d}}}{\sigma_{\mathrm{pl}}}$, evolve as a function of nondimensionalized $\dfrac{t}{\varepsilon_{\mathrm{D}} m / \rho_0 v_{\mathrm{C}}}$, for different impact velocities at $v_{\mathrm{im}} = \tilde{v}\sqrt{\dfrac{\varepsilon_{\mathrm{D}}\sigma_{\mathrm{pl}}}{\rho_0}}$.

2.2.2 Pressure on the mass

Substitute the expression for x as a function of time into Eq. (2), one has

$$u = \frac{\varepsilon_{\mathrm{D}}}{1\text{-}\varepsilon_{\mathrm{D}}}\frac{m}{\rho_{\mathrm{D}}}\left(-1+\sqrt{1+2\tilde{v}\tilde{t}-\tilde{t}^{2}}\right). \tag{23}$$

The pressure on the mass can be calculated from

$$p = -m\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}, \tag{24}$$

which gives

$$\frac{p}{\sigma_{\mathrm{pl}}} = \frac{1+\tilde{v}^2}{\left(1+2\tilde{v}\tilde{t}-\tilde{t}^2\right)^{\frac{3}{2}}}. \tag{25}$$

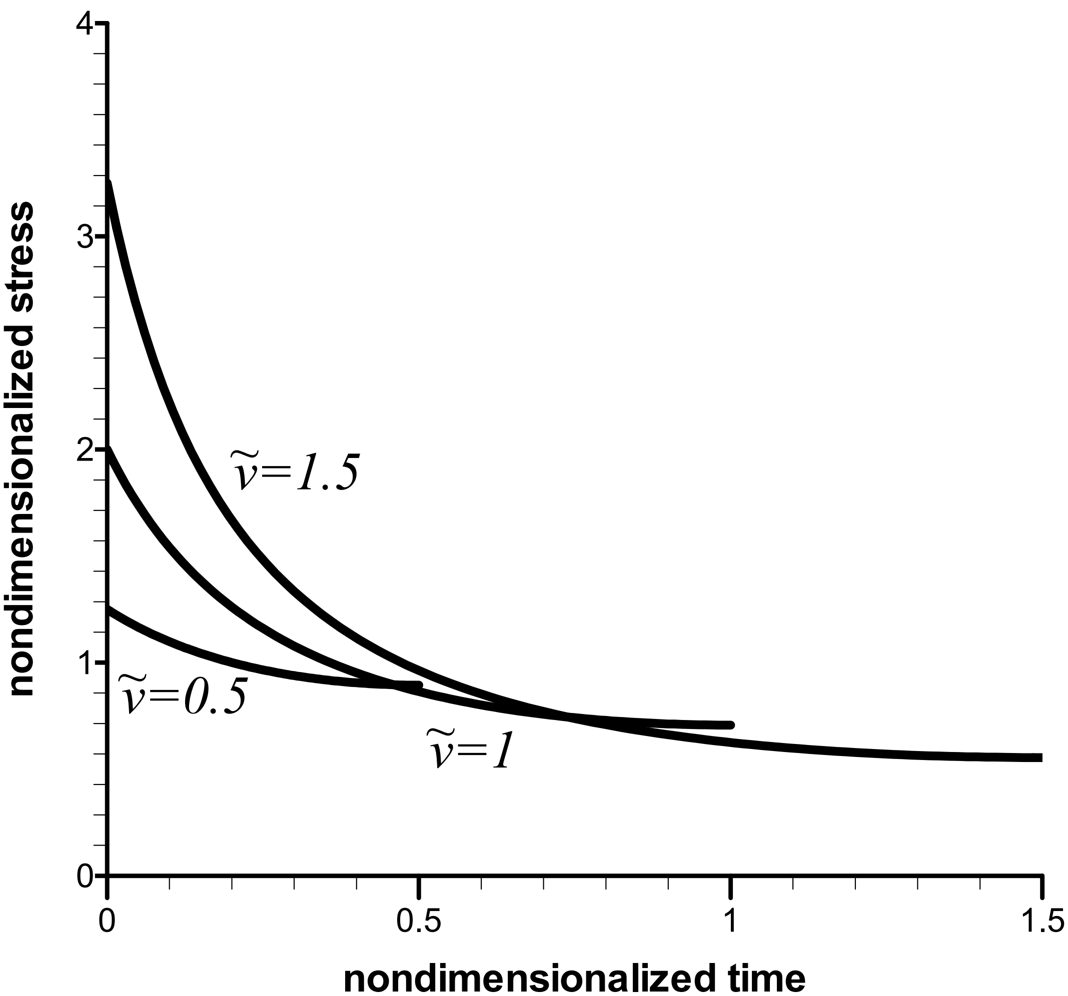


Figure 7. Nondimensionalized stress on the mass, $\frac{p}{\sigma_{\mathrm{pl}}}$, evolve as a function of nondimensionalized $\frac{t}{\varepsilon_{\mathrm{D}} m / \rho_0 v_{\mathrm{C}}}$, for different impact velocities at $v_{\mathrm{im}} = \tilde{v}\sqrt{\frac{\varepsilon_{\mathrm{D}}\sigma_{\mathrm{pl}}}{\rho_0}}$.

3 Shock resistance

There is continued interest in the development of shock resistant structures, in order to maximize survivability both of the structure and of any occupants. In blast protection the absorber must absorb the kinetic energy of the conserving momentum without reaching its densification strain ε_D – then the stress it transmits never exceeds the plateau stress.

3.1 Modelling

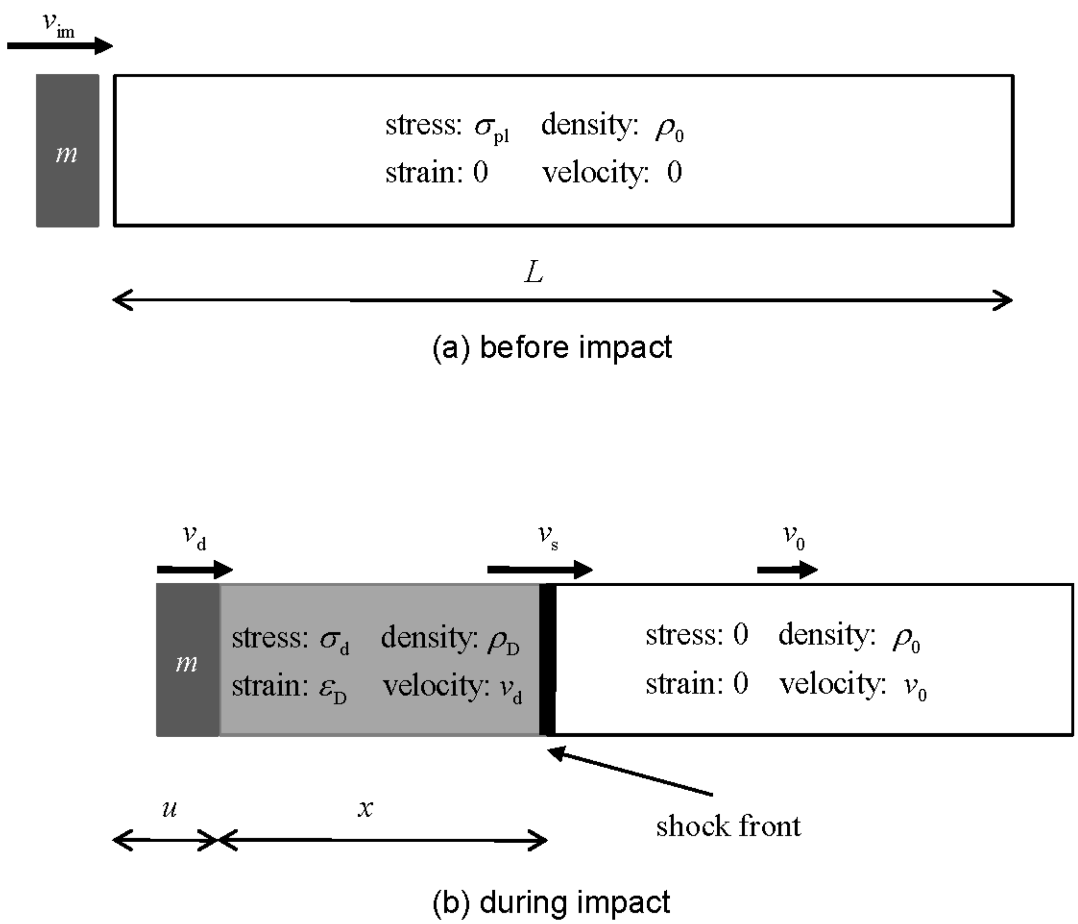


Figure 8. A free stationary rod of cellular material impacted by a massive striker at a velocity v_{im}. Shock wave is generated from the impact end and propagates at a velocity v_s. The shock front separate the rod into two parts, denoted as state "0" (with initial density ρ_0, zero stress and strain, and particle velocity v_0) and state "d" (with condensed strain ε_D, density ρ_0, stress σ_d,

and particle velocity v_d). During the impact, the displacement of the striker is *u*, and the length of the crushed foam is *x*.

For the usage of subscript, capital letter "D" such as that in ρ_D and ε_D denotes material constants during the deformation, and the small letter "d" such as that in v_d and σ_d denotes variables that change during the deformation. The subscript "0" includes both material constants such as ρ_0 and variables such as v_0.

The same as in the last section, $\rho_0 = \rho_D(1-\varepsilon_D)$.

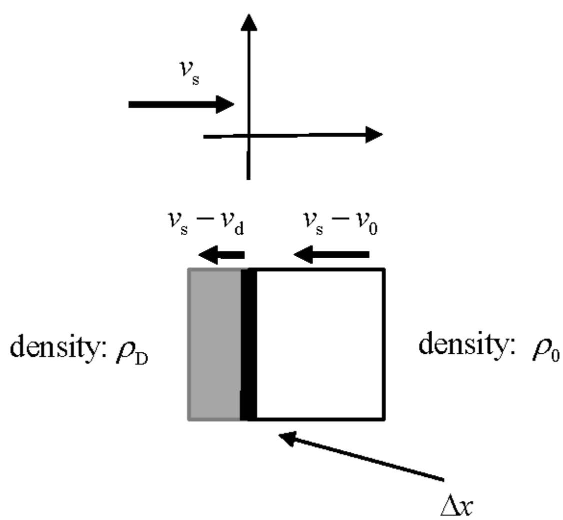


Figure 9. Local coordinate system at the shock front that travels with the shock wave. Viewing in this reference coordinate system, the particles move across the shock front at the speed of $v_s - v_0$ to the left, and comes out to the left at the speed of $v_s - v_d$.

Set a local coordinate system moving with the shock wave front as in Figure 9. Consider a element of length $\Delta X = (v_s - v_0)\Delta t$, shown as a white block in Figure 9. In the local reference coordinate system, the element ΔX moves across the front and reduces the velocity from $v_s - v_0$ to $v_s - v_d$. At the beginning of the time section Δt, the mass of the element is $\rho_0 \Delta X$. At the end of the time section Δt, the element crosses the shock front and reduces the length to $(v_s - v_d)\Delta t$, thus the mass becomes $\rho_D(v_s - v_d)\Delta t$.

The conservation of mass across the shock front provides

$$\rho_D(v_s - v_d) = \rho_0(v_s - v_0),$$

which gives

$$v_{\mathrm{s}} = \frac{v_{\mathrm{d}} - v_0}{\varepsilon_{\mathrm{D}}} + v_0 , \tag{26}$$

The mass of the foam behind the shock is $\rho_{\mathrm{D}} x$, and the mass of the foam ahead the shock is $\rho_0 L - \rho_{\mathrm{D}} x$. Conservation of momentum of the whole system (impact plate and the cellular material) gives

$$\left(m + \rho_{\mathrm{D}} x\right) v_{\mathrm{d}} + \left(\rho_0 L - \rho_{\mathrm{D}} x\right) v_0 = m v_{\mathrm{im}} , \tag{27}$$

where *m* is the areal density of the impact plate.

For the region ahead of the shock wave front, we have

$$\sigma_{\mathrm{pl}} = \left(\rho_0 L - \rho_{\mathrm{D}} x\right) \frac{\mathrm{d} v_0}{\mathrm{d} t} , \tag{28}$$

For the geometry relation,

$$v_{\mathrm{s}} = \frac{\mathrm{d}(x+u)}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} + v_{\mathrm{d}} , \tag{29}$$

From equations (26) to (29) a non-dimensional equation system can be derived as

$$\begin{aligned} &\left(\tilde{m} + \tilde{x}\right)\tilde{v}_{\mathrm{d}} + \left(1 - \tilde{x}\right)\tilde{v}_0 = \tilde{m} \\ &\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{t}} = \tilde{v}_{\mathrm{d}} - \tilde{v}_0 \\ &\frac{\mathrm{d}\tilde{v}_0}{\mathrm{d}\tilde{t}} = \frac{1}{\left(1 - \tilde{x}\right)\eta} \end{aligned} , \tag{30}$$

where the mass coefficient

$$\tilde{m} = \frac{1}{\rho_0 L} m \tag{31}$$

is the ratio of the impact mass versus the foam mass, and the impact coefficient

$$\eta = \frac{\rho_0 v_{\mathrm{im}}^2}{\sigma_{\mathrm{pl}} \varepsilon_{\mathrm{D}}} \tag{32}$$

is a material property combination related to impact energy and absorbing. Time is expressed in a non-dimensional form with respect to the impact velocity, the length of the foam and densification as

$$\tilde{t} = \frac{v_{\text{im}}}{L\varepsilon_{\text{D}}} t . \qquad (33)$$

Three non-dimensional variables that vary with time are

$$\begin{aligned} \tilde{x} &= \frac{x}{\left(1-\varepsilon_{\text{D}}\right)L} \\ \tilde{v}_{\text{d}} &= \frac{v_{\text{d}}}{v_{\text{im}}} \\ \tilde{v}_0 &= \frac{v_0}{v_{\text{im}}} \end{aligned} \quad . \qquad (34)$$

The range for $\tilde{x}$ is $[0,1]$ where $\tilde{x}=0$ corresponds to the commencement for impact, and $\tilde{x}=1$ corresponds to complete densification of the whole foam rod.

As an example, we use Alporas metal foams for the protection from car crashing. For this type of foams, the density is $\rho_0 = 0.2-0.25 \quad \times 10^3 \text{kg/m}^3$, The yield stress $\sigma_{\text{pl}} = 1.6-1.8\text{MPa}$, and the densification strain is $\varepsilon_{\text{D}} = 0.7-0.82$. The impact velocity during the car crash is around 30m/s. For this impact velocity, $\tilde{\sigma}_{\text{pl}}$ is in the range from 5.0 to 8.2.

The equation system (30) generates

$$\eta\left[\tilde{m} - \tilde{v}_0\left(1+\tilde{m}\right)\right]\text{d}\tilde{v}_0 = \left(\frac{1+\tilde{m}}{1-\tilde{x}} - 1\right)\text{d}\tilde{x} .$$

The integration gives

$$\tilde{m}\tilde{v}_0 - \frac{1}{2}\tilde{v}_0^2\left(1+\tilde{m}\right) + \frac{1}{\eta}\left[\tilde{x} + \left(1+\tilde{m}\right)\ln\left(1-\tilde{x}\right)\right] = \text{const} .$$

With the initial condition of $\tilde{v}_0 = 0$ when $\tilde{x} = 0$, the above equation generates

$$\frac{1}{2}\left(1+\tilde{m}\right)\tilde{v}_0^2 - \tilde{m}\tilde{v}_0 - \frac{1}{\eta}\left[\tilde{x} + \left(1+\tilde{m}\right)\ln\left(1-\tilde{x}\right)\right] = 0 .$$

Solving the above equation for $\tilde{v}_0$ one has

$$\tilde{v}_0 = \frac{1}{1+\tilde{m}}\left[\tilde{m} - \sqrt{\tilde{m}^2 + \frac{2}{\eta}\left(1+\tilde{m}\right)\left[\tilde{x} + \left(1+\tilde{m}\right)\ln\left(1-\tilde{x}\right)\right]}\right]. \tag{35}$$

Substituting the expression for $\tilde{v}_0$ into Eq. (30) generates

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{t}} = \frac{1}{\tilde{m}+\tilde{x}}\sqrt{\tilde{m}^2 + \frac{2}{\eta}\left(1+\tilde{m}\right)\left[\tilde{x} + \left(1+\tilde{m}\right)\ln\left(1-\tilde{x}\right)\right]}. \tag{36}$$

The above equation gives the time evolution of the length of the crushed material.

When $\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{t}} = 0$, shock wave disappears since $\tilde{v}_\mathrm{d} = \tilde{v}_0$. The non-dimensional length of crushed foam at this moment is denoted as $\tilde{x}_\mathrm{end}$ and called "shocks-vanish length-coefficient". The shock-vanish length-coefficient $\tilde{x}_\mathrm{end}$ can be determined as the root of the following equation

$$F(\tilde{x}) = \tilde{x} + \left(1+\tilde{m}\right)\ln\left(1-\tilde{x}\right) + \frac{\eta\tilde{m}^2}{2\left(1+\tilde{m}\right)} = 0. \tag{37}$$

For the function $F(\tilde{x})$, the derivative is $\frac{\mathrm{d}F}{\mathrm{d}\tilde{x}} = -\frac{\tilde{x}+\tilde{m}}{1-\tilde{x}}$. In the range $(0,1)$, $F(0) > 0$, $F(1) < 0$ and $\frac{\mathrm{d}F}{\mathrm{d}\tilde{x}} < 0$. So there is a root $\tilde{x}_\mathrm{end}$ for $F(\tilde{x}_\mathrm{end}) = 0$, which correspond to the $\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{t}} = 0$, i.e., the end for the shock wave. The shock wave disappears after the length condensed part of the cellar rod reaches $\tilde{x}_\mathrm{end}\left(1-\varepsilon_\mathrm{D}\right)L$. Figure 10 shows the relation between $\tilde{x}_\mathrm{end}$ and η for several mass coefficients $\tilde{m}$.

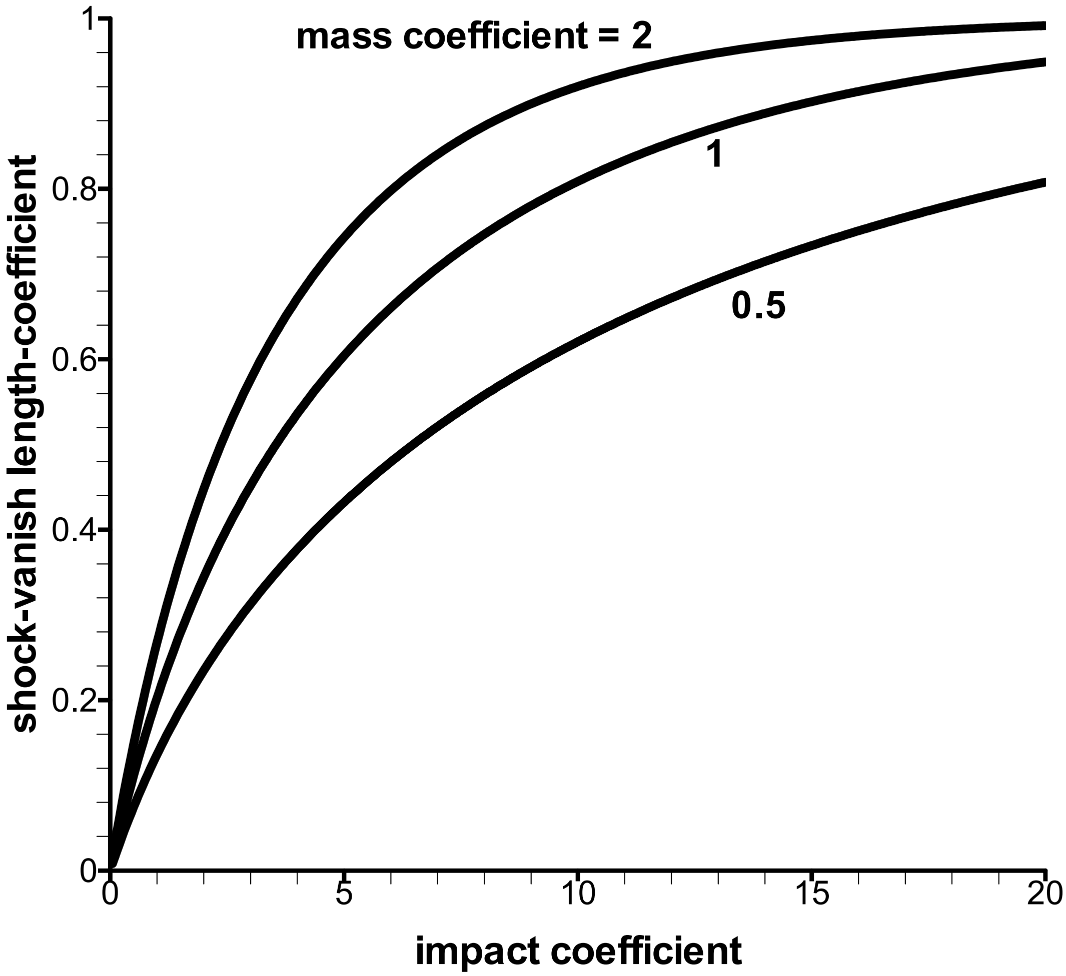


Figure 10. The non-dimensional length of crushed foam at the moment when the shock wave disappears $\tilde{x}_{\mathrm{end}}$ versus impact coefficient η. The curves are drawn for several mass coefficients $\tilde{m}$.

Figure 11 shows the time evolution of $\tilde{x}$ for several mass coefficients ($\tilde{m}$) and impact coefficients ($\tilde{\sigma}_{\mathrm{pl}}$).

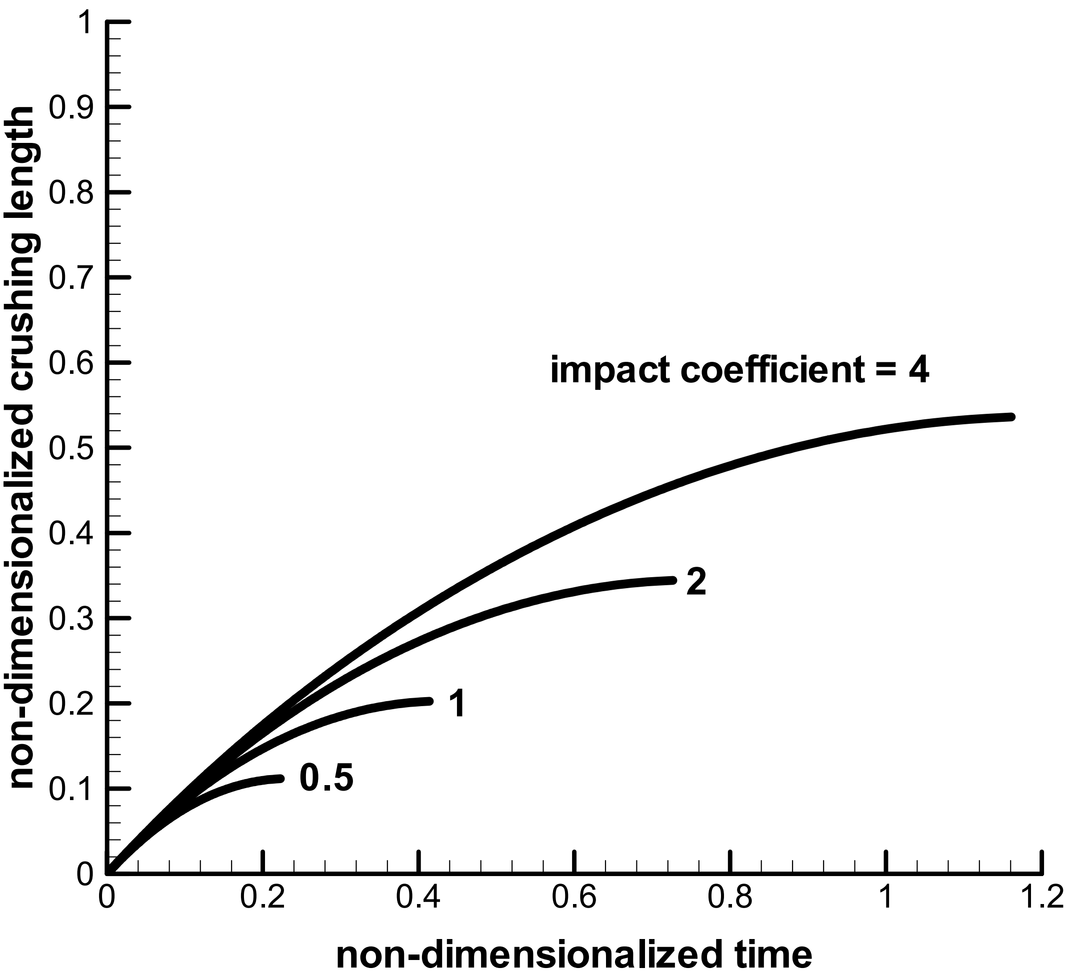


Figure 11. Nondimensionalized length of the crushed cellular material, $\tilde{x} = \frac{x}{(1-\varepsilon_D)L}$, as a function of nondimensionalized time, $\tilde{t} = \frac{v_{im}}{L\varepsilon_D} t$, for several impact coefficients. The mass coefficients is set at $\tilde{m} = 1$.

3.1.1 Pressure on the mass

The pressure on the mass can be calculated as

$$p = m \frac{dv_d}{dt}. \tag{38}$$

The non-dimensional form can be written as

$$\frac{p}{\sigma_{\rm pl}} = \frac{\tilde{m}}{\tilde{\sigma}_{\rm pl}} \frac{{\rm d}\tilde{v}_{\rm d}}{{\rm d}\tilde{t}}, \tag{39}$$

Since $\tilde{v}_{\rm d} = \frac{{\rm d}\tilde{x}}{{\rm d}\tilde{t}} + \tilde{v}_0$, one has

$$\frac{p}{\sigma_{\rm pl}} = -\frac{\tilde{m}}{\tilde{m}+\tilde{x}} \left\{ 2 + \eta \frac{\tilde{m}^2 + 2\tilde{\sigma}_{\rm pl}\left(1+\tilde{m}\right)\left[\tilde{x} + \left(1+\tilde{m}\right)\ln\left(1-\tilde{x}\right)\right]}{\left(\tilde{m}+\tilde{x}\right)^2} \right\}, \tag{40}$$

3.1.2 Stress behind the shock front

During the time period Δt, the impact acting on the mass element ΔX in Figure 9 is $\left(\sigma_{\rm d} - \sigma_{\rm pl}\right)\Delta t$ to the right, which equals the momentum change. Therefore

$$\left(\sigma_{\rm d} - \sigma_{\rm pl}\right)\Delta t = \rho_0 \Delta x \left(v_{\rm d} - v_0\right),$$

which gives

$$\sigma_{\rm d} - \sigma_{\rm pl} = \rho_0 \left(v_{\rm s} - v_0\right)\left(v_{\rm d} - v_0\right),$$

which can be further written as

$$\sigma_{\rm d} = \sigma_{\rm pl} + \rho_0 \frac{\left(v_{\rm d} - v_0\right)^2}{\varepsilon_{\rm D}}. \tag{41}$$

Therefore, we have

$$\sigma_{\rm d} = \sigma_{\rm pl} + \rho_0 \frac{\left(v_{\rm d} - v_0\right)^2}{\varepsilon_{\rm D}}, \tag{42}$$

which in a non-dimensional form gives

$$\frac{\sigma_{\rm d}}{\sigma_{\rm pl}} = 1 + \eta\left(\tilde{v}_{\rm d} - \tilde{v}_0\right)^2, \tag{43}$$

which relates to the length of crushed foam as

$$\frac{\sigma_{\rm d}}{\sigma_{\rm pl}} = 1 + \eta\left(\frac{{\rm d}\tilde{x}}{{\rm d}\tilde{t}}\right)^2. \tag{44}$$

Thus,

$$\frac{\sigma_{\mathrm{d}}}{\sigma_{\mathrm{pl}}} = 1 + \frac{1}{\left(\tilde{m} + \tilde{x}\right)^2} \left\{ \eta \tilde{m}^2 + 2\left(1 + \tilde{m}\right)\left[\tilde{x} + \left(1 + \tilde{m}\right) \ln\left(1 - \tilde{x}\right) \right] \right\}. \tag{45}$$

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