Tilting equivalences: from hereditary algebras to symmetric groups.

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Motif.

Let R be a commutative ring. Let Q be a quiver, whose underlying graph is a tree. We reveal derived equivalences of increasing sophistication, between:

I. The path algebra RQ, and its Koszul dual.

II. The trivial extension algebra of RQ, and the trivial extension algebra of its Koszul dual.

III. The Schur algebra of RQ, and the Schur algebra of its Koszul dual.

IV. A double of the Schur algebra of RQ, and a double of the Schur algebra of its Koszul dual.

Let Q be a Dynkin quiver, of type A. We lift the derived equivalences of IV to equivalences between:

V. A deformation of the double of the Schur algebra of RQ, and a deformation of the double of the Schur algebra of its Koszul dual.

VI. A quotient of the deformation of the double of the Schur algebra of RQ, and a quotient of the deformation of the double of the Schur algebra of its Koszul dual.

Let p be a prime number, and $(K, \mathcal{O}, k)$ a p-modular system. Let Q be a Dynkin quiver, of type A_{p-1}. We conjecture that any block of a symmetric group over $\mathcal{O}$, is equivalent to a quotient of the deformation of the double of the Schur algebra of $\mathcal{O}Q$, and equivalent to a quotient of the deformation of the double of the Schur algebra of the Koszul dual of $\mathcal{O}Q$, as in VI.

History.

Let p be a prime number, and k a field of characteristic p. A conjecture of M. Broué states that every p-block of a finite group of abelian defect is derived equivalent to its Brauer correspondent [4]. This conjecture has been proved for symmetric groups, following a strategy developed by R. Rouquier, by assembly of the following sequence of equivalences:

$$D^b(b_{ab}) \longrightarrow D^b(b_{Rock}) \longrightarrow D^b(b_0 \wr \Sigma_w) \longrightarrow D^b(k(C_p \rtimes C_{p-1}) \wr \Sigma_w).$$

Here, b_{ab} denotes a block of a symmetric group of abelian defect, and weight w. All such blocks have equivalent derived categories, by a theorem of J. Chuang and R. Rouquier [7]. There is a family of distinguished blocks b_{Rock} of weight w, the Rock blocks. By a theorem of J. Chuang and R. Kessar [6], the Rock blocks of abelian defect, and weight w, are all Morita equivalent to the wreath product $b_0 \wr \Sigma_w$ of the principal block b_0 of the symmetric group algebra $k\Sigma_p$ with the symmetric group Σ_w on w letters. By a theorem of J. Rickard, b_0 is derived equivalent to the group algebra $kC_p \rtimes C_{p-1}$ of a semidirect product of cyclic groups [13]. Taking wreath products, we find that $b_0 \wr \Sigma_w$ and $k(C_p \rtimes C_{p-1}) \wr \Sigma_w$ also have equivalent derived categories.

Since the Brauer correspondent of b_{ab} is Morita equivalent to the wreath product $k(C_p \rtimes C_{p-1}) \wr \Sigma_w$, the above sequence of equivalences implies the truth of Broué's abelian defect group conjecture for symmetric groups.

For blocks of nonabelian defect, there is no obvious generalisation of Broué's conjecture. However, it has become apperent that for symmetric groups, a subtle generalisation of the sequence discussed above should hold in arbitrary defect. Chuang and Rouquier's theory applies equally well in nonabelian defect. In the article "Rock blocks", we overturned a conjectural analogue of the Chuang-Kessar equivalence [18], thus suggesting a sequence of equivalences:

$$D^b(b) \longrightarrow D^b(b_{Rock}) \dashrightarrow D^b(\mathcal{D}_Q(w, w)).$$

Here, b denotes a block of a symmetric group, of weight w, and arbitrary defect. The Rock blocks are no longer Morita equivalent to wreath products in nonabelian defect, but there is considerable evidence that they are Morita equivalent to a family of finite dimensional algebras $\mathcal{D}_Q(w, w)$, the Schiver doubles.

The Schiver doubles are defined via a double construction applied to bialgebras of functions on certain quadratic super-algebras $P_Q(n)$. Here, Q is a quiver, obtained by giving an orientation to the Dynkin graph A_{p-1}.

In this paper, we develop further the theory of Schur algebras of quiver algebras, their doubles, and their deformations [19]. One consequence of our work is the existence of a new family of doubles $\mathcal{E}_{Q^{op}}(w,w)$, which are derived equivalent to $\mathcal{D}_Q(w,w)$. Our sequence of equivalences thus extends as follows:

$$D^b(b) \longrightarrow D^b(b_{Rock}) \dashrightarrow D^b(\mathcal{D}_Q(w,w)) \longrightarrow D^b(\mathcal{E}_{Q^{op}}(w,w)).$$

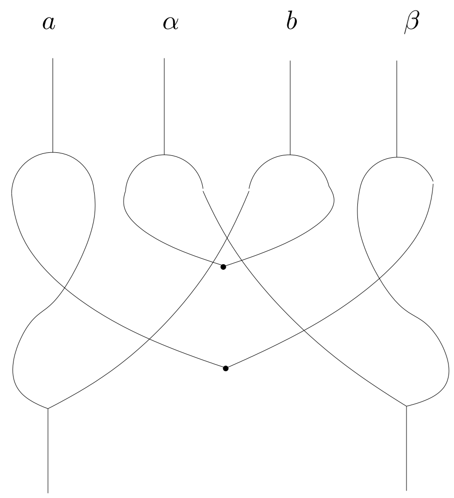
In the special case that Q is a linear orientation of A_{p-1}, and $w < p$, the algebra $\mathcal{E}_Q(w,w)$ is Morita equivalent to $k(C_p \rtimes C_{p-1}) \wr \Sigma_w$, and we recover the sequence of equivalences in abelian defect.

A further novelty of the present article is the consideration of algebras which we expect to describe symmetric group blocks over complete discrete valuation rings, such as the p-adic integers, rather than merely fields of positive characteristic. Such algebras arise as quotients of non-trivial deformations of doubles. Indeed, a suitable deformation $\underline{\mathcal{D}}_Q$ of the double $\mathcal{D}_Q$ can be constructed via a homological duality with the Schur algebra of a preprojective algebra. A deformation of the double $\mathcal{E}_{Q^{op}}$ can then be defined to be the endomorphism ring of certain tilting complex for $\underline{\mathcal{D}}_Q$.

Memories.

Let R be a commutative ring. Unless otherwise stated, all algebras and modules will be defined over R, and free over R. Given R-modules M, N, we write $M \otimes N$ for $M \otimes_R N$. We assume R-modules can be written as a direct sum $M = \bigoplus_{i \in I} M^i$ of R-modules of finite rank. We then write $M^* = \bigoplus_{i \in I} Hom_R(M^i, R)$ for the dual of M.

Let B be a super-bialgebra over R, with dual B^*. The double $D(B) = B \otimes B^*$ attains the structure of a symmetric associative algebra, whose product is described by the following picture (see [20]):



Let A be a finite dimensional super-algebra over R. Let $A(n) = End_A(A^{\oplus n})$. Let

$$\mathcal{S}(A)(n) = \bigoplus_{r \geq 0} \mathcal{S}(A)(n,r),$$

be the Schur super-bialgebra associated to A, where

$$\mathcal{S}(A)(n,r) = (A(n)^{\otimes r})^{\Sigma_r}.$$

Let us write $\mathcal{A}(A)(n)$ for the graded dual of $\mathcal{S}(A)(n)$. The super-bialgebra $\mathcal{A}(A)(n)$ can be thought of as the ring of regular functions on $A(n)$.

The double

$$\mathcal{D}(A)(n) = D(\mathcal{S}(A)(n)) = \mathcal{S}(A)(n) \otimes \mathcal{A}(A)(n),$$

decomposes as a direct sum

$$\mathcal{D}(A)(n) = \bigoplus_{r \geq 0} \mathcal{D}(A)(n,r)$$

of finite dimensional algebras,

$$\mathcal{D}(A)(n,r) = \bigoplus_{r_1 + r_2 = r} \mathcal{S}(A)(n,r_1) \otimes \mathcal{A}(A)(n,r_2).$$

If $\mathcal{C}$ is an abelian category, we write $C(\mathcal{C})$ (resp. $K(\mathcal{C}), D(\mathcal{C})$) for the corresponding category of chain complexes (resp. homotopy category, derived category). We write $X[n]$ for the translation of a chain complex X by n degrees.

Equivalences I. Hereditary algebras.

Let Q be a finite quiver, whose set of vertices is $V = V(Q)$. Let RQ be the path algebra of Q. We write e_v for the idempotent in RQ corresponding to vertex v. Let $P_Q = RP_Q$ be the quiver algebra of Q, modulo the ideal of paths of length ≥ 2.

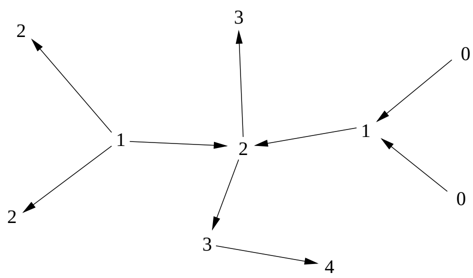
For the length of this section, we assume that the underlying graph of Q is a finite, connected tree. That is to say, Q has finitely many vertices and edges, and contains no circuits. We prove the derived equivalence of P_Q, RQ^{op}.

Lemma 1 *There is a unique map $\eta : V \to \mathbb{Z}_{\geq 0}$ such that,*

(i) $\eta(v) = \eta(w) + 1$, whenever there is an arrow from v to w in Q, for $v, w \in V$.

(ii) $0 \in im(\eta)$. $\square$

Here is an example of such a map η:



Let

$$\mathcal{J}_Q = P_Q \bigotimes_{RV} RQ^{op*}$$

be the Koszul complex for P_Q. Thus, $\mathcal{J}_Q$ is a differential P_Q-RQ^{op}-bimodule, equipped with a homological grading,

$$\mathcal{J}_Q = \bigoplus_{i \geq 0} \mathcal{J}_Q^i,$$

$$\mathcal{J}_Q^i = \bigoplus_{v,w \in V, \eta(v)-\eta(w)=i} \left(P_Q \bigotimes_{RV} e_v RQ^{op*} e_w \right).$$

The grading defines the structure a complex of P_Q-modules on $\mathcal{J}_Q$. This complex defines a projective resolution of the module ${}_{P_Q}RV$, concentrated in degree zero.

In the special situation we are studying, we can shift the grading by η.

Definition 2 *Let $\mathcal{K}_Q$ be the complex of P_Q-RQ^{op}-bimodules,*

$$\mathcal{K}_Q = P_Q \bigotimes_{RV} RQ^{op*},$$

with Koszul differential, and homological grading

$$\mathcal{K}_Q = \bigoplus_{i \geq 0} \mathcal{K}_Q^i,$$

$$\mathcal{K}_Q^i = \bigoplus_{v \in V, \eta(v) = i} \left(P_Q \bigotimes_{RV} e_v RQ^{op*} \right).$$

We have

$$\mathcal{K}_Q \cong \bigoplus_{w \in V} \mathcal{J}_Q e_w[\eta(w)],$$

as complexes of P_Q-modules.

Theorem 3 *$\mathcal{K}_Q$ is a tilting complex of P_Q-RQ^{op}-bimodules. There is a derived equivalence,*

$$D^b(P_Q - mod) \cong D^b(RQ^{op} - mod).$$

This theorem is a consequence of the following more general result.

Theorem 4 *Let A, C be $\mathbb{Z}_+$-graded finite dimensional algebras over R, such that*

$$C \cong Ext_A^*(A^0, A^0).$$

Let $\{\xi_y, y \in \mathcal{Y}\}$ be a collection of orthogonal idempotents in $A^0 \cong C^0$, such that $\sum_{y \in \mathcal{Y}} \xi_y = 1$. Suppose there exists a function,

$$\zeta : \mathcal{Y} \to \mathbb{Z},$$

such that $\xi_y C \xi_z \subseteq C^{\zeta(y) - \zeta(z)}$, for all $y, z \in \mathcal{Y}$. If R is a field, then A has finite global dimension. Furthermore, any perfect complex which is quasi-isomorphic to

$$T = \bigoplus_{y \in \mathcal{Y}} A^0 \xi_y[\zeta(y)]$$

is a tilting complex for A, whose endomorphism ring in the derived category is isomorphic to C. There is an equivalence of derived categories,

$$D^b(A - mod) \cong D^b(C - mod).$$

Proof:

Since A is finite dimensional, and positively graded, its positive part $\oplus_{i>0}A^i$ is a nilpotent ideal, and therefore simple A-modules can be identified with simple A^0-modules. For this reason, T is a generator for $D^b(A)$.

Because B is finite dimensional, $Ext^n_A(A^0, A^0) = 0$, for $n >> 0$. Therefore, A has finite global dimension, whenever R is a field.

Furthermore,

$$Hom_{D^b(A)}(T, T[n]) = \bigoplus_{y,z\in\mathcal{Y}} Hom_{D^b(A)}(A^0\xi_y[\zeta(y)], A^0\xi_z[\zeta(z)+n])$$

$$= \bigoplus_{y,z\in\mathcal{Y}} Ext^{\zeta(y)-\zeta(z)-n}(A^0\xi_y, A^0\xi_z)$$

$$= \xi_y C^{\zeta(y)-\zeta(z)-n}\xi_z = \begin{cases} \xi_y C\xi_z & \text{if } n=0 \\ 0 & \text{if } n\neq 0. \end{cases}$$

Therefore, T is a tilting complex, and $End_{D^b(A)}(T) \cong C$. By Rickard's Morita theory for derived categories, we have $D^b(A) \cong D^b(C)$. □

Proof of theorem 3:

Apply theorem 4, in case $A = P_Q, C = RQ^{op}, \zeta = \eta$. Indeed, $\mathcal{K}_Q$ is a perfect complex of P_Q-modules, which is quasi-isomorphic to

$$\bigoplus_{v\in V} Re_v[\eta(v)].\square$$

Equivalences II. Trivial extension algebras.

For the length of this section, we again assume that Q is a quiver, whose underlying graph is a finite, connected tree.

Let $T(A)$ be the trivial extension algebra of A. Thus, $T(A) \cong A \oplus A^*$, with multiplication given by

$$(a, \phi).(b, \psi) = (ab, a\psi + \phi b).$$

Note that $T(A)$ is a symmetric algebra, with symmetric associative, non-degenerate bilinear form

$$<(a, \phi), (b, \psi)> = <a, \psi> + <\phi, b>.$$

Example 5 Let Γ be a tree. Let $\bar{\Gamma}$ be the double quiver whose vertices are in one-one correspondence with vertices V of Γ, and whose arrows A are in two-one correspondence with the edges of Γ. Thus, an edge joining vertices v_1, v_2 in Γ corresponds to two arrows in $\bar{\Gamma}$, one pointing from v_1 to v_2, the other pointing from v_2 to v_1.

Let Γ have more than one edge. Let v be a vertex of Γ attached to two edges α, β. Let the corresponding arrows in $\bar{\Gamma}$ pointing towards v be labelled α_1, β_1. Let the corresponding arrows pointing away from v be labelled α_2, β_2. Let

$$\mathcal{R}_{\alpha,\beta,v} = \{\alpha_1\beta_2, \beta_1\alpha_2, \alpha_2\alpha_1 - \beta_2\beta_1\},$$

The *zigzag algebra* ZZ_Γ is defined to be the path algebra $R\bar{\Gamma}$, modulo the quadratic ideal generated by $\bigcup_{\alpha,\beta,v} \mathcal{R}_{\alpha,\beta,v}$.

Let Γ have one vertex, and no arrows. Then the zigzag algebra ZZ_Γ is defined to be $R[x]/x^2$.

Let Γ be a Dynkin graph of type A_1. Let the arrows of Γ be denoted α, β. Then the zizag algebra ZZ_Γ is defined to be the path algebra $R\bar{\Gamma}$, modulo the ideal generated by $\alpha\beta\alpha, \beta\alpha\beta$.

Lemma 6 *The trivial extension algebra of P_Q is isomorphic to the zigzag algebra ZZ_Γ.* $\square$

Let U_Q be the trivial extension algebra $T(P_Q)$ of P_Q. By lemma 6 above, U_Q is isomorphic to a zizag algebra, and independent of the orientation of Q. Let V_Q be the trivial extension algebra $T(RQ)$ of RQ.

It is a general result of Rickard that two derived equivalent algebras have equivalent trivial extension algebras [13]. In particular, $U_Q, V_{Q^{op}}$ have equivalent derived categories. As a warm-up to our proof of derived equivalences between doubles which appear later in the paper, let us re-prove the derived equivalence of $U_Q, V_{Q^{op}}$, with our notation.

Definition 7 *Let*

$$T_Q = U_Q \bigotimes_{P_Q} \mathcal{K}_Q,$$

$$E_Q = End_{U_Q-mod}(T_Q).$$

Note that $\mathcal{K}_Q$ is free as a P_Q-module, and so

$$T_Q \cong U_Q \bigotimes_{RV} RQ^{op*},$$

as U_Q-RQ^{op}-modules.

Because T_Q is a complex of U_Q-RQ^{op}-bimodules, E_Q is a dg algebra, and a complex of RQ^{op}-RQ^{op}-bimodules. By the adjunction

$$\left(U_Q \bigotimes_{RV} -, Hom_{U_Q}(U_Q, -)\right),$$

we have the following lemma.

Lemma 8

$$E_Q \cong RQ^{op} \bigotimes_{RV} U_Q \bigotimes_{RV} RQ^{op*},$$

as RQ^{op}-RQ^{op}-bimodules. □

The homological degree of $RQ^{op}e_v \bigotimes_{RV} U_Q \bigotimes_{RV} e_w RQ^{op*}$ is $\eta(w) - \eta(v)$.

Theorem 9 *(Rickard [13]) T_Q is a tilting complex for U_Q. Its endomorphism ring in the homotopy category is isomorphic to $V_{Q^{op}}$. There is a derived equivalence,*

$$D^b(U_Q - mod) \cong D^b(V_{Q^{op}} - mod).$$

Proof:

We show that E_Q has homology concentrated in degree zero, and that $H^0(E_Q)$ is isomorphic to $V_{Q^{op}}$.

There is a direct sum decomposition,

$$U_Q = P_Q \oplus P_Q^*,$$

of P_Q-P_Q bimodules. The differential on E_Q is given by,

$$d_{E_1}(qe_v \otimes x \otimes e_w r) =$$

$$(qe_v \otimes d_{T_1}(x \otimes e_w r)) - (-1)^{\eta(w)-\eta(v)}(d_{T_1^*}(qe_v \otimes x) \otimes e_w r).$$

Consequently, there is a direct sum decomposition,

$$E_Q \cong E_Q^l \oplus E_Q^r,$$

of complices of RQ^{op}-RQ^{op}-bimodules, where

$$E_Q^r = RQ^{op} \bigotimes_{RV} P_Q \bigotimes_{RV} RQ^{op*},$$

$$E_Q^l = RQ^{op} \bigotimes_{RV} P_Q^* \bigotimes_{RV} RQ^{op*}.$$

Indeed, we have isomorphisms,

$$E_Q^r \cong End_{P_Q}(\mathcal{K}_Q),$$
$$E_Q^l \cong \mathcal{K}_Q^* \bigotimes_{P_Q} \mathcal{K}_Q,$$

of complices of RQ^{op}-RQ^{op}-bimodules.

Note that E_Q^l is dual to E_Q^r,

$$E_Q^{l*} = Hom_R(\mathcal{K}_Q^* \bigotimes_{P_Q} \mathcal{K}_Q, R) \cong End_{P_Q}(\mathcal{K}_Q) \cong E_Q^r.$$

By Koszul duality, the map

$$RQ^{op} \to E_Q^r,$$

is a quasi-isomorphism of RQ^{op}-RQ^{op}-bimodules. Therefore, the dual map

$$E_Q^l \to RQ^{op*},$$

is also a quasi-isomorphism of RQ^{op}-RQ^{op}-bimodules.

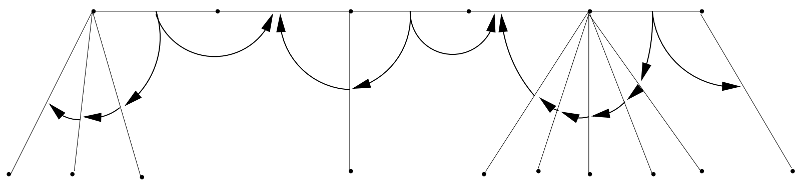
Since E_Q^r, E_Q^l have homology concentrated in degree zero, E_Q itself has homology concentrated in degree, and so T_Q is indeed a tilting complex, as required. By Rickard theory, there is a derived equivalence between U_Q, and $H^0(E_Q)$. However,

$$H^0(E_Q) \cong RQ^{op} \oplus RQ^{op*} \cong V_{Q^{op}},$$

as RQ^{op}-RQ^{op} bimodules. To complete the proof of the theorem, we need only verify that elements of the component RQ^{op*} multiply to zero in $H^0(E_Q)$. This happens to be so, because elements of the component E_Q^l of E_Q represent endomorphisms which map $U_Q \bigotimes_{RV} RQ^{op*}$ to $P_Q^* \bigotimes RQ^{op*}$. Therefore elements of E_Q^l compose to zero, because elements of the component P_Q^* multiply to zero in U_Q. □

Remark 10 When Q, Q' are orientations of the same graph Γ, we have isomorphisms $U_Q \cong ZZ_\Gamma \cong U_{Q'}$. We thus have derived equivalences between ZZ_Γ, V_Q and $V_{Q'}$. In particular, when Γ is a Dynkin graph of type A, we recover some of the equivalences between Brauer tree algebras, first observed by Rickard [13]. As we explain in a separate article, the Brauer trees are all caterpillars, with multiplicity one [20]. Beneath are some pictures, explaining how a caterpillar corresponds to an orientation of Γ.

Caterpillar:



Quiver:

Interlude I: Wreath products.

Let A be a unital super-algebra. Let $n \geq r$.

Definition 11 *Let s_r be the symmetrising map from $A(n)^{\otimes r}$ to $\mathcal{S}(A)(n,r)$,*

$$s_r : a_1 \otimes ... \otimes a_r \mapsto \sum_{\sigma \in \Sigma_r} a_{1^\sigma} \otimes ... \otimes a_{r^\sigma}.$$

Let t_r be the multiplication map from $A(n)^{*\otimes r}$ to $\mathcal{A}(n,r)$,

$$t_r : b_1 \otimes ... \otimes b_r \mapsto b_1 ... b_r.$$

Definition 12 *Let $\{\xi_{ij}, 1 \leq i,j \leq n\}$ be a basis of elementary matrices in $End_R(R^{\oplus n})$.*

Given a subset $J \subset \{1,...,n\}$, let

$$\xi_J = \sum_{\sigma \in \Sigma_J} (\xi_{1^\sigma 1^\sigma} \otimes ... \otimes \xi_{r^\sigma r^\sigma}),$$

an element of $\mathcal{S}_{\cdot}(n,r)$. Let $\xi_{n,r} = \xi_{\{1,...,r\}}$.

According to a primitive form of Schur-Weyl duality [10], $\xi_{n,r}$ is an idempotent in $\mathcal{S}_{\cdot}(n,r)$, such that

$$\xi_{n,r}\mathcal{S}_{\cdot}(n,r)\xi_{n,r} \cong \Sigma_r.$$

The unital embedding of $R = P_{\cdot}$ in A extends to a unital embedding of $P_{\cdot}$ in $A(n)$, and thus to a unital embedding of $\mathcal{S}_{\cdot}(n,r)$ in $\mathcal{S}(A)(n,r)$. Let us identify $\xi_{n,r}$ with its image in $\mathcal{S}(A)(n,r)$, under this embedding.

Lemma 13 *Let A be a super-algebra. Let $n \geq r$. There is an algebra isomorphism,*

$$A \wr \Sigma_r \cong \xi_{n,r}\mathcal{S}(A)(n,r)\xi_{n,r},$$

where the left hand side is the super wreath product of A with Σ_r. The functor,

$$Hom(\mathcal{S}(A)(n,r)\xi_{n,r}, -) : \mathcal{S}(A)(n,r) - mod \to A \wr \Sigma_r - mod,$$

is fully faithful on projective objects.

Proof:

Consider the sequence of natural isomorphisms of R-modules,

$$A \wr \Sigma_r = A^{\bigotimes r} \otimes R\Sigma_r \cong$$

$$\xi_{11} A(n) \xi_{11} \otimes ... \otimes \xi_{rr} A(n) \xi_{rr} \otimes R\Sigma_r \cong$$

$$\xi_{\{1,...,r\}} \mathcal{S}(A)(n,r) \xi_{\{1,...,r\}} = \xi_{n,r} \mathcal{S}(A)(n,r) \xi_{n,r},$$

The second isomorphism here is the map,

$$a_1 \otimes ... \otimes a_r \otimes \theta \mapsto s_r(a_1 \otimes ... \otimes a_r).s_r(\xi_{11^\theta} \otimes ... \otimes \xi_{rr^\theta}).$$

The composition of this sequence of isomorphisms of R-modules defines an algebra isomorphism,

$$A \wr \Sigma_r \cong \xi_{n,r} \mathcal{S}(A)(n,r) \xi_{n,r}.$$

To complete the proof of the lemma, we are required to observe that

$$Hom(\mathcal{S}(A)(n,r)\xi_{n,r}, -) : \mathcal{S}(A)(n,r) - mod \to A \wr \Sigma_r - mod,$$

is fully faithful on projective objects. However, $\mathcal{S}(A)(n,r)\xi_{n,r}$ is isomorphic to $A(n)^{\otimes r}$, as an $A \wr \Sigma_r$-module, and therefore

$$\mathcal{S}(A)(n,r) = End_{A \wr \Sigma_r}(\mathcal{S}(A)(n,r)\xi_{n,r}),$$

by definition. □

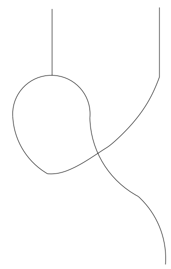
Given a super-algebra A, let $B_A = R \oplus A$ be the super-bialgebra, which is a direct sum of R and A as algebras, with coproduct

$$\Delta(\lambda, a) = (1,0) \otimes (0,a) + (0,a) \otimes (1,0) + (\lambda, 0) \otimes (1,0).$$

Let $B_A(r) = B_A^{\otimes r}$ be the r-fold tensor product of B_A, a super-bialgebra.

The trivial extension algebra $T(A)$ is a super-algebra, with $\mathbb{Z}/2$-grading inherited from that on A. Note the actions of A on A^* involve the introduction of signs. For example, the right action of A on A^* is pictured in the diagram,

$$A^* \qquad A$$



Lemma 14 *The double $D(B_A(r))$ has a component $C_A(r)$, which is isomorphic to the tensor product $T(A)^{\otimes r}$ of r trivial extension super-algebras $T(A)$.*

Proof:

The degree r part of $D(B_A(r))$ has a component,

$$C_A(r) = \bigoplus_{r_1+r_2=r} \left(\bigoplus_{\sigma \in \Sigma_r/\Sigma_{r_1}\times\Sigma_{r_2}} \left((A^{\otimes r_1} \otimes R^{\otimes r_2}) \otimes (R^{*\otimes r_1} \otimes A^{*\otimes r_2})\right)^{\sigma} \right),$$

which is naturally isomorphic to

$$T(A)^{\otimes r} = \bigoplus_{r_1+r_2=r} \left(\bigoplus_{\sigma \in \Sigma_r/\Sigma_{r_1}\times\Sigma_{r_2}} \left(A^{\otimes r_1} \otimes A^{*\otimes r_2}\right)^{\sigma} \right),$$

as an R-module. This R-module isomorphism is in fact an algebra isomorphism. Multiplicativity is easy to check, given there is a unique natural way to put an algebra structure on this super-space. Of course, one must be careful with the signs, and a little thought tells us that there is also a unique natural choice of sign convention. Note that it is enough to check multiplicativity on the subspace

$$A^{\otimes r} \oplus \left(\bigoplus_{\sigma \in \Sigma_r/\Sigma_{r-1}} \left(A^{\otimes r-1} \otimes A^{*}\right)^{\sigma} \right),$$

which generates $T(A)^{\otimes r}$. $\square$

Theorem 15 *Let A be a super-algebra. Let $n \geq r$ There is an algebra isomorphism,*

$$\Lambda : T(A) \wr \Sigma_r \cong \xi_{n,r}\mathcal{D}(A)(n,r)\xi_{n,r},$$

where the left hand side is the super wreath product of $T(A)$ with Σ_r.

Proof:

We have direct sum decompositions,

$$\xi_{n,r}\mathcal{D}(A)(n,r)\xi_{n,r} \cong \bigoplus_{r_1+r_2=r} \xi_{n,r}(\mathcal{S}(A)(n,r_1)\otimes\mathcal{A}(A)(n,r_2))\xi_{n,r}.$$

$$T(A)\wr\Sigma_r \cong \left(\bigoplus_{r_1+r_2=r}\bigoplus_{\sigma\in\Sigma_r/\Sigma_{r_1}\times\Sigma_{r_2}} (A^{\otimes r_1}\otimes A^{*\otimes r_2})^{\sigma}\right)\otimes R\Sigma_r.$$

Here, we are careful to employ the super sign convention, when conjugating by a permutation σ.

We proceed to write down an explicit isomorphism Λ_{r_1,r_2} between the $(r_1,r_2)^{th}$ components of the above decompositions. Indeed, Λ_{r_1,r_2} is defined to be the composition of the sequence of natural isomorphisms,

$$\left(\bigoplus_{\sigma\in\Sigma_r/\Sigma_{r_1}\times\Sigma_{r_2}} (A^{\otimes r_1}\otimes A^{*\otimes r_2})^{\sigma}\right)\otimes R\Sigma_r \cong$$

$$\left(\bigoplus_{\sigma\in\Sigma_r/\Sigma_{r_1}\times\Sigma_{r_2}} (\bigotimes_{i=1}^{r_1}\xi_{i^\sigma i^\sigma}A(n)\xi_{i^\sigma i^\sigma})\otimes(\bigotimes_{i=r_1+1}^{r}\xi_{i^\sigma i^\sigma}A(n)^*\xi_{i^\sigma i^\sigma})\right)\otimes R\Sigma_r \cong$$

$$\bigoplus_{\sigma,\tau\in\Sigma_r/\Sigma_{r_1}\times\Sigma_{r_2}} (\xi_{\{1^\sigma,\dots,r_1^\sigma\}}\mathcal{S}(A)(n,r_1)\xi_{\{1^\tau,\dots,r_1^\tau\}}\otimes$$

$$\xi_{\{(r_1+1)^\sigma,\dots,r^\sigma\}}\mathcal{A}(A)(n,r_1)\xi_{\{(r_1+1)^\tau,\dots,r^\tau\}}) =$$

$$\xi_{\{1,\dots,r\}}\left(\mathcal{S}(A)(n,r_1)\otimes\mathcal{A}(A)(n,r_2)\right)\xi_{\{1,\dots,r\}} =$$

$$\xi_{n,r}(\mathcal{S}(A)(n,r_1)\otimes\mathcal{A}(A)(n,r_2))\xi_{n,r}.$$

The second isomorphism here is the map,

$$(a_1\otimes\dots\otimes a_{r_1})\otimes(b_1\otimes\dots\otimes b_{r_2})\otimes\theta\mapsto$$

$$s_{r_1}(a_1\otimes\dots\otimes a_{r_1})\otimes t_{r_2}(b_1\otimes\dots\otimes b_{r_2}).s_r(\xi_{11^\theta}\otimes\dots\otimes\xi_{rr^\theta}).$$

By summing our isomorphisms, we obtain an isomorphism of R-modules.

$$\Lambda : T(A)\wr\Sigma_r \cong \xi_{n,r}\mathcal{D}(A)(n,r)\xi_{n,r},$$

To complete the proof of the theorem, we prove that Λ is an algebra isomorphism. It is enough to check,

(i) Λ restricted to $R\Sigma_r$ is an isomorphism,
(ii) Λ restricted to $T(A)^{\otimes r}$ is an isomorphism,

(iii) $\Lambda(\theta x) = \Lambda(\theta)\Lambda(x)$, for $x \in T(A)^{\otimes r}, \theta \in \Sigma_r$,
(iv) $\Lambda(x\theta) = \Lambda(x)\Lambda(\theta)$, for $x \in T(A)^{\otimes r}, \theta \in \Sigma_r$.

Statements (i), (iii), (iv) follow from the basic observation,

$$\Lambda(\theta) = s(\xi_{11^\theta} \otimes ... \otimes \xi_{rr^\theta}),$$

for $\theta \in \Sigma_r$. To see the truth of statement (ii), note that the image of Λ restricted to $T(A)^{\otimes r}$ can be naturally identified with the algebra $C_A(r)$ of lemma 14, and that the restriction of Λ to $T(A)^{\otimes r}$ can be identified with the algebra isomorphism of lemma 14. This completes the proof of the theorem. □

Corollary 16 *Suppose R is a field of characteristic zero, and $n \geq r$. The algebras $\mathcal{S}(A)(n,r)$ and $A \wr \Sigma_r$ are Morita equivalent. The algebras $\mathcal{D}(A)(n,r)$ and $T(A) \wr \Sigma_r$ are Morita equivalent.*

Proof:

Let us assume that R is a field, of characteristic zero. The summands of the right $\mathcal{S}_{.}(n,r)$-module $\mathcal{S}_{.}(n,r)$ can all be identified with summands of $\xi_{n,r}\mathcal{S}_{.}(n,r)$ Correspondingly, the indecomposable summands of the right $A \wr \Sigma_r$-module $\mathcal{S}(A)(n,r)\xi_{n,r}$, can be identified with summands of $\xi_{n,r}\mathcal{S}(A)(n,r)\xi_{n,r} \cong A \wr \Sigma_r$. Therefore, $\mathcal{S}(A)(n,r)\xi_{n,r}$ is a progenerator for $A \wr \Sigma_r$, which is Morita equivalent to the endomorphism ring $\mathcal{S}(A)(n,r)$.

The surjection from $T(A) \wr \Sigma_r$ to $A \wr \Sigma_r$ has nilpotent kernel, as does the surjection from $\mathcal{D}(A)(n,r)$ to $\mathcal{S}(A)(n,r)$. Therefore $T(A) \wr \Sigma_r$, $A \wr \Sigma_r$, $\mathcal{S}(A)(n,r)$, and $\mathcal{D}(A)(n,r)$ all have the same number of simple modules. Consequently, $\mathcal{D}(A)(n,r)\xi_{n,r}$ is a progenerator for $\mathcal{D}(A)(n,r)$, which is Morita equivalent to the endomorphism ring $T(A) \wr \Sigma_r$. □

Equivalences III. Schur algebras.

Given a quiver Q, let

$$RQ(n) = End_{RQ} RQ^{\oplus n},$$

$$RP_Q(n) = End_{RP_Q} RP_Q^{\oplus n}.$$

Here, we think of $RQ(n)$ as an ordinary associative algebra, and $RP_Q(n)$ as a super-algebra, where paths of length i in Q have parity $i \in \mathbb{Z}/2$. We write

$$\mathcal{S}_Q(n) = \mathcal{S}(P_Q(n)),$$

$$\mathcal{T}_Q(n) = \mathcal{S}(RQ(n)),$$

$$\mathcal{S}_Q(n,r) = \mathcal{S}(P_Q(n))(r),$$

$$\mathcal{T}_Q(n,r) = \mathcal{S}(RQ(n))(r).$$

Let Q be a quiver, whose underlying graph is a finite, connected tree. Let $n \geq r$. In this section, we prove the derived equivalence of $\mathcal{S}_Q(n,r), \mathcal{T}_{Q^{op}}(n,r)$.

Let $P_Q \wr \Sigma_r$ be the wreath product of the super-algebra P_Q, with Σ_r. Let $RQ^{op} \wr \Sigma_r$ be the wreath product of the associative algebra RQ^{op}, with Σ_r. Wreathing the Koszul differential bimodule $P_Q \bigotimes_{RV} RQ^{op*}$ with Σ_r, we obtain a differential $P_Q \wr \Sigma_r$-$RQ^{op} \wr \Sigma_r$-bimodule,

$$P_Q \wr \Sigma_r \bigotimes_{RV \wr \Sigma_r} RQ^{op*} \wr \Sigma_r,$$

which is isomorphic to

$$P_Q \wr \Sigma_r \bigotimes_{RV \wr \Sigma_r} RQ \wr \Sigma_r.$$

Applying $Hom_{P_Q \wr \Sigma_r}(-, (P_Q^{\oplus n})^{\otimes r})$ functorially on the left, and functorially applying $Hom_{RQ \wr \Sigma_r}((RQ^{\oplus n})^{\otimes r}, -)$ on the right, we obtain a differential $\mathcal{S}_Q(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodule,

$$(P_Q^{\oplus n})^{\otimes r} \bigotimes_{RV \wr \Sigma_r} (RQ^{op*\oplus n})^{\otimes r},$$

which is isomorphic to

$$(P_Q^{\oplus n})^{\otimes r} \bigotimes_{RV \wr \Sigma_r} (RQ^{\oplus n})^{\otimes r} \cong$$

$$\mathcal{S}_Q(n,r)\xi_r \bigotimes_{RV \wr \Sigma_r} \xi_r \mathcal{T}_Q(n,r),$$

as a complex of $\mathcal{S}_Q(n,r)$-$\mathcal{S}_V(n,r)$-bimodules. By lemma 13, this bimodule is isomorphic to a differential $\mathcal{S}_Q(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodule,

$$\mathcal{J}_Q(n,r) = \mathcal{S}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r).$$

The differential bimodule $\mathcal{J}_Q(n,r)$ inherits a homological grading from the homological grading on $\mathcal{J}_Q \wr \Sigma_r$. In this way, $\mathcal{J}_Q(n,r)$ is a complex of $\mathcal{S}_Q(n,r)$-$\mathcal{S}_V(n,r)$-bimodules.

Definition 17 *Given a subquiver O of Q, let f_Q be the unit of the subalgebra $\mathcal{S}_{V(O)}(n)$ of $\mathcal{S}_{V(Q)}(n)$. In particular, if $v \in V(Q)$, let f_v be the unit of the subalgebra $\mathcal{S}_v(n)$ of $\mathcal{S}_{V(Q)}(n)$*

For the rest of this section, we assume Q is a quiver, whose underlying graph is a finite, connected tree.

Proposition 18 *The complex $\mathcal{J}_Q(n,r)$ of $\mathcal{S}_Q(n,r)$-$\mathcal{S}_V(n,r)$-bimodules defines a projective resolution of the bimodule $\mathcal{S}_V(n,r)$,*

$$\mathcal{S}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r) \twoheadrightarrow \mathcal{S}_V(n,r).$$

Proof:

We proceed by induction on the number of vertices of Q. If Q has no edges, the proposition is obvious. Otherwise, assume that Q' is a finite quiver whose underlying graph is a finite connected tree, and assume the proposition is known to be true for all such quivers with fewer edges than Q'. We demonstrate the truth of the proposition for the quiver Q'.

In Q', there exists a vertex v with no arrows pointing into v. Let Q be the quiver obtained by removing v, and all arrows connected to v, from Q'. By assumption, the proposition is true for the quiver Q.

Let $V = V(Q), V' = V(Q')$. We wish to show that $\mathcal{J}_{Q'}(n,r)$ defines a resolution of $\mathcal{S}_{V'}(n,r)$. It is enough for us to show that $\mathcal{J}_{Q'}(n,r)f_x$ defines a resolution of $\mathcal{S}_x(n,r)$, for $x \in V'$, because the inductive hypothesis then tells us that

$$\mathcal{J}_{Q'}(n,r) \cong \mathcal{J}_{Q'}(n,r)(f_x \otimes f_Q) \cong \bigoplus_{j \geq 0} \mathcal{J}_{Q'}(n,j)f_x \otimes \mathcal{J}_Q(n,r-j)$$

defines a resolution of $\mathcal{S}_{V'}(n,r) \cong \bigoplus_{j \geq 0} \mathcal{S}_x(n,j) \otimes \mathcal{S}_V(n,r-j)$, for all j.

There are now three cases to consider:

(i) $x = v$
(ii) $x \in V$, and there is no path from v to x in Q.
(iii) $x \in V$, and there is some path from v to x in Q.

Case (i) is easy to face down: $\mathcal{S}_v(n,r) \cong \mathcal{J}_{Q'}(n,r)f_v$ is a projective $\mathcal{S}_{Q'}(n,r)$-module.

Case (ii) is similarly elementary: $\mathcal{J}_{Q'}(n,r)f_x \cong \mathcal{J}_Q(n,r)f_x$, because there is no path from x to v in Q^{op}. However, $\mathcal{J}_Q(n,r)f_x$ is known to define a projective resolution of $\mathcal{S}_x(n,r)$, by the inductive hypothesis.

Case (iii). Note that the path from v to x in Q' is necessarily unique, because the underlying graph of Q' is a tree. Let us write this path as a composition $a.p$, where a is the arrow at the beginning of the path whose source is v, and p is a path in Q.

We have,

$$P_{Q'} = P_Q \oplus Ra \oplus Rv,$$

$$\mathcal{J}_{Q'}x = Rv \to C,$$

where

$$C = P_{Q'} \bigotimes_{RV} \mathcal{J}_Q f_x \cong \mathcal{J}_Q \oplus (Ra \otimes p).$$

In this way, we have a direct sum decomposition of complexes,

$$\mathcal{J}_{Q'} f_x \cong (Rv \to Ra) \bigoplus \mathcal{J}_Q f_x.$$

The component $(Rv \to Ra)$ is acyclic, whilst the component $\mathcal{J}_Q f_x$ is quasi-isomorphic to Rx.

Analogously,

$$\mathcal{J}_{Q'}(n,r)f_x =$$

$$\mathcal{S}_v(n,r) \to \mathcal{S}_v(n,r-1)\otimes C_1 \to \mathcal{S}_v(n,r-2)\otimes C_2 \to \ldots \to \mathcal{S}_v(n,1)\otimes C_{r-1} \to C_r,$$

where

$$C_j = \mathcal{S}_{Q'}(n,j) \bigotimes_{\mathcal{S}_V(n,j)} \mathcal{J}_Q(n,j)f_x$$

$$\cong \bigoplus_{i=0}^{j} \mathcal{J}_Q(n,i)f_x \otimes \left(\bigvee_a (n,j-i) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{S}_p(n,r) \right).$$

Here, $\bigvee_a(n)$ is our notation for the fixed points of Σ_r on the super tensor product $(End_R(Ra^{\oplus n}))^{\otimes r}$. In this way, we have a direct sum decomposition of complices,

$$\mathcal{J}_{Q'}(n,r)f_x \cong \bigoplus_{i=0}^{r} W_i,$$

where

$$W_0 = \mathcal{J}_Q(n,r)f_x,$$

and

$$W_i \cong$$

$$\left(\mathcal{S}_v(n,i) \to \mathcal{S}_v(n,i-1) \otimes \bigvee_a(n,1) \to ... \to \bigvee_a(n,i)\right) \otimes \mathcal{J}_Q(n,r-i)f_x,$$

for $i > 0$.

Whilst $i > 0$, the complex W_i can be thought of as a tensor product of the Koszul complex

$$\left((M_n(Rv) \to M_n(Ra))^{\otimes i}\right)^{\Sigma_i} \cong$$

$$\mathcal{S}_v(n,i) \to \mathcal{S}_v(n,i-1) \otimes \bigvee_a(n,1) \to ... \to \bigvee_a(n,i),$$

for the space $M_n(Ra) = End_R(Ra^{\oplus n})$, with $\mathcal{J}_Q(n,r-i)$. The Koszul complex is acyclic, and thus W_i is acyclic, for $i > 0$.

Therefore, $\mathcal{J}_{Q'}(n,r)f_x$ is quasi-isomorphic to $W_0 = \mathcal{J}_Q(n,r)f_x$, which defines a resolution of $\mathcal{S}_x(n,r)$, by the induction hypothesis. This completes the proof of the proposition. □

Corollary 19

$$Ext^*_{\mathcal{S}_Q(n,r)}(\mathcal{S}_V(n,r), \mathcal{S}_V(n,r)) \cong \mathcal{T}_{Q^{op}}(n,r). \square$$

Let us write $\mathcal{I}$ for the set of V-tuples $\underline{i} = (i_v)_{v\in V}$ of elements $i_v \in \mathbb{Z}_{\geq 0}$, such that $\sum_{v\in V} i_v = r$.

We have a direct sum decomposition of algebras,

$$\mathcal{S}_V(n,r) \cong \bigoplus_{\underline{i}\in\mathcal{I}, \sum_{v\in V} i_v = r} \left(\bigotimes_{v\in V} \mathcal{S}_.(n,i_v)\right).$$

We write $\xi_{\underline{i}}$ for the unit element of the component $\left(\bigotimes_{v\in V} \mathcal{S}_.(n,i_v)\right)$ of $\mathcal{S}_V(n,r)$.

By definition, the set V embeds in Q. Correspondingly, $\mathcal{S}_V(n,r)$ embeds as a unital subalgebra in $\mathcal{S}_Q(n,r)$. We may therefore think of $\xi_{\underline{i}}$ as an element of $\mathcal{S}_Q(n,r)$.

Definition 20 *Let $\mathcal{K}_Q(n,r)$ be the complex of $\mathcal{S}_Q(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodules,*

$$\mathcal{K}_Q(n,r) = \mathcal{S}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r),$$

with Koszul differential, and homological grading

$$\mathcal{K}_Q(n,r) = \bigoplus_{j\geq 0} \mathcal{K}^j_Q(n,r),$$

$$\mathcal{K}_Q^j(n,r) = \bigoplus_{\underline{i}\in\mathcal{I}, \sum_V \eta(v)i_v = j} \left(\mathcal{S}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \xi_{\underline{i}} \mathcal{T}_Q(n,r) \right).$$

We have

$$\mathcal{K}_Q(n,r) \cong \bigoplus_{\underline{i}\in\mathcal{I}} \mathcal{J}_Q(n,r)\xi_{\underline{i}}[\sum_V \eta(v)i_v],$$

as complices of $\mathcal{S}_Q(n,r)$-$\mathcal{S}_V(n,r)$-bimodules.

Theorem 21 *$\mathcal{K}_Q(n,r)$ is a tilting complex of $\mathcal{S}_Q(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodules. There is a derived equivalence,*

$$D^b(\mathcal{S}_Q(n,r) - mod) \cong D^b(\mathcal{T}_{Q^{op}}(n,r) - mod).$$

Proof:

This is an application of theorem 4. We put $A = \mathcal{S}_Q(n,r)$. We assume the grading on A is inherited from the grading on P_Q which places vertices in degree zero, and arrows in degree one. Thus, $A^0 = \mathcal{S}_V(n,r)$. We put $C = \mathcal{T}_{Q^{op}}(n,r)$, and $\mathcal{Y} = \mathcal{I}$. We define

$$\zeta : \mathcal{I} \to \mathbb{Z},$$
$$\underline{i} \mapsto \sum_V \eta(v)i_v,$$

and identify the $\xi_{\underline{i}}$ defined above with idempotents $\xi_y, y \in \mathcal{Y}$. The hypotheses of theorem 4 apply, and consequently the present theorem holds. □

Corollary 22 *The action of $\mathcal{T}_{Q^{op}}(n,r)$ on $\mathcal{K}_Q(n,r)$ defines a quasi-isomorphism of complices of $\mathcal{T}_{Q^{op}}(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodules,*

$$\Upsilon(n,r) : \mathcal{T}_{Q^{op}}(n,r) \to Hom_{\mathcal{S}_Q(n,r)-mod}(\mathcal{K}_Q(n,r), \mathcal{K}_Q(n,r)).$$

The dual map defines a quasi-isomorphism of complexes of $\mathcal{T}_{Q^{op}}(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodules,

$$\Upsilon(n,r)^* : \mathcal{K}_Q(n,r)^* \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{K}_Q(n,r) \to \mathcal{B}_{Q^{op}}(n,r).\square$$

Equivalences IV. Doubles.

Let Q be a quiver. We write $\mathcal{A}_Q(n)$ for the graded dual of $\mathcal{S}_Q(n)$, and $\mathcal{B}_Q(n)$ for the graded dual of $\mathcal{T}_Q(n)$.

We write $\mathcal{D}_Q(n)$ for $D(P_Q)(n)$. We write $\mathcal{E}_Q(n)$ for $D(RQ)(n)$. Thus,

$$\mathcal{D}_Q(n) = \mathcal{S}_Q(n) \otimes \mathcal{A}_Q(n),$$

$$\mathcal{E}_Q(n) = \mathcal{T}_Q(n) \otimes \mathcal{B}_Q(n).$$

We have algebra direct sum decompositions,

$$\mathcal{D}_Q(n) = \bigoplus_{r \geq 0} \mathcal{D}_Q(n, r),$$

$$\mathcal{E}_Q(n) = \bigoplus_{r \geq 0} \mathcal{E}_Q(n, r).$$

Let $n \geq r$. For the rest of this section, we again assume that Q be a quiver, whose underlying graph is a finite, connected tree. In this section, we prove the derived equivalence of $\mathcal{D}_Q(n, r), \mathcal{E}_{Q^{op}}(n, r)$.

Definition 23 *Let $T_Q(n, r)$ be the complex of $\mathcal{D}_Q(n, r)$-modules, given by*

$$T_Q(n, r) = \mathcal{D}_Q(n, r) \bigotimes_{\mathcal{S}_Q(r,r)} \mathcal{K}_Q(n, r).$$

Remark 24 Since $\mathcal{T}_Q(n, r)$ is a projective $\mathcal{S}_V(n, r)$-module, we have,

$$T_Q(n, r) \cong \mathcal{D}_Q(n, r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n, r),$$

as $\mathcal{D}_Q(n, r)$-$\mathcal{T}_{Q^{op}}(n, r)$-bimodules.

Taking the endomorphism ring of $T_Q(n, r)$ in the category of modules, we obtain a dg algebra,

$$E_Q(n, r) = End_{\mathcal{D}_Q(n,r)-mod}(T_Q(n, r)).$$

Lemma 25

$$E_Q(n, r) \cong \mathcal{T}_{Q^{op}}(n, r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{D}_Q(n, r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n, r),$$

as $\mathcal{T}_{Q^{op}}(n, r)$-$\mathcal{T}_{Q^{op}}(n, r)$-bimodules.

Proof:

$$E_Q(n, r) = End_{\mathcal{D}_Q(n,r)}(T_Q(n, r)) \cong$$

$$Hom_{\mathcal{D}_Q(n,r)} \left(\mathcal{D}_Q(n, r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n, r), \mathcal{D}_Q(n, r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n, r) \right) \cong$$

$$Hom_{\mathcal{S}_V(n,r)}\left(\mathcal{T}_Q(n,r), Hom_{\mathcal{D}_Q(n,r)}(\mathcal{D}_Q(n,r), \mathcal{D}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r))\right) \cong$$

$$Hom_{\mathcal{S}_V(n,r)}\left(\mathcal{T}_Q(n,r), \mathcal{D}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r)\right) \cong$$

$$\mathcal{T}_{Q^{op}}(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{D}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r).\square$$

Definition 26 *Let $r_1 + r_2 = r$. Let*

$$E_Q(n, r_1, r_2) = \mathcal{T}_{Q^{op}}(n,r) \bigotimes_{\mathcal{S}_V(n,r)} (\mathcal{S}_Q(n,r_1) \otimes \mathcal{A}_Q(n,r_2)) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r).$$

Lemma 27 *There is a direct sum decomposition of complexes of $\mathcal{T}_{Q^{op}}(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodules,*

$$E_Q(n,r) \cong \bigoplus_{r_1+r_2=r} E_Q(n, r_1, r_2).$$

Proof:

By definition,

$$\mathcal{D}_Q(n,r) = \bigoplus_{r_1+r_2=r} \mathcal{S}_Q(n,r_1) \otimes \mathcal{A}_Q(n,r_2)$$

as $\mathcal{S}_Q(n,r)$-$\mathcal{S}_Q(n,r)$-bimodules. Therefore,

$$E_Q(n,r) \cong \bigoplus_{r_1+r_2=r} E_Q(n, r_1, r_2),$$

as an $\mathcal{S}_Q(n,r)$-$\mathcal{S}_Q(n,r)$-bimodule. We need to check that the differential on $E_Q(n,r)$ honours this direct sum decomposition. It is enough to check this over $\mathbb{Z}$, and therefore over its field of fractions $\mathbb{Q}$. Note that $\mathcal{S}_V(n,r)$ is Morita equivalent to $\mathbb{Q}V \wr \Sigma_r$, over $\mathbb{Q}$. It is therefore enough to observe a corresponding decomposition for the complex $E_Q \wr \Sigma_r$ of $\mathbb{Q}V \wr \Sigma_r$-$\mathbb{Q}V \wr \Sigma_r$-bimodules. However, we know that

$$E_Q = E_Q^l \oplus E_Q^r,$$

as complexes, and therefore we have isomorphisms of complexes,

$$E_Q \wr \Sigma_r = E_Q^{\otimes r} \otimes \Sigma_r \cong$$

$$\bigoplus_{r_1+r_2=r} \left(\left(\bigoplus_{\sigma \in \Sigma_r/(\Sigma_{r_1} \times \Sigma_{r_2})} (E_Q^{l \otimes r_1} \otimes E_Q^{r \otimes r_2})^\sigma \right) \otimes \Sigma_r \right),$$

where the (r_1, r_2) summand corresponds to $\mathcal{S}_Q(n, r_1) \otimes \mathcal{A}_Q(n, r_2)$. This completes the proof of the lemma. $\square$

The following technical lemma, and its corollary, can be interpreted thus: when you tensor up Schur algebras, the resulting bimodule is the only thing it could possibly be.

Lemma 28 *Let $r \in \mathbb{Z}_{\geq 0}$. Let $r_i, s_i \in \mathbb{Z}_{\geq 0}$, for $i = 1, ..., k$, such that $\sum_i r_i = \sum_i s^i = r$. Then there is an isomorphism,*

$$\left(\bigotimes_{i=1}^{k} \mathcal{S}(n, r_i) \right) \bigotimes_{\mathcal{S}(n,r)} \left(\bigotimes_{i=1}^{k} \mathcal{S}(n, s^i) \right) \cong$$

$$\bigoplus_{\substack{t_j^i \in \mathbb{Z}_{\geq 0}, i,j=1,...,r, \\ \sum_i t_j^i = r_j, \sum_j t_j^i = s^i}} \left(\bigotimes_{i,j=1}^{k} \mathcal{S}(n, t_j^i) \right),$$

as $\left(\bigotimes_{i=1}^{k} \mathcal{S}(n, r_i)\right)$- $\left(\bigotimes_{i=1}^{k} \mathcal{S}(n, s^i)\right)$-bimodules. $\square$

Corollary 29

$$(\mathcal{S}_V(n, r_1) \otimes \mathcal{S}_V(n, r_2)) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n, r) \cong \mathcal{T}_Q(n, r_1) \otimes \mathcal{T}_Q(n, r_2),$$

as $\mathcal{S}_V(n, r)$-$\mathcal{T}_{Q^{op}}(n, r)$-bimodules.

$$\mathcal{T}_{Q^{op}}(n, r) \bigotimes_{\mathcal{S}_V(n,r)} (\mathcal{S}_V(n, r_1) \otimes \mathcal{S}_V(n, r_2)) \cong \mathcal{T}_{Q^{op}}(n, r_1) \otimes \mathcal{T}_{Q^{op}}(n, r_2),$$

as $\mathcal{T}_{Q^{op}}(n, r)$-$\mathcal{S}_V(n, r)$-bimodules. $\square$

Lemma 30 *There is an isomorphism of complexes of $\mathcal{T}_{Q^{op}}(n, r)$-$\mathcal{T}_{Q^{op}}(n, r)$-bimodules,*

$$E_Q(n, r_1, r_2) \cong$$

$$\left(End_{\mathcal{S}_Q(n,r_1)}(\mathcal{K}_Q(n, r_1))\right) \otimes \left(\mathcal{K}_Q(n, r_2)^* \bigotimes_{\mathcal{S}_Q(n,r_2)} \mathcal{K}_Q(n, r_2) \right),$$

where the lower expression is thought of as a tensor product of complexes,

$$Hom_{\mathcal{S}_Q(n,r_1)}(\mathcal{K}_Q(n, r_1), \mathcal{K}_Q(n, r_1)) \quad , \quad \mathcal{K}_Q(n, r_2)^* \bigotimes_{\mathcal{S}_Q(n,r_2)} \mathcal{K}_Q(n, r_2),$$

whose differentials are inherited from the differentials on $\mathcal{K}_Q(n, r_1), \mathcal{K}_Q(n, r_2)$.

Proof:

We have

$$E_Q(n,r_1,r_2) =$$

$$\mathcal{T}_{Q^{op}}(n,r) \bigotimes_{\mathcal{S}_V(n,r)} (\mathcal{S}_Q(n,r_1)\otimes\mathcal{A}_Q(n,r_2)) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r) \cong$$

$$\mathcal{T}_{Q^{op}}(n,r) \bigotimes_{\mathcal{S}_V(n,r)} (\mathcal{S}_V(n,r_1)\otimes\mathcal{S}_V(n,r_2)) \bigotimes_{\mathcal{S}_V(n,r_1)\otimes\mathcal{S}_V(n,r_2)} (\mathcal{S}_Q(n,r_1)\otimes\mathcal{A}_Q(n,r_2))$$

$$\bigotimes_{\mathcal{S}_V(n,r_1)\otimes\mathcal{S}_V(n,r_2)} (\mathcal{S}_V(n,r_1)\otimes\mathcal{S}_V(n,r_2)) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r) \cong$$

$$(\mathcal{T}_{Q^{op}}(n,r_1)\otimes\mathcal{T}_{Q^{op}}(n,r_2)) \bigotimes_{\mathcal{S}_V(n,r_1)\otimes\mathcal{S}_V(n,r_2)} (\mathcal{S}_Q(n,r_1)\otimes\mathcal{A}_Q(n,r_2))$$

$$\bigotimes_{\mathcal{S}_V(n,r_1)\otimes\mathcal{S}_V(n,r_2)} (\mathcal{T}_Q(n,r_1)\otimes\mathcal{T}_Q(n,r_2)) \cong$$

$$\left(\mathcal{T}_{Q^{op}}(n,r_1) \bigotimes_{\mathcal{S}_V(n,r_1)} \mathcal{S}_Q(n,r_1) \bigotimes_{\mathcal{S}_V(n,r_1)} \mathcal{T}_Q(n,r_1)\right) \otimes$$

$$\left(\mathcal{T}_{Q^{op}}(n,r_2) \bigotimes_{\mathcal{S}_V(n,r_2)} \mathcal{A}_Q(n,r_2) \bigotimes_{\mathcal{S}_V(n,r_2)} \mathcal{T}_Q(n,r_2)\right) \cong$$

$$\left(End_{\mathcal{S}_Q(n,r_1)-mod}(\mathcal{K}(n,r_1))\right) \otimes \left(\mathcal{K}(n,r_2)^* \bigotimes_{\mathcal{S}_Q(n,r_2)} \mathcal{K}(n,r_2)\right) .\square$$

Lemma 31 *The homology of $E_Q(n,r_1,r_2)$ is concentrated in degree zero. There is an isomorphism of $\mathcal{T}_{Q^{op}}(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodules,*

$$\Phi_Q(n,r_1,r_2) : \mathcal{T}_{Q^{op}}(n,r_1)\otimes\mathcal{B}_{Q^{op}}(n,r_2) \to H^0(E_Q(n,r_1,r_2)).$$

Proof:

The complexes

$$Hom_{\mathcal{S}_Q(n,r_1)}(\mathcal{K}_Q(n,r_1),\mathcal{K}_Q(n,r_1)) \quad , \quad \mathcal{K}_Q(n,r_2)^* \bigotimes_{\mathcal{S}_Q(n,r_2)} \mathcal{K}_Q(n,r_2),$$

both have homology concentrated in degree zero, and their zeroth homologies are isomorphic to

$$\mathcal{T}_{Q^{op}}(n,r_1) \quad , \quad \mathcal{B}_{Q^{op}}(n,r_2),$$

respectively. Lemma 30 implies the existence of $\Phi_Q(n,r_1,r_2)$. $\square$

Corollary 32 *$T_Q(n,r)$ is a tilting complex for $\mathcal{D}_Q(n,r)$. There is an isomorphism of $\mathcal{T}_{Q^{op}}(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodules,*

$$\Phi_Q(n,r) : \mathcal{E}_{Q^{op}}(n,r) \to H^0(E_Q(n,r)).$$

Proof:

By definition,

$$\mathcal{E}_{Q^{op}}(n,r) = \bigoplus_{r_1+r_2=r} \mathcal{T}_{Q^{op}}(n,r_1) \otimes \mathcal{B}_{Q^{op}}(n,r_2).$$

By lemma 27,

$$E_Q(n,r) \cong \bigoplus_{r_1+r_2=r} E_Q(n,r_1,r_2),$$

as complices of $\mathcal{T}_{Q^{op}}(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodules. By lemma 31, $E_Q(n,r_1,r_2)$ has homology concentrated in degree zero. Therefore, $E_Q(n,r)$ has homology concentrated in degree zero, and $T_Q(n,r)$ is a tilting complex. We define $\Phi_Q(n,r)$ to be the sum of isomorphisms

$$\Phi_Q(n,r_1,r_2) : \mathcal{T}_{Q^{op}}(r,r_1) \otimes \mathcal{B}_Q(r,r_2) \to H^0(E_Q(n,r_1,r_2)).$$

□

We wish to show that $\Phi_Q(n,r)$ is an algebra homomorphism. The following lemma allows us to reduce the pursuit of algebra homomorphisms from $\mathcal{E}_{Q^{op}}(n,r)$ to the pursuit of algebra homomorphisms from $V_{Q^{op}} \wr \Sigma_r$.

Lemma 33 *Let A be an algebra, containing a subalgebra S. Let $\xi \in S$ be an idempotent, such that A is generated by the subalgebras $\xi A \xi, S$. Suppose that*

$$\phi : A \to B$$

is a morphism of S-S-bimodules, and that

$$\phi_{\xi A \xi} : \xi A \xi \to \phi(\xi) B \phi(\xi)$$

is an algebra homomorphism. Then ϕ is an algebra homomorphism. □

If A is a super-algebra, we denote by $A \wr \Sigma_r$ the super-wreath product of A with Σ_r. Note that $A \wr \Sigma_r$ is not isomorphic to the wreath product of the associative algebra A with Σ_r.

Lemma 34 *Suppose that A, B are super-algebras over K, that X is a tilting complex of A-supermodules, and that*

$$\Psi : B \to End_{D^b(A-mod)}(X)$$

is an isomorphism of super-algebras. Then the map,

$$\Psi \wr \Sigma_r : B \wr \Sigma_r \to End_{D^b(A \wr \Sigma_r - mod)}(X \wr \Sigma_r)$$

is an isomorphism of super-algebras.

Proof:

$$\begin{aligned} B \wr \Sigma_r \to & Hom_{D^b(A \wr \Sigma_r)}(A \wr \Sigma_r \bigotimes_{A^{\otimes r}} X^{\otimes r}, A \wr \Sigma_r \bigotimes_{A^{\otimes r}} X^{\otimes r}) \\ & \cong Hom_{D^b(A^{\otimes r})}(X^{\otimes r}, A \wr \Sigma_r \bigotimes_{A^{\otimes r}} X^{\otimes r}) \\ & \cong Hom_{D^b(A^{\otimes r})}(X^{\otimes r}, X^{\otimes r} \otimes R\Sigma_r) \\ & \cong Hom_{D^b(A^{\otimes r})}(X^{\otimes r}, X^{\otimes r}) \otimes R\Sigma_r \\ & \cong B \wr \Sigma_r . \square \end{aligned}$$

Theorem 35 *There is an equivalence of derived categories,*

$$D^b(\mathcal{D}_Q(r,r) - mod) \cong D^b(\mathcal{E}_{Q^{op}}(n,r) - mod).$$

Proof:

We know that $T_Q(n,r)$ is a tilting complex for $\mathcal{D}_Q(n,r)$, and we have an isomorphism

$$\Phi_Q(n,r) : \mathcal{D}_Q(n,r) \cong End_{D^b(\mathcal{D}_Q(n,r))}(T_Q(n,r)).$$

So far, it is only clear that this is an isomorphism of $\mathcal{T}_{Q^{op}}(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodules. To prove the theorem, we ought to show that Φ is an algebra homomorphism. Indeed, assuming the multiplicativity of Φ, these algebras must have equivalent derived categories by Rickard theory, since $T_Q(n,r)$ is a tilting complex for $\mathcal{D}_Q(n,r)$, whose endomorphism ring in the derived category is isomorphic to $\mathcal{E}_{Q^{op}}(n,r)$.

To prove that Φ is an algebra homomorphism, we may assume $R = \mathbb{Z}$, since Φ is compatible with base change. In fact, since $\mathbb{Z}$ is a subring of $\mathbb{Q}$, we may assume $R = \mathbb{Q}$. We know that the map

$$\mathcal{E}_{Q^{op}}(1,1) \wr \Sigma_r \cong End_{D^b(U_Q \wr \Sigma_r)}(T_Q \wr \Sigma_r)$$

is an algebra homomorphism, by proposition 34. That is to say, the map,

$$\xi_{n,r}\mathcal{E}_{Q^{op}}(n,r)\xi_{n,r} \cong End_{D^b(\xi_{n,r}\mathcal{D}_Q(n,r)\xi_{n,r})}(\xi_{n,r}T_Q(n,r)\xi_{n,r})$$

is an algebra homomorphism. However, because $R = \mathbb{Q}$, we know that $\mathcal{E}_{Q^{op}}(n,r)$ is Morita equivalent to $V_{Q^{op}} \wr \Sigma_r$, by corollary 16. $\mathcal{E}_{Q^{op}}(n,r)$ is therefore generated by the subalgebras $\mathcal{S}_V(n,r)$, and $\xi_{n,r}\mathcal{E}_{Q^{op}}(n,r)\xi_{n,r}$. By lemma 33, the map

$$\lambda^{-1}\mu : \mathbb{Q}\mathcal{E}_{Q^{op}}(n,r) \cong End_{D^b(\mathcal{D}_Q(n,r))}(T_Q(n,r))$$

is an algebra homomorphism, as required. □

We have the following theorem:

Theorem 36 *([18], theorem 154) Let Q, Q' be finite quivers, with the same underlying graph Γ. Then, $\mathcal{D}_Q(n,r) \cong \mathcal{D}_{Q'}(n,r)$.* □

Corollary 37 *Let Q, Q' be finite quivers, with the same underlying graph Γ. Suppose that Γ is a tree. Then,*

$$D^b(\mathcal{E}_Q(n,r) - mod) \cong D^b(\mathcal{E}_{Q'}(n,r) - mod).\square$$

Equivalences V. Deformations of doubles.

Let $n \geq r$, and let Q be a finite Dynkin quiver, of type A. We define one parameter polynomial deformations $\underline{\mathcal{D}}_Q(n,r), \underline{\mathcal{E}}_Q(n,r)$ of $\mathcal{D}_Q(n,r), \mathcal{E}_Q(n,r)$. We prove the derived equivalence of $\underline{\mathcal{D}}_Q(n,r), \underline{\mathcal{E}}_{Q^{op}}(n,r)$.

We have conjectured that the algebras $\mathcal{D}_Q(n,r)$ possess certain deformations, and proved the existence of such deformations, in type A [19]. We summarise our construction here. It is based on a pair of theorems, which we restate below as theorems 38 and 40.

Given a graded algebra $A = \bigoplus_{i\in\mathbb{Z}^+} A^i$, let $A^{>0} = \bigoplus_{i>0} A^i$, and let $A^! = Ext^*_A(A^{0*}, A^{0*})$.

Theorem 38 *Let $\tilde{\Gamma}$ be an affine Dynkin graph of type A. Let $\Pi_{\tilde{\Gamma}}(n)$ be the Schur algebra of the preprojective algebra of $\tilde{\Gamma}$. Let $\tilde{Q}$ be an orientation of $\tilde{\Gamma}$, and let $n \geq r$. Then*

$$\Pi_{\tilde{\Gamma}}(n,r)^! \cong \mathcal{D}_{\tilde{Q}}(n,r),$$

$$\mathcal{D}_{\tilde{Q}}(n,r)^! \cong \Pi_{\tilde{\Gamma}}(n,r).\square$$

Here, the degree zero parts of $\Pi_{\tilde{\Gamma}}(n,r), \mathcal{D}_{\tilde{Q}}(n,r)$ are both isomorphic to $\mathcal{S}_{\tilde{V}}(n,r)$, where $\tilde{V}$ is the set of vertices of $\tilde{\Gamma}$. When one forms its Schur algebra, one thinks of $\Pi_{\tilde{\Gamma}}$ as concentrated in parity zero.

Definition 39 *Let A and B be $\mathbb{Z}_+$-graded k-algebras. An algebra C is a* graded multiplicative extension *of A by B if we have a graded algebra embedding*

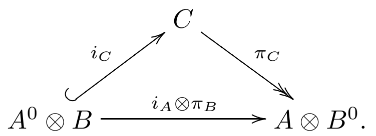
$$i_C : A^0 \otimes B \hookrightarrow C,$$

and a graded algebra epimorphism

$$\pi_C : C \twoheadrightarrow A \otimes B^0,$$

such that

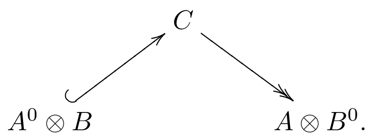
1. *The following diagram commutes:*



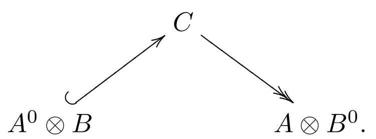
where $i_A : A^0 \hookrightarrow A$ denotes the natural embedding, and $\pi_B : B \to B^0$ the natural projection.

2. *The left and right actions of B on C are free, and commute.*
3. *We have $C \underset{B}{\otimes} B^{>0} = B^{>0} \underset{B}{\otimes} C = ker(\pi_C)$.*

We draw a graded multiplicative extension of A by B thus:



Theorem 40 *Let C be a graded multiplicative extension,*



Suppose that $A_{A^0}, B_{B^0}, C_{A^0 \otimes B}$ are projective modules, and that ${}_A A^{0*}$, ${}_B B^{0*}$ possess linear resolutions of the form,

$$... \to A \otimes_{A^0} A^{2!*} \to A \otimes_{A^0} A^{1!*} \to A \otimes_{A^0} A^{0!*} \to A^{0*},$$

$$... \to B \otimes_{B^0} B^{2!*} \to B \otimes_{B^0} B^{1!*} \to B \otimes_{B^0} B^{0!*} \to B^{0*},$$

where $A^{i!}, B^{i!}$ are finite dimensional modules over A^0, B^0, for $i \geq 0$. Then ${}_CC^{0*}$ possesses a linear resolution of the form

$$... \to C \otimes_{C^0} C^{2!*} \to C \otimes_{C^0} C^{1!*} \to C \otimes_{C^0} C^{0!*} \to C^{0*},$$

and $C^!$ is a graded multiplicative extension,

$$\begin{array}{ccc} & C^! & \\ \nearrow & & \searrow \\ B^{!0} \otimes A^! & & B^! \otimes A^{!0}.\square \end{array}$$

To deform $\mathcal{D}_{\tilde{Q}}(n,r)$, one first observes the presence of a distinguished central quadratic element a in $\Pi_{\tilde{\Gamma}}(n,r)$. One then defines a multiplicative extension $\underline{\Pi}_{\tilde{\Gamma}}(n,r) = R[\lambda] \otimes \mathcal{S}(\Pi_{\tilde{\Gamma}})(n,r)/(\lambda^2 - a)$ of $\Pi_{\tilde{\Gamma}}(n,r)$ by the Koszul algebra $R[\lambda]/\lambda^2$. Here, $\Pi_{\tilde{\Gamma}}$ and $R[\lambda]$ are thought of as super-algebras, whose $\mathbb{Z}/2$-gradings are inherited from their $\mathbb{Z}_+$-gradings.

Definition 41

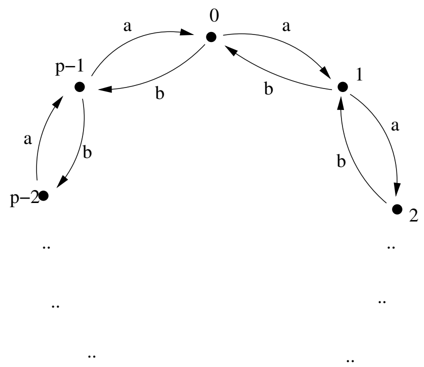
$$\underline{\mathcal{D}}_{\tilde{Q}}(n,r) = \underline{\Pi}_{\tilde{\Gamma}}(n,r)^!$$

The Koszul dual of $R[\lambda]/\lambda^2$ is a polynomial ring $R[\zeta]$ in one variable. By theorem 40, $\underline{\mathcal{D}}_{\tilde{Q}}(n,r)$ is a multiplicative extension of $R[\zeta]$ by $\mathcal{D}_{\tilde{Q}}(n,r)$. Since λ super-commutes with $\underline{\Pi}_{\tilde{\Gamma}}(n,r)$, the variable ζ commutes with $\underline{\mathcal{D}}_{\tilde{Q}}(n,r)$. Therefore, $\underline{\mathcal{D}}_{\tilde{Q}}(n,r)$ is a one-parameter deformation of $\mathcal{D}_{\tilde{Q}}(n,r)$.

Removing a vertex v from the graph $\tilde{\Gamma}$, one obtains an ordinary Dynkin graph Γ, of type A. Removing v, from the quiver $\tilde{Q}$, one obtains an orientation Q of Γ. Cutting $\mathcal{D}_{\tilde{Q}}(n,r)$ at the corresponding idempotent f_V, one obtains $\mathcal{D}_Q(n,r)$. Cutting $\tilde{\mathcal{D}}_{\tilde{Q}}(n,r)$ at f_V, one obtains a deformation $\underline{\mathcal{D}}_Q(n,r)$ of $\mathcal{D}_Q(n,r)$.

Remark 42 We are lucky that we can define a deformation of $\mathcal{D}_Q(w,w)$ so easily. Fortunately, $\mathcal{D}_{\tilde{Q}}(n,r)$ has a homological dual, $\Pi_{\tilde{\Gamma}}(n,r)$, which is an associative algebra. In general, the homological dual of an algebra is an A_∞-algebra.

Example 43 Let $\tilde{Q}$ have p vertices. In case $w = 1$, the $R[\zeta]$-algebra $\underline{\mathcal{D}}_{\tilde{Q}}(1,1)$ is isomorphic to the the $R[\zeta]$-algebra generated by the quiver,



modulo relations $a_i b_i + b_{i-1}a_{i-1} + \zeta^2 v_i = 0$. Here, v_i represents the vertex i, and a_i (respectively b_i) represents the arrow from vertex i to vertex $i+1$ (respectively vertex $i+1$ to vertex i), given $i \in \mathbb{Z}/p$.

Lemma 44 *Let Q be a Dynkin quiver, of type A. The algebra embedding,*

$$\mathcal{S}_Q(n,r) \hookrightarrow \mathcal{D}_Q(n,r)$$

lifts to an algebra embedding,

$$\mathcal{S}_Q(n,r) \hookrightarrow \underline{\mathcal{D}}_Q(n,r).$$

We have,

$$\underline{\mathcal{D}}_Q(n,r) \cong R[\zeta] \otimes \mathcal{D}_Q(n,r),$$

as $R[\zeta]$-$\mathcal{S}_Q(n,r)$-$\mathcal{S}_Q(n,r)$-trimodules.

Proof:

It is sufficient for us to prove this theorem for the affine quiver $\tilde{Q}$, of type A. One approach to this uses homological algebra. There is a commutative diagram of algebra homomorphisms,

$$\begin{array}{ccc} \Pi_{\tilde{\Gamma}}(n,r) & \hookrightarrow & \underline{\Pi}_{\tilde{\Gamma}}(n,r) \\ & \searrow \quad \swarrow & \\ & \mathcal{T}_{\tilde{Q}}(n,r) & \end{array}$$

giving rise to a commutative diagram of exact functors

$$\begin{array}{ccc} \Pi_{\tilde{\Gamma}}(n,r) - mod & \longleftarrow & \underline{\Pi}_{\tilde{\Gamma}}(n,r) - mod \\ & \nwarrow \quad \nearrow & \\ & \mathcal{T}_{\tilde{Q}}(n,r) - mod & \end{array}$$

which extend to exact functors

$$\begin{array}{ccc} D(\Pi_{\tilde{\Gamma}}(n,r)) & \longleftarrow & D(\underline{\Pi}_{\tilde{\Gamma}}(n,r)) \\ & \nwarrow \quad \nearrow & \\ & D(\mathcal{T}_{\tilde{Q}}(n,r)) & \end{array}$$

The degree zero parts of our three algebras are all isomorphic to $\mathcal{S}_{\tilde{V}}(n,r)$, and $Ext^i_A(M,N) \cong Hom_D(A)(M,N[i])$. Therefore, we have a commutative diagram

$$\begin{array}{ccc} \Pi_{\tilde{\Gamma}}(n,r)^! & \twoheadleftarrow & \underline{\Pi}_{\tilde{\Gamma}}(n,r)^! \\ & \nwarrow \quad \nearrow & \\ & \mathcal{T}_{\tilde{Q}}(n,r)^! & \end{array}$$

which is a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{\tilde{Q}}(n,r) & \twoheadleftarrow & \underline{\mathcal{D}}_{\tilde{Q}}(n,r) \\ & \nwarrow \quad \nearrow & \\ & \mathcal{S}_{\tilde{Q}}(n,r) & \end{array}$$

as required. Since its construction is entirely homological, the $R[\zeta]$-module decomposition,

$$\underline{\mathcal{D}}_{\tilde{Q}}(n,r) \cong R[\zeta] \otimes \mathcal{D}_{\tilde{Q}}(n,r),$$

can also be taken to be a decomposition of $\mathcal{S}_{\tilde{Q}}(n,r)$-$\mathcal{S}_{\tilde{Q}}(n,r)$-bimodules. □

Definition 45 *Let*

$$\underline{\mathcal{T}}_Q(n,r) = \underline{\mathcal{D}}_Q(n,r) \bigotimes_{\mathcal{S}_Q(n,r)} \mathcal{K}_Q(n,r).$$

Theorem 46 *Let Q be an ordinary Dynkin quiver, of type A. Then $\underline{\mathcal{T}}_Q(n,r)$ is a tilting complex for $\underline{\mathcal{D}}_Q(n,r)$. Its endomorphism ring in the derived category, $End_{D^b(\underline{\mathcal{D}}_Q(n,r)-mod)}$, is a deformation $\underline{\mathcal{E}}_{Q^{op}}(n,r)$ of $\mathcal{E}_{Q^{op}}(n,r)$. There is an equivalence of derived categories,*

$$D^b(\underline{\mathcal{D}}_Q(r,r) - mod) \cong D^b(\underline{\mathcal{E}}_{Q^{op}}(n,r) - mod).$$

□

Proof:

By lemma 25, and lemma 44, we have

$$\underline{E}_Q(n,r) = End_{\underline{\mathcal{D}}_Q(n,r)}(\underline{\mathcal{T}}_Q(n,r)) \cong$$

$$\mathcal{T}_{Q^{op}}(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \underline{\mathcal{D}}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r) \cong R[\zeta] \otimes E_Q(n,r),$$

as dg $R[\zeta]$-$\mathcal{T}_{Q^{op}}(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-trimodules. This complex has homology concentrated in degree zero, and $R[\zeta]$ acts freely. Modulo ζ, we have the complex $E_Q(n,r)$, which is quasi-isomorphic to $\mathcal{E}_{Q^{op}}(n,r)$. Therefore, its endomorphism ring in the derived category is a deformation

$$\underline{\mathcal{E}}_{Q^{op}}(n,r) = R[\zeta] \otimes \mathcal{E}_{Q^{op}}(n,r),$$

of $\mathcal{E}_{Q^{op}}(n,r)$. $\square$

Dreams and reflections.

Let p be a prime number. Let $(K, \mathcal{O}, k)$ be a p-modular system.

We have made a detailed study of Rock blocks of symmetric groups [18]. We made the following conjecture...

Conjecture 47 *Let Q be an orientation of the Dynkin quiver A_{p-1}. Every Rock block of a symmetric group, of weight w, is Morita equivalent to $\mathcal{D}_Q(w,w)$, over k.*

Thanks to the work of Chuang and Rouquier [7], we have the following equivalent conjecture:

Conjecture 48 *Let Q be an orientation of the Dynkin quiver A_{p-1}. Every symmetric group block of weight w is derived equivalent to $\mathcal{D}_Q(w,w)$, over k.*

After theorem 35, we now have the following equivalent conjecture.

Conjecture 49 *Let Q be an orientation of the Dynkin quiver A_{p-1}. Every symmetric group block of weight w is derived equivalent to $\mathcal{E}_Q(w,w)$, over k.*

Originally, we defined the deformations $\underline{\mathcal{D}}_Q(n,r)$ in order to compare the algebras $\mathcal{D}_Q(n,r)$ with the Cubist algebras, another family of algebras which are also related to blocks of symmetric groups [8]. However, it appears these deformations may play a more fundamental role: they provide interesting $\mathcal{O}$-forms for the algebras $k\mathcal{D}_Q(n,r)$. We assume that $\sqrt{p} \in \mathcal{O}$.

Conjecture 50 *Let Q be an orientation of the Dynkin quiver A_{p-1}. Every Rock block of a symmetric group, of weight w, is Morita equivalent to*

$$\underline{\mathcal{D}}_Q(w,w)/(\zeta-\sqrt{p}),$$

over $\mathcal{O}$. Every symmetric group block of weight w, is derived equivalent to

$$\underline{\mathcal{D}}_Q(w,w)/(\zeta-\sqrt{p}),$$

over $\mathcal{O}$.

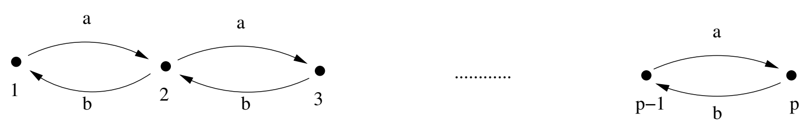
Thanks to theorem 46, we have the following equivalent conjecture:

Conjecture 51 *Let Q be an orientation of the Dynkin quiver A_{p-1}. Every symmetric group block of weight w is derived equivalent to*

$$\underline{\mathcal{E}}_Q(w,w)/(\zeta-\sqrt{p}),$$

over $\mathcal{O}$.

Remark 52 In case $w=1$, blocks of symmetric groups of weight one are Morita equivalent over $\mathcal{O}$ to the path algebra of the quiver,



modulo relations $a_ib_i+b_{i-1}a_{i-1}+pv_i=0$, where v_i represents the vertex i, and a_i (respectively b_i) represents the arrow from vertex i to vertex $i+1$ (respectively vertex $i+1$ to vertex i), for $i=1,...,p-1$.

By comparison with example 43, we see that over $\mathcal{O}$, this algebra is isomorphic to $\underline{\mathcal{D}}_Q(1,1)/(\zeta-\sqrt{p})$, whenever Q is an orientation of A_{p-1}. Therefore, conjectures 50, and 51 hold in case $w=1$. By the work of Chuang and Kessar [6], the conjectures also hold in case $w<p$.

Remark 53 It is possible to define subalgebras of $\underline{\mathcal{D}}_Q(n,r),\underline{\mathcal{E}}_Q(n,w)$ which are multiplicative extensions of $\mathcal{D}_Q(n,r),\mathcal{E}_Q(n,r)$ by $\mathcal{O}[\zeta^2]$ [19]. Working with these deformations instead, we can remove the assumption that $\sqrt{p}\in\mathcal{O}$.

Remark 54 Let $\mathcal{X}=\mathcal{D}_Q(w,w)$ (respectively $\mathcal{E}_Q(w,w)$) and let $\underline{\mathcal{X}}=\underline{\mathcal{D}}_Q(w,w)$ (respectively $\underline{\mathcal{E}}_Q(w,w)$). Let us choose a splitting $\underline{\mathcal{X}}=\mathcal{O}[z]\otimes\mathcal{X}$. Given $x,y\in\underline{\mathcal{X}}$, we write $xy = \sum \zeta^i \otimes (xy)_i$, where $(xy)_i \in \mathcal{X}$. We can lift the nondegenerate, associative, symmetric bilinear form $<,>$ on $\mathcal{X}$, with values in $\mathcal{O}$, to an associative form $(,)$ on $\underline{\mathcal{X}}$, with values in $\mathcal{O}[\zeta]$, via the formula

$$(x, y) = \sum < 1, (xy)_i > z^i.$$

Passing to the quotient $\mathcal{O}[\zeta]/(\zeta - \sqrt{p}) \cong \mathcal{O}$, we obtain an associative bilinear form $(,)_p$ on $\mathcal{X}_p = \underline{\mathcal{X}}/(\zeta - \sqrt{p})$. Over k, the forms $(,)_p, <,>$ are identical forms on $\mathcal{X}_p$. Therefore, $(,)_p$ is non-degenerate over k, and consequently non-degenerate over $\mathcal{O}$.

In conclusion, we have defined a bilinear form on $\mathcal{X}_p$ which is non-degenerate, and associative. Thus, $\mathcal{X}_p$ is a Frobenius algebra, over $\mathcal{O}$.

Remark 55 When Q is the quiver of type A_2, the Brauer tree algebras U_Q, and $V_{Q^{op}}$ are isomorphic. However, $\mathcal{D}_Q(n, r)$ and $\mathcal{E}_{Q^{op}}(n, r)$ are not Morita equivalent in this case, for $r > 1$.

Remark 56 There ought to be braid group actions on the derived categories of $\mathcal{D}_Q(n, r), \mathcal{E}_Q(n, r)$, generalising those of Rouquier-Zimmermann [16], and Seidel-Thomas [17].

Remark 57 Let A be a block of a symmetric group G. Let D be a defect group of A. Then D contains an elementary abelian p-subgroup E, such that $N_G(E) > N_G(D)$. Let B be the Brauer correspondent of A in $N_G(E)$. The truth of conjecture 50 would imply the existence of an algebra $\mathcal{E}$, containing an idempotent e, such that A is derived equivalent to $\mathcal{E}$, and B is Morita equivalent to $e\mathcal{E}e$.

Remark 58 In this paper, we have considered Schur bialgebras of the form,

$$\mathcal{S}(A) = \bigoplus_{r \geq 0} (A^{\otimes r})^{\Sigma_r},$$

where A is an associative algebra. Another way to generalise the classical Schur bialgebra is the following:

Let B be a bialgebra, and let V be a B-module. Let $\phi_r : B \to End(V^{\otimes r})$ be the natural map corresponding to the action of B on $V^{\otimes r}$. Let $\mathcal{S}(B, V)(r) =$

$im(\phi_r)$, and let

$$\mathcal{S}(B,V) = \bigoplus_{r\geq 0} \mathcal{S}(B,V)(r).$$

Given r_1, r_2, such that $r_1 + r_2 = r$, we have $V^{\otimes r} = V^{\otimes r_1} \otimes V^{\otimes r_2}$. There is consequently an algebra homomorphism

$$\Delta_{r_1,r_2} : \mathcal{S}(B,V)(r) \to \mathcal{S}(B,V)(r_1) \otimes \mathcal{S}(B,V)(r_2).$$

The map

$$\Delta = \sum_{r_1,r_2\geq 0} \Delta_{r_1,r_2} : \mathcal{S}(B,V) \to \mathcal{S}(B,V) \otimes \mathcal{S}(B,V)$$

is a coproduct, giving $\mathcal{S}(B,V)$ the structure of a bialgebra.

The bialgebra $\mathcal{S}(U(\mathfrak{gl}_n), E)$ associated to the universal enveloping algebra of $\mathfrak{gl}_n$ and its n dimensional irreducible module E, is the classical Schur bialgebra $\mathcal{S}_{.}(n)$.

So long as V is finite dimensional, one can take the double of $\mathcal{S}(B,V)$, and obtain a symmetric algebra with finite dimensional components. Are these of any interest, for example when B is the enveloping algebra of a classical Lie algebra, and V its natural module ?

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