On seven families of algebras.

W. Turner

January 17, 2007

Mathematical Institute, University of Oxford, 24-29 St. Giles', Oxford, England.
Email: turnerw@maths.ox.ac.uk

Abstract

This paper encircles the Schiver doubles, introduced in the article "Rock blocks" [26]. The doubles are a gang of symmetric algebras $\mathcal{D}_\Gamma(n)$, an algebra being associated to each graph Γ, and natural number n.

We discuss algebraic relations between the Schiver doubles, and six other families of algebras... blocks of symmetric groups, Zigzag algebras, Preprojective algebras, Rhombal algebras, symplectic reflection algebras, and blocks of category $\mathcal{O}$.

One unifying concept is that of a multiplicative extension of algebras, which is an analogue, in the category of algebras, of the concept of a group extension, in the category of groups. A second is homological duality. We prove that, under favourable conditions, the homological dual of a multiplicative extension is a multiplicative extension of homological duals.

Summary.

An investigation into blocks of symmetric groups over fields of finite characteristic encouraged the definition of certain finitely generated algebras, named "Schiver doubles" [26]. Conjecturally, every block of a symmetric group is derived equivalent to a Schiver double, corresponding to an ordinary Dynkin graph of type A. We review this conjecture, and make a precise comparison between Schiver doubles, and wreath products of zigzag algebras. We consequently describe a homological duality between Schiver doubles of affine type A, and Schur algebras associated to the corresponding preprojective algebra.

Following these results, we propose the existence of a pair of generalisations.

First, algebras enveloping M. Peach's Rhombal algebras [20] at one limit, and realising the Schiver doubles at a separate limit. These essentially correspond to polynomial deformations of the Schiver doubles. Such deformations are analogous to certain deformations of $S(V+V^*)\rtimes G$ which define symplectic reflection algebras.

We next define Koszul algebras over any field, which are equivalent, over $\mathbb{C}$, to blocks of category $\mathcal{O}$. We exhibit a relation between these algebras, and tensor powers of zigzag algebras. Indeed, our second generalisation, describes extensions of such tensor powers by algebraic representatives of blocks of $\mathcal{O}$.

Introduction.

The Rock blocks are a collection of blocks known to be a set of derived representatives for all blocks of symmetric groups [5]. J. Chuang and R. Kessar proved a structure theorem for Rock blocks of symmetric groups of abelian defect, in [4]. In [26], I proposed a generalisation of Chuang and Kessar's theorem to arbitrary defect, and made steps towards proving it. I conjectured the blocks should be equivalent to Schiver doubles, finite dimensional algebras which resemble fat wreath products. The Schiver doubles, and a soap opera of related families of algebras, are the subject of this note.

It is not immediately clear to me how Schiver doubles should be judged. Should they be weighed as an interesting quirk of symmetric group blocks, a momentary glimpse into the murky world of nonabelian defect whose general consequence is slight ? Or can they be perceived to be of weightier significance, genuine Rocks upon which a solid foundation can be built ?

Is my conjecture correct ? Some scepticism was expressed by experts at a recent conference I attended in Banff [2], their suspicion being aroused by the fact the Schiver doubles are $\mathbb{Z}_+$-graded in a natural way. The existence of such gradings on blocks of symmetric groups of nonabelian defect would be surprising. However, I know of no counterexample, and all the evidence of which I am aware suggests such a grading. Indeed, naively accepting the truth of this conjecture, we can innocently ask further questions. Is it possible to describe more sophisticated finite dimensional algebras, which are Morita equivalent to more general blocks of symmetric groups ? The representation of Rock blocks by Schiver doubles could be one small aspect of a greater phenomenon.

In this paper, we develop this possibility, encouraged by results of M. Peach [20], who has defined a collection of "Rhombal algebras", which are related to weight two blocks of symmetric groups. To be precise, the unramified region of a weight two symmetric group block, is Morita equivalent to a region of some Rhombal algebra. We seek algebras which generalise both the Schiver doubles, and the Rhombal algebras. Such would be certain subquotients of polynomial deformations of Schiver doubles, whose existence we predict. We prove their existence in type A. These deformations suggest a union between Schiver doubles, and symplectic reflection algebras in the "wreath product case" [13].

In early sections, we discuss relations between Schiver doubles and wreathed Zigzag algebras, and an interesting homological duality involving the Schur algebra of a preprojective algebra. These make our new definition possible.

We would like yet more subtle generalisations of the Schiver doubles. Such are not apparent to me at the moment. However, given the sympathy between Schiver doubles and tensor powers of Zigzag algebras, generalisations of these latter algebras may be suggestive.

In the final section of the paper we develop this idea. We unveil extensions of tensor powers of Zigzag algebras, by algebraic analogues of principal blocks of category $\mathcal{O}$. Here, $\mathcal{O}$ is the highest weight category of modules for a semisimple Lie algebra defined by J. Bernstein, I. Gelfan'd, and S. Gelfan'd.

We wish always, if possible, to work over fields of positive characteristic. With this in mind, in a latter section of this paper we define algebras, over fields of positive characteristic, whose module categories behave structurally like categories $\mathcal{O}$. It is only possible to perform this sculpture, after we have plundered deep geometric results of W. Soergel concerning category $\mathcal{O}$.

The methods of this paper are mostly homological. The general idea which unites our constructions is that of a multiplicative extension of algebras. Examples of such extensions are group algebras of group extensions, and algebra deformations. Our definition of multiplicative extension possesses an intrinsic symmetry, which allows us to prove that under agreeable circumstances, the homological dual of a multiplicative extension is a multiplicative extension of homological duals.

Our paper is concerned with, in writing, a septuple of familiar algebras: Schiver doubles, blocks of symmetric groups, Zigzag algebras, Preprojective algebras, Rhombal algebras, symplectic reflection algebras, and blocks of category $\mathcal{O}$.

I am most grateful to Joe Chuang for introducing me to the Rhombal algebras, and for many interesting discussions.

Recollections.

We recall some of the principal definitions of the Schiver doubles [26], as well as the definition of the preprojective algebra, and the zigzag algebra.

Let k be a field. Let Γ be a graph, whose set of vertices is denoted V, and whose set of edges is denoted E. Let Q be an orientation of Γ.

Let kQ be the path algebra of Q. Let $kQ(n) = End_{kQ}(kQ^{\oplus n})$. The set of regular functions on the affine variety $kQ(n)$, is a bialgebra. Its polynomial dual decomposes as a direct sum of algebras,

$$\mathcal{T}_Q(n,r) = \bigoplus_{r\geq 0} \mathcal{T}_Q(n,r).$$

Here, the r^{th} homogeneous component $\mathcal{T}_Q(n,r) = (kQ(n)^{\otimes r})^{\Sigma_r}$ is the fixed point set of the r-fold tensor product of $kQ(n)$, under the action of the symmetric group Σ_r on r letters.

Let P_Q be the path algebra of Q, modulo all quadratic relations. We define on P_Q the structure of a super-algebra, giving vertices parity 0, and arrows parity 1. Let $P_Q(n) = End_{P_Q}(P_Q^{\oplus n})$. The set of regular functions $\mathcal{A}_Q(n)$ on the affine super-variety, $P_Q(n)$, is a super-bialgebra. We name the polynomial dual of this bialgebra a *Schiver bialgebra*. It decomposes as a direct sum of algebras,

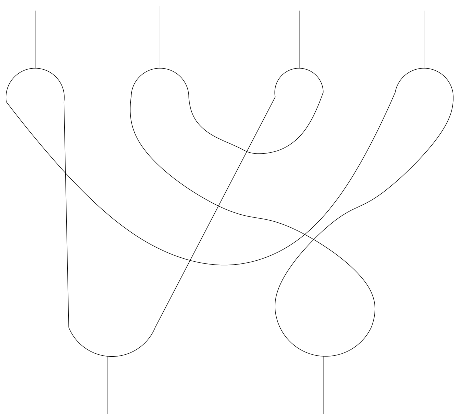
$$\mathcal{S}_Q(n) = \bigoplus_{r \geq 0} \mathcal{S}_Q(n,r).$$

where $\mathcal{S}_Q(n,r) = (P_Q(n)^{\otimes r})^{\Sigma_r}$.

Performing an algebraic double construction on $\mathcal{S}_Q(n)$ ([26], chapter 8), we obtain the *Schiver double* associated to Q,

$$\mathcal{D}_Q(n) = \mathcal{S}_Q(n) \otimes \mathcal{A}_{Q^{op}}(n).$$

The product is described by the diagram,



This is a symmetric, associative algebra, which is independent of the orientation Q of Γ ([26], theorem 154). It decomposes as a direct sum of algebras,

$$\mathcal{D}_\Gamma(n) = \bigoplus_{r \geq 0} \mathcal{D}_\Gamma(n,r).$$

Here, $\mathcal{D}_Q(n,r) = \bigoplus_{r_1+r_2=r} \mathcal{S}_Q(n,r_1) \otimes \mathcal{A}_{Q^{op}}(n,r_2)$.

The algebra $\mathcal{D}_\Gamma(n)$ is $\mathbb{Z}_+$-graded ([26], remark 153), its degree zero part being,

$$\mathcal{S}_V(n) \cong \bigotimes_{v \in V} \mathcal{S}_v(n).$$

Here, $\mathcal{S}_v(n) \cong \mathcal{S}_.(n)$ is a classical Schur algebra [14].

Let l be a prime number. Let Σ_r be the symmetric group on r letters. Any defect group of a symmetric group l-block, is isomorphic to the Sylow l-subgroup of $\Sigma_l \wr \Sigma_w$, for some unique number w. The definition of $\mathcal{D}_\Gamma(n)$ was made, with the expectation that the following conjecture holds.

Conjecture 1 *(see [26], pg. 96) Let B be an l-block of a symmetric group, whose defect group is equal to the Sylow l-subgroup of $\Sigma_l \wr \Sigma_w$. Then B is derived equivalent to $\mathcal{D}_{A_{l-1}}(w, w)$.*

Let $D(\Gamma)$ be the double quiver whose vertices are in one-one correspondence with vertices V of Γ, and whose arrows A are in two-one correspondence with the edges of Γ. Thus, an edge joining vertices v_1, v_2 in Γ corresponds to two arrows in Q, one pointing from v_1 to v_2, the other pointing from v_2 to v_1.

Given $a \in A$, we denote its source $s(a)$, and its tail $t(a)$.

Let v be a vertex of Γ attached to two edges α, β. Let the corresponding arrows in $D(\Gamma)$ pointing towards v be labelled α_1, β_1. Let the corresponding arrows pointing away from v be labelled α_2, β_2.

$$\Gamma : \qquad \circ \overset{\alpha}{\text{———}} v \overset{\beta}{\text{———}} \circ$$

$$D(\Gamma) : \qquad \begin{tikzcd} \circ \arrow[r, bend left, "\alpha_1"] & v \arrow[l, bend left, "\alpha_2"] \arrow[r, bend left, "\beta_2"] & \circ \arrow[l, bend left, "\beta_1"] \end{tikzcd}$$

Let

$$R\Pi^{\pm}_{\alpha,\beta,v} = \alpha_2\alpha_1 \pm \beta_2\beta_1$$

$$RZZ^{\pm}_{\alpha,\beta,v} = \{\alpha_1\beta_2, \beta_1\alpha_2, \alpha_2\alpha_1 \mp \beta_2\beta_1\},$$

dual sets of quadratic elements of $kD(\Gamma)$.

The *preprojective algebra* $\Pi^{\pm}_\Gamma$ is the algebra generated by the path algebra $kD(\Gamma)$, modulo the quadratic ideal generated by $\bigcup_{\alpha,\beta,v} R\Pi^{\pm}_{\alpha,\beta,v}$ (I.M. Gelfan'd, V.A. Ponomarev, see [21]).

The *zigzag algebra* $ZZ^{\pm}_\Gamma$ is the algebra generated by the path algebra $kD(\Gamma)$, modulo the quadratic ideal generated by $\bigcup_{\alpha,\beta,v} RZZ^{\pm}_{\alpha,\beta,v}$ (R.S. Huerfano, M. Khovanov, see [15]).

When Γ is not a Dynkin graph, the algebras Π^+_Γ, and ZZ^+_Γ are in Koszul duality, as are the algebras Π^-_Γ, and ZZ^-_Γ [18].

If Γ is bipartite, then $\Pi^+_\Gamma \cong \Pi^-_\Gamma$, and $ZZ^+_\Gamma \cong ZZ^-_\Gamma$.

Let $\Pi^{\pm}_\Gamma(n, 1) = End_{\Pi_\Gamma}(\Pi^{\pm \oplus n}_\Gamma)$. The set of regular functions on the affine variety $\Pi^{\pm}_\Gamma(n, 1)$, is a bialgebra. We call its polynomial dual the *Schur preprojective algebra*. It decomposes as a direct sum of algebras,

$$\Pi^{\pm}_\Gamma(n) = \bigoplus_{r \geq 0} \Pi^{\pm}_\Gamma(n, r),$$

where $\Pi_\Gamma^\pm(n,r) = (\Pi_\Gamma^\pm(n,1)^{\otimes r})^{\Sigma_r}$. We will write $\Pi_\Gamma(n,r)$ for $\Pi_\Gamma^+(n,r)$ in the sequel.

Schiver doubles and Wreath products of Zigzag algebras.

We prove that the Schiver doubles $\mathcal{D}_\Gamma(r,r)$ can be thought of as chubby wreath products of the zigzag algebra on Γ, with a symmetric group.

Let us view ZZ_Γ^+ as a super-algebra whose $\mathbb{Z}/2$-grading is inherited from the natural $\mathbb{Z}_+$-grading by quiver path length.

For $r \geq 1$, we denote by $ZZ_\Gamma^+ \wr \Sigma_r$, the wreath product of the super-algebra ZZ_Γ^+, with the symmetric group Σ_r. It is a simple exercise to write down generators and relations for this algebra:

Lemma 2 *The super wreath product $ZZ_\Gamma^+ \wr \Sigma_r$ may be described by generators,*

$$\{e_{\underline{v}} | \underline{v} \in V^r\} \cup \Sigma_r,$$

in degree zero; by generators,

$$\{q_{\underline{v},a,i} | \underline{v} \in V^{r-1}, a \in A, 1 \leq i \leq r\},$$

in degree one; and by relations,

$$e_{\underline{v}}.e_{\underline{w}} = \delta_{\underline{v},\underline{w}}.e_{\underline{v}},$$

for $\underline{v}, \underline{w} \in V^r$

$$\sigma.e_{\underline{v}} = e_{\sigma(\underline{v})}.\sigma,$$

for $\underline{v} \in V^r, \sigma \in \Sigma_r$.

$$e_{\underline{v}}.q_{\underline{w},a,i} = \delta_{\underline{v},(w_1,..,w_{i-1},s(a),w_i,...,w_{r-1})}.q_{\underline{w},a,i},$$

$$q_{\underline{w},a,i}.e_{\underline{v}} = \delta_{(w_1,..,w_{i-1},t(a),w_i,...,w_{r-1}),\underline{v}}.q_{\underline{w},a,i},$$

for $\underline{v} \in V^r, \underline{w} \in V^{r-1}, a \in A, 1 \leq i \leq r$.

$$\sigma.q_{\underline{v},a,i} = q_{(v_{\sigma 1},...,v_{\sigma(\sigma^{-1}i-1)},v_{\sigma(\sigma^{-1}i+1)},...,v_{\sigma r}),a,\sigma i},$$

for $a \in A, \underline{v} \in V^{r-1}, \sigma \in \Sigma_r, 1 \leq i \leq r$.

Let v be a vertex of Γ attached to two edges α, β. Let the corresponding arrows in $D(\Gamma)$ pointing towards v be labelled α_1, β_1. Let the corresponding arrows pointing away from v be labelled α_2, β_2. Let $\underline{v} \in V^{r-1}$, and $1 \leq i \leq r$. Then,

$$q_{\underline{v},\alpha_1,i}.q_{\underline{v},\beta_2,i} = 0,$$

$$q_{\underline{v},\beta_1,i}.q_{\underline{v},\alpha_2,i} = 0,$$

$$q_{\underline{v},\alpha_2,i} \cdot q_{\underline{v},\alpha_1,i} - q_{\underline{v},\beta_2,i} \cdot q_{\underline{v},\beta_1,i} = 0.$$

Let $a, b \in A, \underline{v}, \underline{w} \in V^{r-1}, i \neq j$.

$$q_{\underline{v},a,i} \cdot q_{\underline{w},b,j} + q_{\underline{w},b,j} \cdot q_{\underline{v},a,i} = 0.$$

□

As we have already noted, $\mathcal{D}_\Gamma(n)$ is $\mathbb{Z}_+$-graded, its degree zero part being a tensor product of classical Schur algebras,

$$\mathcal{S}_V(n) \cong \bigotimes_{v \in V} \mathcal{S}_v(n).$$

Let E_v denote the unique irreducible $\mathcal{S}_v(n,1)$-module. Let $n \geq r$. Upon fixing a basis for E, inside the direct sum,

$$\mathcal{S}_v(n, \leq r) = \bigoplus_{i=0}^{r} \mathcal{S}_v(n,i)$$

an idempotent $\xi_v = \sum_{i=0}^{r} \xi_{(1^i)}$ can readily be identified ([14], section 6), such that,

$$\mathcal{S}_v(n, \leq r)\xi_v \cong \bigoplus_{i=0}^{r} E_v^{\otimes i}.$$

Thus, $\xi_V = \bigotimes_{v \in V} \xi_v$ is an idempotent in $\mathcal{S}_V(n)$. Let $\xi_V(n,r)$ be the component of ξ_V in $\mathcal{S}_V(n,r)$. Upon writing $E_V = \oplus_V E_v$, we find that,

$$\mathcal{S}_V(n,r)\xi_V(n,r) \cong E_V^{\otimes r}.$$

The theorem below generalises the classical isomorphism between $End_{\mathcal{S}(n,r)}(E^{\otimes r})$ and $k\Sigma_r$.

Theorem 3 *Let $n \geq r$. Then,*

$$\xi_V(n,r)\mathcal{D}_\Gamma(n,r)\xi_V(n,r) \cong ZZ_\Gamma^+ \wr \Sigma_r.$$

If k has characteristic zero, or characteristic greater than r, then $\mathcal{D}_\Gamma(n,r)$ is Morita equivalent to $ZZ_\Gamma^+ \wr \Sigma_r$.

Proof:

We have proved the isomorphism of this theorem in a more general form, in a separate article. For a complete proof, we refer the reader to that paper [27]. In order to be explicit however, let us define elements of $\xi_\Gamma(n,r)\mathcal{D}_\Gamma(n,r)\xi_\Gamma(n,r)$ which correspond to the generators of $ZZ_\Gamma^+ \wr \Sigma_r$ described in Lemma 2. If the reader wishes, he may check that these elements are generators of the subalgebra

$\xi_\Gamma(n,r)\mathcal{D}_\Gamma(n,r)\xi_\Gamma(n,r)$, that the relations of Lemma 2 hold inside $\mathcal{D}_\Gamma(n,r)$, and that $\xi_\Gamma(n,r)\mathcal{D}_\Gamma(n,r)\xi_\Gamma(n,r)$ has the same dimension as $ZZ_\Gamma^+ \wr \Sigma_r$. This is enough to confirm the isomorphism, $\xi_\Gamma(n,r)\mathcal{D}_\Gamma(n,r)\xi_\Gamma(n,r) \cong ZZ_\Gamma^+ \wr \Sigma_r$.

The symmetric group Σ_r acts as permutations on $E_V^{\otimes r} \cong \mathcal{S}_V(n,r)\xi_V(n,r)$. This provides the elements Σ_r of degree zero in $\xi_V(n,r)\mathcal{D}_\Gamma(n,r)\xi_V(n,r)$. The elements $e_{(v_1,...,v_r)}$ we identify with tensors of r idempotents $\xi_{v_i}(n,1) \in End_{\mathcal{S}_V(n,1)}(E_{v_i})$.

Let $a \in A$. Let us fix an orientation Q of Γ, whose set of arrows contains a. In this way, $\mathcal{D}_\Gamma(n,r) = \mathcal{D}_Q(n,r)$ contains a subalgebra $\mathcal{S}_Q(n,r)$. If we identify the source and tail of a with s,t, we have,

$$\mathcal{S}_Q(n,r)[1]\xi_V(n,r)e_{(v_1,...,v_{i-1},t,v_{i+1},...,v_r)}$$

$$\cong E_{v_1} \otimes ... \otimes E_{v_{i-1}} \otimes E_s \otimes E_{v_{i+1}} ... \otimes E_{v_r}.$$

where $\mathcal{S}_Q(n,r)[1]$ is the degree one part of $\mathcal{S}_Q(n,r)$. This implies that in degree one,

$$e_{(v_1,...,v_{i-1},s,v_{i+1},...,v_r)}\xi_V(n,r)\mathcal{S}_Q(n,r)[1]\xi_V(n,r)e_{(v_1,...,v_{i-1},t,v_{i+1},...,v_r)}$$

$$\cong k_{v_1} \otimes ... \otimes k_{v_{i-1}} \otimes k_s \otimes k_{v_{i+1}} ... \otimes k_{v_r} \cong k,$$

Our fixed basis for E identifies a natural generator for this copy of k. We identify $q_{(v_1,...,v_{i-1},v_{i+1},...,v_r),a,i}$ with this generator.

The finite dimensional algebra $\mathcal{D}_\Gamma(n,r)$ can be positively graded, with degree zero part

$$\mathcal{S}_V(n,r) \cong \bigoplus_{r_v \geq 0, \sum r_v = r} \left(\bigotimes_{v \in V} \mathcal{S}_v(n,r_v) \right).$$

Simple $\mathcal{S}_v(n,r_v)$-modules are in bijection with the set of partitions of r_v. Therefore, simple $\mathcal{D}_\Gamma(n,r)$-modules are in natural bijection with the set

$$\Lambda = \{(\lambda_i)_{i \in V}, \lambda_i \vdash r_i, \sum r_i = r\}.$$

If k has characteristic zero, or characteristic greater than r, then the simple $ZZ_\Gamma^+ \wr \Sigma_r$-modules are also in natural bijection with Λ [6]. It follows that $\mathcal{D}_\Gamma(n,r)\xi_V(n,r)$ is a progenerator for $\mathcal{D}_\Gamma(n,r)$, and the two algebras are Morita equivalent. $\square$

Homological duality.

We discuss homological duality for $\mathbb{Z}_+$-graded k-algebras $A = \bigoplus_{i \geq 0} A^i$, whose degree zero part A^0 is not necessarily semisimple. We assume that A^i is finite dimensional, for $i \geq 0$.

Definition 4 *Let $M = (M_i, d_i)$ be a complex of modules for a $\mathbb{Z}_+$-graded algebra A. Then M is said to be linear if the differential*

$$d_i : M_i \to M_{i+1}$$

is a degree one map, for every i.

Given a left/right module M, let $M^* = Hom(M, k)$ denote the k-dual of M, a right/left module.

Definition 5 *For a $\mathbb{Z}_+$-graded algebra A, let $A^! = Ext^*_A(A^{0*}, A^{0*})$ denote the homological dual of A, taken with respect to A^{0*}.*

Lemma 6 *Suppose that A is a $\mathbb{Z}_+$-graded algebra, such that A_{A^0} is a projective module, and that ${}_AA^{0*}$ possesses a linear resolution,*

$$... \to A \otimes_{A^0} A^{2!*} \to A \otimes_{A^0} A^{1!*} \to A \otimes_{A^0} A^{0!*} \to A^{0*},$$

where $A^{i!}$ is a finite dimensional A^0-module, for $i \geq 0$. Then $A^{!j} \cong A^{j!}$.

Proof:

Let $\widehat{A^{i!*}}$ denote a projective resolution of ${}_{A^0}A^{i!*}$. Since ${}_{A^0}A$ is flat, $A \otimes_{A^0} \widehat{A^{i!*}}$ is a projective resolution of $A \otimes_{A^0} A^{i!*}$, and we can therefore manufacture a complex of complexes

$$... \to A \otimes_{A^0} \widehat{A^{2!*}} \to A \otimes_{A^0} \widehat{A^{1!*}} \to A \otimes_{A^0} \widehat{A^{0!*}},$$

whose total complex is quasi-isomorphic to A^{0*}. Applying $Hom_A(-, A^{0*})$ to our projective resolution of A^{0*} gives a complex, with zero differential, of complexes

$$... \to Hom_{A^0}(\widehat{A^{2!*}}, A^{0*}) \to Hom_{A^0}(\widehat{A^{1!*}}, A^{0*}) \to Hom_{A^0}(\widehat{A^{0!*}}, A^{0*}).$$

Because A^{0*} is an injective A^0-module, the functor $Hom_{A^0}(-, A^{0*})$ is exact. Our cohomology complex is therefore quasi-isomorphic to the complex

$$... \to Hom_{A^0}(A^{2!*}, A^{0*}) \to Hom_{A^0}(A^{1!*}, A^{0*}) \to Hom_{A^0}(A^{0!*}, A^{0*}),$$

all of whose differentials are zero. This is isomorphic to the complex

$$... \to A^{2!} \to A^{1!} \to A^{0!},$$

all of whose differentials are zero. For this reason $A^{j!} \cong A^{!j}$. $\square$

Definition 7 *For a $\mathbb{Z}_+$-graded algebra A, let $A^\dagger = Ext^*_A(A^0, A^0)$ denote the homological dual of A, taken with respect to A^0.*

Lemma 8 *([29], Theorem 2.5) Suppose that A is a $\mathbb{Z}_+$-graded algebra, and that ${}_A A^0$ possesses a linear resolution,*

$$\to A \otimes_{A^0} A^{2\dagger} \to A \otimes_{A^0} A^{1\dagger} \to A \otimes_{A^0} A^{0\dagger} \to A^0,$$

where $A^{i\dagger}$ is a finite dimensional projective A^0-module, for $i \geq 0$. Then $A^{\dagger j} \cong A^{j\dagger}$, and $A^\dagger$ is generated in degrees zero and one, modulo quadratic relations. $\square$

Recall a $\mathbb{Z}_+$-graded algebra A is *Koszul* if A^0 is semisimple, and possesses a linear projective resolution. For a Koszul algebra A, the algebra $A^! \cong A^\dagger$ is also a Koszul algebra, known as the Koszul dual of A. In this case, $A^{!!} = A$ [3].

Homological duality for Schiver doubles.

When Γ is an affine Dynkin graph of type A, and $n \geq r$, we prove a homological duality between $\mathcal{D}_\Gamma(n,r)$, and $\Pi_\Gamma(n,r)$.

For a quiver Q, the algebra P_Q is Koszul, its dual being the path algebra kQ^{op}. Therefore, there exists a differential kQ-$P_{Q^{op}}$-bimodule, the Koszul differential bimodule,

$$kQ \bigotimes_{kV} P^*_{Q^{op}}.$$

We may identify $P^*_{Q^{op}}$ with P_Q, as a kV-kV-bimodule. Our differential bimodule can therefore otherwise be written $kQ \bigotimes_{kV} P_Q$. Restricting on the left, we obtain a complex which provides a linear projective resolution of kQ-modules,

$$kQ \bigotimes_{kV} P_Q \twoheadrightarrow kV.$$

In the same way, there is a differential P_Q-kQ^{op}-bimodule $P_Q \bigotimes_{kV} kQ$, restriction of which provides a linear projective resolution of P_Q-modules,

$$P_Q \bigotimes_{kV} kQ \twoheadrightarrow kV.$$

We have the following generalisation of these relations. Here, we write $\bigvee_V(n,r)$ (dual to $\bigwedge_V(n,r)$) for the invariant subspace of the action of Σ_r on $(kV^{\oplus n})^{\otimes r}$ via signed permutations:

Lemma 9 *Let $n \geq r$. Let Q be an orientation of a connected graph Γ, so that each vertex is the tail and source of precisely one arrow. Thus, Γ is an affine Dynkin graph of type A.*

Then there exists a differential $\mathcal{T}_Q(n,r)$-$\mathcal{S}_{Q^{op}}(n,r)$-bimodule,

$$\mathcal{T}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{S}_Q(n,r).$$

Restricting on the left, we obtain a complex which provides a linear resolution of left $\mathcal{T}_Q(n,r)$-modules,

$$\mathcal{T}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{S}_Q(n,r) \twoheadrightarrow \bigvee{}_V(n,r).$$

There exists a differential $\mathcal{S}_Q(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodule,

$$\mathcal{S}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r).$$

Restricting on the left, we obtain a complex which provides a linear projective resolution of left $\mathcal{S}_Q(n,r)$-modules,

$$\mathcal{S}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r) \twoheadrightarrow \mathcal{S}_V(n,r).$$

Proof:

We prove the existence of the second of these bimodules. The first can be observed similarly.

Let $P_Q \wr \Sigma_r$ be the wreath product of the super-algebra P_Q, with Σ_r. Let $kQ \wr \Sigma_r$ be the wreath product of the associative algebra kQ, with Σ_r. Wreathing the Koszul complex for P_Q with Σ_r, we obtain a differential bimodule,

$$P_Q \wr \Sigma_r \bigotimes_{kV \wr \Sigma_r} kQ \wr \Sigma_r,$$

providing a left resolution of $kV \wr \Sigma_r$. Applying $Hom_{P_{Q^{op}} \wr \Sigma_r}(-, (P_{Q^{op}}^{\oplus n})^{\otimes r})$ functorially on the left, and $Hom_{kQ^{op} \wr \Sigma_r}((kQ^{op \oplus n})^{\otimes r}, -)$ on the right, we obtain a differential $\mathcal{S}_Q(n,r)$-$\mathcal{T}_{Q^{op}}(n,r)$-bimodule,

$$(P_Q^{\oplus n})^{\otimes r} \bigotimes_{kV \wr \Sigma_r} (kQ^{\oplus n})^{\otimes r},$$

providing a left resolution of $\mathcal{S}_V(n,r)$. This differential bimodule is isomorphic to the differential bimodule,

$$\mathcal{S}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r).$$

Its restriction on the left is a complex with a filtration whose top section is $\mathcal{S}_V(n,r)$, concentrated in degree zero, and whose other sections are sums of tensor products over k of exact Koszul complexes of the classical form

$$\bigoplus_{j=0}^{r_v} \Big(\mathcal{S}_v(n,j) \otimes \bigvee{}_v(n, r_v - j)\Big) \cong \bigoplus_{j=0}^{r_v} \Big(S^i(M) \otimes \bigwedge{}^{r_v - j}(M)\Big)^*,$$

where $M = End_k(k^n)$ (cf. [26], Theorem 180). $\square$

Recall that $\mathcal{D}_\Gamma(n,r)$ is realised as a component of a double of the superbialgebra,

$$\mathcal{S}_Q(n) = \bigoplus_{r\geq 0} \mathcal{S}_Q(n,r),$$

where Q is some orientation of the graph Γ. The following lemma describes a triangular decomposition for Schiver doubles associated to affine Dynkin quivers of type A.

Lemma 10 *Let $n \geq r$. Let Q be an orientation of a connected graph Γ, so that each vertex is the tail and source of precisely one arrow. Let Q^{op} be the orientation of Γ, opposite to that of Q. Then,*

$$\mathcal{D}_\Gamma(n,r) \cong \mathcal{S}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{S}_{Q^{op}}(n,r),$$

$$\Pi_\Gamma(n,r) \cong \mathcal{T}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_{Q^{op}}(n,r),$$

as $\mathcal{S}_V(n,r)$-$\mathcal{S}_V(n,r)$-bimodules.

Proof:

The hypotheses of the lemma allow us to identify V, E.

There is a natural subalgebras P_Q (respectively $P_{Q^{op}}$) of ZZ_Γ, generated by all the arrows pointing clockwise (respectively anticlockwise). Multiplication between these subalgebras brings forth a triangular decomposition,

$$ZZ_\Gamma = P_Q \bigotimes_{kV} P_{Q^{op}}.$$

Wreathing this up, we obtain a decomposition,

$$ZZ_\Gamma \wr \Sigma_r = P_Q \wr \Sigma_r \bigotimes_{kV\wr\Sigma_r} P_{Q^{op}} \wr \Sigma_r.$$

by Theorem 3, we have an isomorphism

$$\phi : \xi\mathcal{D}_\Gamma(n,r)\xi \cong \xi\mathcal{S}_Q(n,r)\xi \bigotimes_{\xi\mathcal{S}_V(n,r)\xi} \xi\mathcal{S}_{Q^{op}}(n,r)\xi,$$

where $\xi = \xi_V(n,r)$. We may now observe that the summand

$$\bigotimes_{v\in V} \left(\mathcal{S}(n,a_v) \otimes \bigvee(n,b_v) \otimes \mathcal{A}(n,c_v) \otimes \bigwedge(n,d_v)\right),$$

of $\mathcal{D}_\Gamma(n,r)$ corresponds under ϕ to the summand

$$\left(\bigotimes_{v\in V} \mathcal{S}(n,a_v+d_v) \otimes \bigvee(n,c_v+b_v)\right) \bigotimes_{\mathcal{S}_V(n,r)}$$

$$\left(\bigotimes_{v\in V}\mathcal{S}(n,a_v+b_v)\otimes\bigvee(n,c_v+d_v)\right),$$

of $\mathcal{S}_Q(n,r)\bigotimes_{\mathcal{S}_V(n,r)}\mathcal{S}_{Q^{op}}(n,r)$, where $\sum a_v+c_v+b_v+d_v=r$. But these two summands are isomorphic, since

$$\bigvee(n,b_v)\cong\bigvee(n,b_v)\bigotimes_{\mathcal{S}(n,b_v)}\mathcal{S}(n,b_v),$$

$$\mathcal{A}(n,c_v)\cong\bigvee(n,c_v)\bigotimes_{\mathcal{S}(n,c_v)}\bigvee(n,c_v),$$

$$\bigwedge(n,d_v)\cong\mathcal{S}(n,d_v)\bigotimes_{\mathcal{S}(n,d_v)}\bigvee(n,d_v),$$

as $\mathcal{S}_V$-$\mathcal{S}_V$-bimodules. The second of these isomorphisms is the dual of the Ringel self-duality isomorphism

$$\mathcal{S}(n,c_v)\cong Hom_{\mathcal{S}(n,c_v)}(\bigwedge(n,c_v),\bigwedge(n,c_v)),$$

due to S. Donkin ([9], chapter 4).

The triangular decompostion for $\Pi_\Gamma(n,r)$ is proven in the same manner. It is in fact easier, since there is no need to invoke Ringel duality. □

Lemma 11 *Let Q be an orientation of a connected graph Γ, so that each vertex is the tail and source of precisely one arrow. Then there is an isomorphism of $\mathcal{S}_V(n,r)$-$\mathcal{S}_V(n,r)$-bimodules,*

$$\mathcal{T}_Q(n,r)\bigotimes_{\mathcal{S}_V(n,r)}\mathcal{S}_{Q^{op}}(n,r)\cong\mathcal{S}_{Q^{op}}(n,r)\bigotimes_{\mathcal{S}_V(n,r)}\mathcal{T}_Q(n,r).$$

Proof:

Since each vertex is the tail and source of precisely one arrow, we have $kQ\bigotimes_{kV}P_{Q^{op}}\cong kQ^{\oplus 2}$, and $P_{Q^{op}}\bigotimes_{kV}kQ\cong kQ^{\oplus 2}$, as kV-kV-bimodules. Therefore

$$kQ\bigotimes_{kV}P_{Q^{op}}\cong P_{Q^{op}}\bigotimes_{kV}kQ,$$

as kV-kV-bimodules. Schurifying this isomorphism, by wreathing it with Σ_r, before applying the functor $Hom_{kV\wr\Sigma_r}(-,(kV^{\oplus n})^{\otimes r})$ on the left, and the functor $Hom_{kV\wr\Sigma_r}((kV^{\oplus n})^{\otimes r},-)$ on the right, we obtain an isomorphism,

$$(kQ^{\oplus n})^{\otimes r}\bigotimes_{kV\wr\Sigma_r}(P_{Q^{op}}^{\oplus n})^{\otimes r}\cong(P_{Q^{op}}^{\oplus n})^{\otimes r}\bigotimes_{kV\wr\Sigma_r}(kQ^{\oplus n})^{\otimes r},$$

which can be identified as an isomorphism,

$$\mathcal{T}_Q(n,r)\bigotimes_{\mathcal{S}_V(n,r)}\mathcal{S}_{Q^{op}}(n,r)\cong\mathcal{S}_{Q^{op}}(n,r)\bigotimes_{\mathcal{S}_V(n,r)}\mathcal{T}_Q(n,r).\square$$

Under the specialised hypotheses of the above lemma, we prove a homological duality for the Schiver doubles $\mathcal{D}_\Gamma(n,r)$.

Proposition 12 *Let $n \geq r$. Let Γ be an affine Dynkin graph of type A. Then there exists a differential $\Pi_\Gamma(n,r)$- $\mathcal{D}_\Gamma(n,r)$-bimodule,*

$$\Pi_\Gamma(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{D}_\Gamma(n,r)$$

Restricting on the left, we obtain a linear resolution

$$\Pi_\Gamma(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{D}_\Gamma(n,r) \twoheadrightarrow \mathcal{A}_V(n,r),$$

of $\mathcal{A}_V(n,r)$ by left $\Pi_\Gamma(n,r)$-modules.

There exists a differential $\mathcal{D}_\Gamma(n,r)$- $\Pi_\Gamma(n,r)$-bimodule,

$$\mathcal{D}_\Gamma(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \Pi_\Gamma(n,r)$$

Restricting on the left, we obtain a linear projective resolution

$$\mathcal{D}_\Gamma(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \Pi_\Gamma(n,r) \twoheadrightarrow \mathcal{S}_V(n,r),$$

of $\mathcal{S}_V(n,r)$ by left $\mathcal{D}_\Gamma(n,r)$-modules.

Proof:

We prove the existence of the differential bimodule $\Pi_\Gamma(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{D}_\Gamma(n,r)$, and prove that its left restriction gives a linear projective resolution. The analogous facts for $\mathcal{D}_\Gamma(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \Pi_\Gamma(n,r)$ can be proven entirely analogously.

I. There exists such a differential $\mathcal{S}_V(n,r)$-$\mathcal{S}_V(n,r)$-bimodule.

We have a differential $\mathcal{S}_V(n,r)$-$\mathcal{S}_V(n,r)$-bimodule,

$$\mathcal{T}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{S}_Q(n,r).$$

Tensoring the above module with the corresponding one for Q^{op}, we obtain a differential bimodule,

$$\left(\mathcal{T}_{Q^{op}}(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{S}_{Q^{op}}(n,r)\right) \bigotimes_{\mathcal{S}_V(n,r)} \left(\mathcal{T}_Q(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{S}_Q(n,r)\right).$$

By Lemma 11, this is isomorphic to a differential bimodule,

$$\left(\mathcal{T}_{Q^{op}}(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{T}_Q(n,r)\right) \bigotimes_{\mathcal{S}_V(n,r)} \left(\mathcal{S}_{Q^{op}}(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{S}_Q(n,r)\right)$$

which is isomorphic, by Lemma 10, to the differential bimodule,

$$\Pi_\Gamma(n,r) \bigotimes_{\mathcal{S}_V(n,r)} \mathcal{D}_\Gamma(n,r). \tag{1}$$

This maps onto the $\mathcal{S}_V(n,r)$-$\mathcal{S}_V(n,r)$-bimodule $\bigvee_V(n,r)\bigotimes_{\mathcal{S}_V(n,r)}\bigvee_V(n,r)$. Therefore, by the dual of the Ringel self-duality isomorphism, 1 maps onto $\mathcal{A}_V(n,r)$.

Why is the complex a resolution of $\mathcal{A}_V(n,r)$? Because for $r = r_1 + r_2$, the $\mathcal{S}_V(n,r)$-module $(\bigwedge_V(n,r_1)\otimes\mathcal{S}_V(n,r_2))$ is filtered by standard modules. Therefore the functor

$$(\bigwedge{}_V(n,r_1)\otimes\mathcal{S}_V(n,r_2))\bigotimes_{\mathcal{S}_V(n,r)} -$$

is exact on the category of Δ-filtered $\mathcal{S}_V(n,r)$-modules ([9], appendix $A4$), and both the rows and columns of the double complex

$$\left(\mathcal{T}_{Q^{op}}(n,r)\bigotimes_{\mathcal{S}_V(n,r)}\mathcal{S}_{Q^{op}}(n,r)\right)\bigotimes_{\mathcal{S}_V(n,r)}\left(\mathcal{T}_Q(n,r)\bigotimes_{\mathcal{S}_V(n,r)}\mathcal{S}_Q(n,r)\right)$$

are exact in nonzero degrees. Thus the total complex provides a resolution of $\mathcal{A}_V(n,r)$, by the acyclic assembly lemma ([28], 2.7.3).

II. The natural actions of $\Pi_\Gamma(n,r)$ and $\mathcal{D}_\Gamma(n,r)$ commute with the differential on $*$. Therefore, 1 is a differential $\Pi_\Gamma(n,r)$-$\mathcal{D}_\Gamma(n,r)$-bimodule.

It is enough to check this over a complete discrete valuation ring R, lifting k (all we have said carries over to such a ring); it is therefore enough to check it over the field of fractions K of R, a field of characteristic zero; by the Morita equivalence of theorem 3, it is enough to check it for the wreath products, $ZZ_\Gamma^+\wr\Sigma_r$, and $\Pi_\Gamma^+\wr\Sigma_r$. However, the complex in this case,

$$\Pi_\Gamma^+\wr\Sigma_r\bigotimes_{KV\wr\Sigma_r}ZZ_\Gamma^{+*}\wr\Sigma_r,$$

is just a wreathing of the the classical Koszul bimodule,

$$\Pi_\Gamma^+\bigotimes_{KV}ZZ_\Gamma^{+*},$$

where the actions of Π_Γ^+, ZZ_Γ^+ certainly commute with the differential.

III. 1 defines a projective resolution of right $\mathcal{D}_\Gamma(n,r)$-modules, because $\Pi_\Gamma(n,r)$ is a projective $\mathcal{S}_V(n,r)$-module. $\square$

We release the following consequences of the above proposition.

Corollary 13 *Let $n\geq r$. Let Γ be an affine Dynkin graph of type A. Then $\mathcal{D}_\Gamma(n,r)$ is generated in degrees zero and one, modulo only quadratic relations.* $\square$

Corollary 14 *Let $n \geq r$. Let Γ be an affine Dynkin graph of type A. Then we have algebra isomorphisms $\Pi_\Gamma(n,r) \cong \mathcal{D}_\Gamma(n,r)^\dagger$, and $\mathcal{D}_\Gamma(n,r) \cong \Pi_\Gamma(n,r)^!$.*

Proof:

The first formula follows from Lemma 8 and Proposition 12. The second follows from Lemma 6 and Proposition 12. □

Remark 15 Applying the same argument as we did to prove Corollary 14, only using right resolutions instead of left resolutions, and applying duality, we find that $\Pi_\Gamma(n,r) \cong \mathcal{D}_\Gamma(n,r)^!$, and $\mathcal{D}_\Gamma(n,r) \cong \Pi_\Gamma(n,r)^\dagger$.

Theorem 16 *Let $n \geq r$. Let Γ be an affine Dynkin graph of type A. There is an equivalence of derived categories,*

$$D(mod - \mathcal{D}_\Gamma(n,r)) \to D_{dg}(\Pi_\Gamma(n,r)),$$

where $\Pi_\Gamma(n,r)$ is considered to be a dg algebra, whose grading is inherited from the grading on Π_Γ by path length, and whose differential is zero.

Proof:

The action of $\Pi_\Gamma(n,r)$ on the complex

$$Hom_{\mathcal{S}_V(n,r)}(\Pi_\Gamma(n,r), \mathcal{D}_\Gamma(n,r)),$$

resolving the $\mathcal{D}_\Gamma(n,r)$-module $\mathcal{S}_V(n,r)$, defines a quasi-isomorphism of dg algebras,

$$\Pi_\Gamma(n,r) \to$$

$$Hom^\bullet_{\mathcal{D}_\Gamma(n,r)}\left(Hom_{\mathcal{S}_V(n,r)}(\Pi_\Gamma(n,r), \mathcal{D}_\Gamma(n,r)), Hom_{\mathcal{S}_V(n,r)}(\Pi_\Gamma(n,r), \mathcal{D}_\Gamma(n,r))\right).$$

Thus, $Hom^\bullet_{\mathcal{D}_\Gamma(n,r)}$ above is a formal A_∞-algebra, and by B. Keller's theory ([16], section 3), there is a derived equivalence,

$$D_\infty(\mathcal{D}_\Gamma(n,r)) \to D_\infty(\Pi_\Gamma(n,r)),$$

taking the regular representation $\mathcal{D}_\Gamma(n,r)$ to

$$Hom_{\mathcal{D}_\Gamma(n,r)}(\mathcal{S}_V(n,r), \mathcal{D}_\Gamma(n,r)) \cong \mathcal{A}_V(n,r).$$

Here, $\mathcal{D}_\Gamma(n,r)$ is considered to be a unital dg algebra, concentrated entirely in degree zero. The derived equivalences,

$$D(mod - \mathcal{D}_\Gamma(n,r)) \to D_\infty(\mathcal{D}_\Gamma(n,r)),$$

$$D_{dg}(\Pi_\Gamma(n,r)) \to D_\infty(\Pi_\Gamma(n,r)),$$

due to K. Lefèvre ([17], 4.1.3.1, 4.1.3.9), complete the proof of the theorem. □

Remark 17 If k has characteristic zero, or characteristic greater than r, then $\mathcal{S}_V(n,r)$ is semisimple, and the algebras $\mathcal{D}_\Gamma(n,r)$ and $\Pi_\Gamma(n,r)$ are in Koszul duality. If the characteristic of k is smaller than r, then $\mathcal{S}_V(n,r)$ is not semisimple, so the algebras above cannot be Koszul.

Remark 18 The above theorems should hold under the more general hypothesis that Γ is not an ordinary Dynkin graph. In this situation, R. Martinez-Villa has proved proposition 12 in case $r=1$ [18].

Remark 19 We may define a negative version of the Schiver double, $\mathcal{D}^-_\Gamma(n,r) = \Pi^-_\Gamma(n,r)^!$. This is isomorphic to $\mathcal{D}_\Gamma(n,r)$ as a vector space, but only isomorphic as an algebra if Γ is bipartite. When we form deformations in the sequel, it will be more convenient to work with $\mathcal{D}^-_\Gamma(n,r)$.

Multiplicative extensions.

We give a definition of a multiplicative extension. Examples are group algebras of group extensions, and algebra deformations. We prove that, under favourable conditions, the homological dual of a multiplicative extension is a multiplicative extension of homological duals.

Definition 20 *Let A and B be k-algebras. An algebra C is a* multiplicative extension *of A by B if we have algebra embeddings*

$$i_A : A^0 \hookrightarrow A, \qquad i_C : A^0 \otimes B \hookrightarrow C,$$

and algebra epimorphisms

$$\pi_B : B \twoheadrightarrow B^0 \qquad \pi_C : C \twoheadrightarrow A \otimes B^0,$$

such that

1. *The following diagram commutes:*

$$\begin{tikzcd} & C \arrow[dr, two heads, "\pi_C"] & \\ A^0 \otimes B \arrow[ur, hook, "i_C"] \arrow[rr, "i_A \otimes \pi_B"] & & A \otimes B^0. \end{tikzcd}$$

2. *The left and right actions of B on C are free, and commute.*
3. *We have $C \underset{B}{\otimes} ker(\pi_B) = ker(\pi_B) \underset{B}{\otimes} C = ker(\pi_C)$.*

Remark 21 All our algebras, and algebra homomorphisms are assumed to be unital.

The embeddings of $C \underset{B}{\otimes} ker(\pi_B)$ and $ker(\pi_B) \underset{B}{\otimes} C$ in C implicit in 3, are obtained by applying the exact functors $C \underset{B}{\otimes} -$ and $- \underset{B}{\otimes} C$ to the embedding of $kerB$ in B.

We draw a multiplicative extension of A by B as follows:

$$\begin{tikzcd} & C \arrow[dr, two heads] & \\ A^0 \otimes B \arrow[ur, hook] & & A \otimes B^0. \end{tikzcd}$$

Example 22 Let G be a group containing a normal subgroup N. The group algebra kG is a flat multiplicative extension of $A = kG/N$ by $B = kN$. Thus, kN acts freely on $C = kG$, whilst $i_A : k \hookrightarrow kG/N$ is the unital embedding, and $\pi_B : kG \twoheadrightarrow k$ is the algebra epimorphism, which takes all group elements to $1 \in k$.

Example 23 An infinitesimal (respectively polynomial, or formal) deformation of a unital k-algebra A is a multiplicative central extension of A by $k[x]/x^2$ (respectively by $k[x]$, or $k[[x]]$).

In this paper, we would like to consider multiplicative extensions of positively graded algebras $A = \bigoplus_{i \in \mathbb{Z}_+} A^i$.

Definition 24 *Let A and B be $\mathbb{Z}_+$-graded k-algebras. A $\mathbb{Z}_+$-graded algebra C is a* graded multiplicative extension *of A by B, if it is a multiplicative extension, where A^0, B^0 denote the degree zero parts of A, B, where i_A, π_B are the natural algebra homomorphisms, and where i_C, π_C are graded algebra homomorphisms.*

Remark 25 If C is a graded multiplicative extension of A by B, then the embedding of $A^0 \otimes B$ in C, and the algebra epimorphism $C \to A \otimes B^0$, induce an algebra isomorphism $A^0 \otimes B^0 \cong C^0$.

Theorem 26 *Let C be a graded multiplicative extension,*

$$\begin{tikzcd} & C \arrow[dr, two heads] & \\ A^0 \otimes B \arrow[ur, hook] & & A \otimes B^0. \end{tikzcd}$$

Suppose that $A_{A^0}, B_{B^0}, C_{A^0 \otimes B}$ are projective modules, and that ${}_A A^{0*}$, ${}_B B^{0*}$ possess linear resolutions of the form,

$$... \to A \otimes_{A^0} A^{2!*} \to A \otimes_{A^0} A^{1!*} \to A \otimes_{A^0} A^{0!*} \to A^{0*},$$

$$... \to B \otimes_{B^0} B^{2!*} \to B \otimes_{B^0} B^{1!*} \to B \otimes_{B^0} B^{0!*} \to B^{0*},$$

where $A^{i!}, B^{i!}$ are finite dimensional modules over A^0, B^0, for $i \geq 0$. Then ${}_C C^{0*}$ possesses a linear resolution of the form

$$... \to C \otimes_{C^0} C^{2!*} \to C \otimes_{C^0} C^{1!*} \to C \otimes_{C^0} C^{0!*} \to C^{0*},$$

and $C^!$ is a graded multiplicative extension,

$$\begin{tikzcd} & C^! \arrow[dr, two heads] & \\ B^{!0} \otimes A^! \arrow[ur, hook] & & B^! \otimes A^{!0}. \end{tikzcd}$$

Proof:

Let $\widehat{A^{i!*}}$ denote a projective resolution of ${}_{A^0} A^{i!*}$, and let $\widehat{B^{i!*}}$ denote a projective resolution of ${}_{B^0} B^{i!*}$.

We can form a complex of complexes

$$.. \to A \otimes_{A^0} \widehat{A^{2!*}} \to A \otimes_{A^0} \widehat{A^{1!*}} \to A \otimes_{A^0} \widehat{A^{0!*}}$$

whose total complex is quasi-isomorphic to ${}_A A^{0*}$. Tensoring with $\widehat{B^{0!*}}$, we obtain a complex of double complexes

$$\begin{aligned} ... \to (A \otimes B^0) \underset{A^0 \otimes B^0}{\otimes} (\widehat{A^{2!*}} \otimes \widehat{B^{0!*}}) \to (A \otimes B^0) \underset{A^0 \otimes B^0}{\otimes} (\widehat{A^{1!*}} \otimes \widehat{B^{0!*}}) \\ \to (A \otimes B^0) \underset{A^0 \otimes B^0}{\otimes} (\widehat{A^{0!*}} \otimes \widehat{B^{0!*}}), \end{aligned} \tag{2}$$

of $A \otimes B^0$-modules, whose total complex is quasi-isomorphic to $A^{0!*} \otimes B^{0*}$.

We can also form a complex of complexes

$$.. \to B \otimes_{B^0} \widehat{B^{2!*}} \to B \otimes_{B^0} \widehat{B^{1!*}} \to B \otimes_{B^0} \widehat{B^{0!*}}$$

whose total complex is quasi-isomorphic to ${}_B B^{0*}$. Tensoring with $\widehat{A^{i!*}}$, we obtain a complex of double complexes

$$\begin{aligned} .. \to (A^0 \otimes B) \underset{A^0 \otimes B^0}{\otimes} (\widehat{A^{i!*}} \otimes \widehat{B^{2!*}}) \to (A^0 \otimes B) \underset{A^0 \otimes B^0}{\otimes} (\widehat{A^{i!*}} \otimes \widehat{B^{1!*}}) \\ \to (A^0 \otimes B) \underset{A^0 \otimes B^0}{\otimes} (\widehat{A^{i!*}} \otimes \widehat{B^{0!*}}), \end{aligned}$$

of $A^0 \otimes B$-modules, whose total complex is quasi-isomorphic to $A^{i!*} \otimes B^{0*}$. Applying the exact functor $C \otimes_{A^0 \otimes B} -$, we obtain a complex

$$.. \to \{\widehat{A^{i!*}} \otimes \widehat{B^{2!*}}\} \to \{\widehat{A^{i!*}} \otimes \widehat{B^{1!*}}\} \to \{\widehat{A^{i!*}} \otimes \widehat{B^{0!*}}\} \tag{3}$$

of double complexes of C-modules, where $\{M\} = C \underset{A^0 \otimes B^0}{\otimes} M$. Since

$$C \underset{A^0 \otimes B}{\otimes} (A^0 \otimes B^0) \cong A \otimes B^0,$$

the total complex is quasi-isomorphic to $(A \otimes B^0) \underset{A^0 \otimes B^0}{\otimes} (\widehat{A^{i!*}} \otimes B^{0*})$.

Splicing together complexes 2 and 3, we form a double complex with linear differentials,

$$\begin{tikzcd}
\cdot\cdot \arrow[r] & \{\widehat{A^{2!*}} \otimes \widehat{B^{0!*}}\} \arrow[r] & \{\widehat{A^{1!*}} \otimes \widehat{B^{0!*}}\} \arrow[r] & \{\widehat{A^{0!*}} \otimes \widehat{B^{0!*}}\}\ , \\
\cdot\cdot \arrow[r] & \{\widehat{A^{2!*}} \otimes \widehat{B^{1!*}}\} \arrow[r] \arrow[u] & \{\widehat{A^{1!*}} \otimes \widehat{B^{1!*}}\} \arrow[r] \arrow[u] & \{\widehat{A^{0!*}} \otimes \widehat{B^{1!*}}\} \arrow[u] \\
\cdot\cdot \arrow[r] & \{\widehat{A^{2!*}} \otimes \widehat{B^{2!*}}\} \arrow[r] \arrow[u] & \{\widehat{A^{1!*}} \otimes \widehat{B^{2!*}}\} \arrow[r] \arrow[u] & \{\widehat{A^{0!*}} \otimes \widehat{B^{2!*}}\} \arrow[u] \\
\cdot\cdot & \cdot\cdot \arrow[u] & \cdot\cdot \arrow[u] & \cdot\cdot \arrow[u]
\end{tikzcd}$$

of double complexes of C-modules, whose total complex is quasi-isomorphic to $C^{0*} = A^{0*} \otimes B^{0*}$. Applying the functor $Hom_C(-, C^{0*})$, we obtain a double complex

$$\begin{tikzcd}
\cdot\cdot & {[\widehat{A^{2!*}} \otimes \widehat{B^{0!*}}]} \arrow[l] \arrow[d] & {[\widehat{A^{1!*}} \otimes \widehat{B^{0!*}}]} \arrow[l] \arrow[d] & {[\widehat{A^{0!*}} \otimes \widehat{B^{0!*}}]} \arrow[l] \arrow[d] \\
\cdot\cdot & {[\widehat{A^{2!*}} \otimes \widehat{B^{1!*}}]} \arrow[l] \arrow[d] & {[\widehat{A^{1!*}} \otimes \widehat{B^{1!*}}]} \arrow[l] \arrow[d] & {[\widehat{A^{0!*}} \otimes \widehat{B^{1!*}}]} \arrow[l] \arrow[d] \\
\cdot\cdot & {[\widehat{A^{2!*}} \otimes \widehat{B^{2!*}}]} \arrow[l] \arrow[d] & {[\widehat{A^{1!*}} \otimes \widehat{B^{2!*}}]} \arrow[l] \arrow[d] & {[\widehat{A^{0!*}} \otimes \widehat{B^{2!*}}]} \arrow[l] \arrow[d] \\
\cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot
\end{tikzcd}$$

of double complexes, where $[M]$ denotes $Hom_{A^0 \otimes B^0}(M, C^{0*})$. The cohomology of the total complex is, by definition, $Ext^*_C(C^{0*}, C^{0*})$. Since the original resolutions of A^{0*}, B^{0*} were linear, so are the differentials in the above diagram. However, all terms in the above double complex are concentrated in graded degree zero, and therefore the differentials are all zero. Furthermore,

$$[\widehat{A^{i!*}} \otimes \widehat{B^{j!*}}] \cong Hom_{A^0 \otimes B^0}(\widehat{A^{i!*}} \otimes \widehat{B^{j!*}}, C^{0*}),$$

which is quasi-isomorphic to

$$Hom_{A^0 \otimes B^0}(A^{i!*} \otimes B^{j!*}, A^{0*} \otimes B^{0*}) \cong$$

$$Hom_{A^0 \otimes B^0}(A^0 \otimes B^0, A^{i!} \otimes B^{j!}) \cong A^{i!} \otimes B^{j!},$$

since $A^{0*} \otimes B^{0*}$ is injective as an $A^0 \otimes B^0$-module. It follows that this double complex is quasi-isomorphic to the double complex

$$\begin{tikzcd}
\cdot\cdot & A^{2!} \otimes B^{0!} \arrow[l] \arrow[d] & A^{1!} \otimes B^{0!} \arrow[l] \arrow[d] & A^{0!} \otimes B^{0!} \arrow[l] \arrow[d] \\
\cdot\cdot & A^{2!} \otimes B^{1!} \arrow[l] \arrow[d] & A^{1!} \otimes B^{1!} \arrow[l] \arrow[d] & A^{0!} \otimes B^{1!} \arrow[l] \arrow[d] \\
\cdot\cdot & A^{2!} \otimes B^{2!} \arrow[l] \arrow[d] & A^{1!} \otimes B^{2!} \arrow[l] \arrow[d] & A^{0!} \otimes B^{2!} \arrow[l] \arrow[d] \\
\cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot
\end{tikzcd} \tag{4}$$

all of whose differentials are zero. Taking the homology of the total complex, we realise that $Ext^*_C(C^{0*}, C^{0*})$ is isomorphic (as a vector space) to

$$\bigoplus_{i,j \in \mathbb{Z}_{\geq 0}} A^{i!} \otimes B^{j!} \cong Ext^*_A(A^{0*}, A^{0*}) \otimes Ext^*_B(B^{0*}, B^{0*}),$$

as the statement of the theorem predicts.

What about multiplicative structure ? We have $A^{0!} \cong End_{A^0}(A^{0*}) \cong A^0$, and $B^{0!} \cong B^0$. The surjection $C \twoheadrightarrow A \otimes B^0$ defines an exact functor

$$F : A \otimes B^0 - mod \to C - mod$$

between module categories, which extends to an exact functor between derived categories

$$D(A \otimes B^0 - mod) \to D(C - mod).$$

We have $F(A^{0*} \otimes B^{0*}) = C^{0*}$. For any algebra $\mathcal{A}$, we have an isomorphism

$$Ext^i_{\mathcal{A}}(M, M) \cong Hom_{D(\mathcal{A})}(M, M[i]),$$

and we therefore recover an algebra homomorphism,

$$A^! \otimes B^{!0} \cong Ext^*_{A \otimes B^0}(A^{0*} \otimes B^{0*}, A^{0*} \otimes B^{0*}) \hookrightarrow Ext^*_C(C^{0*}, C^{0*}) \cong C^!.$$

Note that $A^!$ acts along horizontal lines $A^! \otimes B^{i!}$ in double complex 4, whose total homology is $C^!$. Since $A^!_{A^!}$ and ${}_{A^!}A^!$ are free, $C^!_{A^!}$ and ${}_{A^!}C^!$ are also free.

Restriction from C to $A^0 \otimes B$ defines an exact functor between module categories, which extends to an exact functor between derived categories

$$D(C - mod) \to D(A^0 \otimes B - mod),$$

taking C^{0*} to $A^{0*} \otimes B^{0*}$. We therefore have an algebra homomorphism

$$C^! = Ext^*_C(C^0, C^0) \to Ext^*_{A^0 \otimes B}(A^{0*} \otimes B^{0*}, A^{0*} \otimes B^{0*}) \cong A^{!0} \otimes B^!.$$

From our homology double complex, it is visible that this map is surjective, and the kernel is equal to $A^{!>0} \otimes B^!$. This completes the proof of the theorem. $\square$

Corollary 27 *Suppose that an algebra C is a graded multiplicative extension of A by B, where A and B are Koszul algebras. Then C is Koszul, and its Koszul dual is a multiplicative extension of the Koszul dual of B, by the Koszul dual of A.* $\square$

Given an algebra A, let $\mathcal{J}(A)$ denote the Jacobson radical of A. The following theorem is proved entirely analogously to Theorem 26, only using minimal resolutions instead of linear ones.

Theorem 28 *Let A, B be finite dimensional algebras, and*

$$A^0 = A/\mathcal{J}(A) \qquad B^0 = B/\mathcal{J}(B).$$

Suppose that C is a multiplicative extension of A by B; let $C^0 = A^0 \otimes B^0$. Then there is a dg algebra D, which is a multiplicative extension of $Ext^*_B(B^0, B^0)$ by $Ext^*_A(A^0, A^0)$, whose cohomology is isomorphic to $Ext^*_C(C^0, C^0)$. $\square$

We obtain the following corollary, by considering the case when C is the group algebra of an extension of two p-groups.

Corollary 29 *Let*

$$1 \to P_1 \to Q \to P_2 \to 1$$

be an extension of p-groups. Then there is a dg algebra D, which is a multiplicative extension of $H^*(P_1)$ by $H^*(P_2)$, whose cohomology is isomorphic to $H^*(Q)$. $\square$

Remark 30 We have stated the results of this sections for algebras over a field k. The same results hold, with suitable refinements, for R-free algebras over a commutative ring R.

On M. Peach's Rhombal algebras.

Rhombal algebras are locally finite dimensional algebras, defined by quiver and relations, geometrically described by planar rhombi [20]. They are radically graded symmetric algebras of Loewy length 5. There are infinite families of Rhombal algebras, the members of which are all derived equivalent. The derived equivalences are non-trivial, given by compositions of two-term tilting complexes. The "unramified region" of a weight two symmetric group block, is Morita equivalent to a region of some Rhombal algebra.

We do not describe Peach's definition of a Rhombal algebra here, but a more general construction, which generalises his notion from weight two to arbitrary weight $r \in \mathbb{N}$. In a paper with Joseph Chuang, we study these algebras, and the associated combinatorics, in detail [7].

The Heisenberg Lie super-algebra $\mathcal{H}_r$ is isomorphic to $kz + W$, where W is an orthogonal vector space of dimension $2r$ and parity one, z is a central element of parity zero, and there is the commutation relation $[w_1, w_2] =< w_1, w_2 > z$. Its universal enveloping algebra $U(\mathcal{H}_r)$ can be described as follows.

Generators $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_r, z$

Relations $\alpha_i\alpha_j = \beta_i\beta_j = 0$, all i, j; $\alpha_i\beta_j + \beta_j\alpha_i = \delta_{i,j}z$, all i, j.

Thus $U(\mathcal{H}_r)$ is a noncommutative deformation of the super-polynomial ring $k[z|a_1, ..., a_r, b_1, ..., b_r]$.

Over the complex numbers, the torus $T(r) = (\mathbb{R}/\mathbb{Z})^r$ acts on W, inducing an action on $\mathcal{H}_r$, and consequently on $U(\mathcal{H}_r)$, by

$$(x_1, ..., x_r).\alpha_i = e^{2\pi x_i i}\alpha_i,$$

$$(x_1, ..., x_r).\beta_i = e^{-2\pi x_i i}\beta_i.$$

Since $T(r)$ is a compact group, it acts semisimply on complex representations. Its group of characters is $Hom(T(r), \mathbb{C}^*) \cong \mathbb{Z}^r$.

Lemma 31 *The category of finite dimensional graded representations of the semidirect product $U(\mathcal{H}_r) \rtimes T(r)$ is equivalent to the category of finite dimensional graded modules over the algebra $\mathcal{U}_r$, given by generators,*

$$\{e_x, x \in \mathbb{Z}^r\},$$

in degree zero; by generators,

$$\{\alpha_{x,i}, \beta_{x,i}, x \in \mathbb{Z}^r, 1 \leq i \leq r\},$$

in degree one, by a single generator z, in degree two; and relations,

$$e_x\alpha_{y,i} = \delta_{x,y}\alpha_{y,i},$$

$$e_x\beta_{y,i} = \delta_{x,y}\beta_{y,i},$$

$$\alpha_{y,i}e_x = \delta_{x,(y_1,...,y_{i-1},y_i+1,y_{i+1},...,y_r)}\alpha_{y,i},$$

$$\beta_{y,i}e_x = \delta_{x,(y_1,...,y_{i-1},y_i-1,y_{i+1},...,y_r)}\beta_{y,i},$$

$$\alpha_{x,i}\alpha_{(x_1,...,x_{i-1},x_i+1,x_{i+1},...,x_r),j} + \alpha_{x,j}\alpha_{(x_1,...,x_{j-1},x_j+1,x_{j+1},...,x_r),i} = 0,$$

$$\beta_{x,i}\beta_{(x_1,...,x_{i-1},x_i-1,x_{i+1},...,x_r),j} + \beta_{x,j}\beta_{(x_1,...,x_{j-1},x_j-1,x_{j+1},...,x_r),i} = 0,$$

$$\alpha_{x,i}\beta_{(x_1,...,x_{i-1},x_i+1,x_{i+1},...,x_r),j} + \beta_{x,j}\alpha_{(x_1,...,x_{j-1},x_j+1,x_{j+1},...,x_r),i} = 0,$$

for $i \neq j$,

$$\alpha_{x,i}\beta_{(x_1,...,x_{i-1},x_i+1,x_{i+1},...,x_r),i} + \beta_{x,i}\alpha_{(x_1,...,x_{i-1},x_i+1,x_{i+1},...,x_r),i} = e_x z,$$

$$\alpha_{x,i}\alpha_{(x_1,...,x_{i-1},x_i+1,x_{i+1},...,x_r),i} = 0,$$

$$\beta_{x,i}\beta_{(x_1,...,x_{i-1},x_i-1,x_{i+1},...,x_r),i} = 0,$$

and the condition that z *is central.* $\Box$

The action of the symmetric group Σ_r has a natural action on $\mathbb{Z}^r$, which lifts to an action on $\mathcal{U}_r$ as super-algebra isomorphisms. In this way, we have $e_x^\sigma = e_{x^\sigma}$, $\alpha_{x,i}^\sigma = \alpha_{x^\sigma,i^\sigma}$, and $\beta_{x,i}^\sigma = \beta_{x^\sigma,i^\sigma}$.

Let us define a subset $\mathcal{X}$ of $\mathbb{Z}^r$ to be *connected*, if for any $x, y \in \mathcal{X}$, there exists a sequence $x = x_0, x_1, ..., x_l = y$, such that $x_i \in \mathcal{X}$ and for each $i \geq 1$, we have $x_{i+1} = (x_{i,1}, ..., x_{i,j-1}, x_{i,j} \pm 1, x_{i,j+1}, ..., x_{i,r})$, for some $j = j(i)$.

For $x \in \mathbb{Z}^r$, let us define $x[i] = (x_1 + i, ..., x_r + i) \in \mathbb{Z}^r$.

Definition 32 *A subset* $\mathcal{X} \subset \mathbb{Z}^r$ *is Cubist, if it is a maximal connected subset, such that* $x \in \mathcal{X}$ *implies* $x[i]$ *is not in* $\mathcal{X}$, *for* $i \neq 0$.

For any Cubist subset $\mathcal{X}$ *of* $\mathbb{Z}^r$, *we define the Cubist algebra* $U_{\mathcal{X}}$, *to be the algebra* $\mathcal{U}_r$, *modulo the ideal generated by* $\{e_y, y \in \mathbb{Z}^r \backslash \mathcal{X}\}$.

The Cubist algebra $U_{\mathcal{X}}$ has graded simple modules in one-one correspondence with $\mathcal{X}$. In case $r = 3$, we recover the Rhombal algebras.

Polynomial deformations of Schiver doubles.

Observing the definition of Rhombal algebras, in the awareness that Rhombal algebras in case $r = 3$, as well as Schiver doubles, resemble blocks of symmetric groups, the sensitive reader will ask himself whether there is a general construction giving both families of algebras at certain limits. This is indeed the case, as the following theorem demonstrates. Let $\mathcal{U}_r \rtimes \Sigma_r$ denote the semidirect product of the super-algebra $\mathcal{U}_r$ with Σ_r.

Theorem 33 *Let* $n \geq r$. *Let* Γ *be an ordinary, or affine Dynkin graph, of type A. There exist natural polynomial deformations* $\widetilde{\mathcal{D}}_\Gamma^-(n,r) \subset \underline{\mathcal{D}}_\Gamma^-(n,r)$ *of* $\mathcal{D}_\Gamma^-(n,r)$. *When* Γ *is the infinite affine Dynkin diagram of type A, we have*

$$\xi_V(n,r)\widetilde{\mathcal{D}}_\Gamma^-(n,r)\xi_V(n,r) \cong \mathcal{U}_r \rtimes \Sigma_r.$$

Our proof of this theorem is homological, using the theory of multiplicative extensions. It follows from Proposition 36 below.

Remark 34 The polynomial deformations of the above theorem should generalise to an arbitrary finite graph Γ.

The passage from an algebra to an associative formal deformation is often known as quantisation. The above deformation of $\mathcal{D}_\Gamma^-(n)$ should thus be perceived as a generalisation of the quantisation of the commutative Lie superalgebra in $2r$ odd dimensions, into the super-Heisenberg algebra $\mathcal{H}_r$.

Remark 35 The deformations $\widetilde{\mathcal{D}}_\Gamma^-(n,r)$ are related to symplectic reflection algebras in the wreath product case [10], [13]. The deformations of doubles $\mathcal{D}^-_{\tilde{A}_{p-1}}$, localised at the idempotent $\xi_V(n,r)$, correspond to rational Cherednik super-algebras for the group $C_p \wr \Sigma_r$.

We do not consider here the specialisation of central elements $i.z$, for idempotents i in $\mathcal{S}_V(n,r)$, to elements of the ground field. It is such specialisations, which provide multiparametric deformations of the classical wreath products, which are apparently so valuable to students of these algebras.

Furthermore, there is an extra parameter (denoted ν in [13]) which does not play any rôle in our exposition. There might be interesting generalisations in this direction.

Let us introduce a square root ζ of z into the algebra $\mathcal{U}_r$, a degree one element which super-commutes with odd elements in $\mathcal{U}_r$. The quadratic dual of the resulting algebra $\mathcal{U}_r(\zeta)$ is the commutative algebra,

$$k[x_1, ...x_r, y_1, ..., y_r, \lambda]/(\lambda^2 = \sum x_i y_i).$$

After Remark 36, this algebra is Koszul; its Koszul dual is $\mathcal{U}_r(\zeta)$. It is now clear how deformations of $\mathcal{D}_\Gamma(n,r)$ ought to be defined, when Γ is not a Dynkin graph.

Let Π_Γ^- be the preprojective algebra on Γ. Let a be the central quadratic element of Π_Γ^- in degree two, $a = \sum_V x_v y_v$, where x_v, y_v are the arrows corresponding to a single edge in Γ emanating from v. Let $\Pi_\Gamma^-(n,1) = End_{\Pi_\Gamma}(\Pi_\Gamma^{\oplus n})$. Streamlining notation, let $a \in \Pi_\Gamma^-(n,1)$ be the element corresponding to the central action of $a \in \Pi_\Gamma^-$. Let $\alpha = \sum_i 1^{\otimes i} \otimes a \otimes 1^{\otimes r-i-1} \in \Pi_\Gamma^-(n,r)$. Let

$$\underline{\Pi}_\Gamma^-(n,r) = \Pi_\Gamma^-(n,r)/(\lambda^2 - \alpha),$$

$$\underline{\mathcal{D}}_\Gamma^-(n,r) = \underline{\Pi}_\Gamma^-(n,r)^!.$$

$$\widetilde{\mathcal{D}}_\Gamma^-(n,r) = (\Pi_\Gamma^-(n,r)/a)^!.$$

We will see these algebras form deformations of $\mathcal{D}_\Gamma^-$, after some generalities concerning multiplicative extensions.

Given an algebra A, containing a central element α, let us define $\underline{A}$ to be the quotient $\underline{A} = A \otimes k[\lambda]/(\lambda^2 - \alpha)$ of the super tensor product of A with $k[\lambda]$, a multiplicative extension of A by the exterior algebra in one variable.

Proposition 36 *Suppose that A is a $\mathbb{Z}_+$-graded algebra, such that A_{A^0} is projective. Suppose that there exists a Koszul complex $\bigoplus_{i,j\geq 0} A^i \otimes_{A^0} A^{!j*}$ for A, a differential A-$A^!$ bimodule whose left restriction defines a linear resolution*

$$... \to A \otimes_{A^0} A^{2!*} \to A \otimes_{A^0} A^{1!*} \to A \otimes_{A^0} A^{0!*} \to A^{0*}$$

of A^{0*}, and the dual of whose right restriction defines a linear resolution

$$... \to A^! \otimes_{A^{!0}} A^2 \to A^! \otimes_{A^{!0}} A^1 \to A^! \otimes_{A^{!0}} A^0 \to A^{!0}$$

of $A^{!0}$. Suppose that $\alpha \in A^2$ is a central element which acts freely on A. Then $\underline{A}^!$ and $(A/\alpha)^!$ are polynomial deformations of $A^!$,

$$\begin{tikzcd} & \underline{A}^! \arrow[dr, two heads] & \\ A^0 \otimes k[\zeta] \arrow[ur, hook] & & A^!, \end{tikzcd}$$

$$\begin{tikzcd} & (A/\alpha)^! \arrow[dr, two heads] & \\ A^{!0} \otimes k[\zeta^2] \arrow[ur, hook] & & A^! \otimes k. \end{tikzcd}$$

Proof:

By Theorem 26, the homological dual of $\underline{A}$ is a multiplicative extension

$$\begin{tikzcd} & \underline{A}^! \arrow[dr, two heads] & \\ A^0 \otimes k[\zeta] \arrow[ur, hook] & & A^! \end{tikzcd}$$

of $A^!$ by the polynomial ring in a single variable ζ. In fact, since λ supercommutes with A, the dual variable ζ commutes with $\underline{A}^!$. Therefore, $\underline{A}^!$ is a polynomial deformation of $A^!$.

Suppose now that $\alpha \in A^2$ is a central element which acts freely on A. Then $\underline{A}/\lambda \cong A/\alpha$, and we have a multiplicative extension,

$$\begin{tikzcd} & \underline{A} \arrow[dr, two heads] & \\ A^0 \otimes k[\lambda] \arrow[ur, hook] & & (A/\alpha) \otimes k. \end{tikzcd}$$

Let

$$\cdots \longrightarrow (\underline{A}/\lambda)e_2 \xrightarrow{\bar{f}_2} (\underline{A}/\lambda)e_1 \xrightarrow{\bar{f}_1} (\underline{A}/\lambda)e_0 \longrightarrow A^{0*}$$

denote a projective resolution of the $\underline{A}/\lambda$-module A^{0*}. We have a quasi-isomorphism between $\underline{A} \to^{\lambda} \underline{A}$ and $\underline{A}/\lambda$. We thus have a double complex of projective $\underline{A}$-modules,

$$\begin{tikzcd}
\cdots \arrow[r] & \underline{A}e_2 \arrow[r, "f_2"] & \underline{A}e_1 \arrow[r, "f_1"] & \underline{A}e_0 \\
\cdots \arrow[r] & \underline{A}e_2 \arrow[r, "f_2"] \arrow[u, "\lambda"] & \underline{A}e_1 \arrow[r, "f_1"] \arrow[u, "\lambda"] & \underline{A}e_0, \arrow[u, "\lambda"]
\end{tikzcd}$$

whose total complex is quasi-isomorphic to A^{0*}. Applying $Hom(-, A^{0*})$, we obtain a double complex

$$\begin{tikzcd}
\cdots & Hom(\underline{A}e_2, A^{0*}) \arrow[l] \arrow[d, "0"] & Hom(\underline{A}e_1, A^{0*}) \arrow[l, "f_2^*"'] \arrow[d, "0"] & Hom(\underline{A}e_0, A^{0*}) \arrow[l, "f_1^*"'] \arrow[d, "0"] \\
\cdots & Hom(\underline{A}e_2, A^{0*}) \arrow[l] & Hom(\underline{A}e_1, A^{0*}) \arrow[l, "f_2^*"'] & Hom(\underline{A}e_0, A^{0*}), \arrow[l, "f_1^*"']
\end{tikzcd}$$

which can be identified with the double complex,

$$\begin{tikzcd}
\cdots & Hom((\underline{A}/\lambda)e_2, A^{0*}) \arrow[l] \arrow[d, "0"] & Hom((\underline{A}/\lambda)e_1, A^{0*}) \arrow[l, "\bar{f}_2^*"'] \arrow[d, "0"] & Hom((\underline{A}/\lambda)e_0, A^{0*}) \arrow[l, "\bar{f}_1^*"'] \arrow[d, "0"] \\
\cdots & Hom((\underline{A}/\lambda)e_2, A^{0*}) \arrow[l] & Hom((\underline{A}/\lambda)e_1, A^{0*}) \arrow[l, "\bar{f}_2^*"'] & Hom((\underline{A}/\lambda)e_0, A^{0*}), \arrow[l, "\bar{f}_1^*"']
\end{tikzcd}$$

Taking homology, we observe that $\underline{A}^! \cong (\underline{A}/\lambda)^! \oplus (\underline{A}/\lambda)^![1]$, where

$$(\underline{A}/\lambda)^! = Ext^*_{\underline{A}/\lambda}(A^{0*}, A^{0*}).$$

We thus have a multiplicative extension,

$$\begin{tikzcd}
 & \underline{A}^! \arrow[dr, two heads] & \\
k \otimes (\underline{A}/\lambda)^! \arrow[ur, hook] & & \bigwedge(\zeta) \otimes A^{0!}.
\end{tikzcd}$$

Thus, $\underline{A}^!$ is a multiplicative extension of $(\underline{A}/\lambda)^!$ by $\bigwedge(\zeta)$. Thanks to the existence of the Koszul complex, we know that $\underline{A}^!$ is the quadratic dual of $\underline{A}$. Also $\lambda^2 = 0 \in (\underline{A}/\lambda)^2$. Therefore, in the quadratic dual $\underline{A}^!$, we have $\zeta^2 \in (\underline{A}/\lambda)^{!2}$, and so $(\underline{A}/\lambda)^!$ is a multiplicative extension,

$$\begin{tikzcd}
 & (\underline{A}/\lambda)^! \arrow[dr, two heads] & \\
A^{!0} \otimes k[\zeta^2] \arrow[ur, hook] & & A^! \otimes k.\square
\end{tikzcd}$$

Proof of theorem 33:

Let $\tilde{\Gamma}$ be affine of type A. Removing a vertex from $\tilde{\Gamma}$, one obtains an ordinary Dynkin diagram Γ. Algebraically, this means that $\mathcal{D}_\Gamma(n,r)$ is obtained from $\mathcal{D}_{\tilde{\Gamma}}(n,r)$ by cutting at an idempotent, the unit of $\mathcal{S}_{V(\Gamma)}(n,r)$ in $\mathcal{S}_{V(\tilde{\Gamma})}(n,r)$.

By Proposition 12, and Proposition 36, the algebras $\widetilde{\mathcal{D}}^-_{\tilde{\Gamma}}(n,r) \subset \underline{\mathcal{D}}^-_{\tilde{\Gamma}}(n,r)$ are deformations of $\mathcal{D}^-_{\tilde{\Gamma}}(n,r)$. Cutting at the appropriate idempotent, we reduce deformations for affine Dynkin diagrams to deformations for ordinary Dynkin diagrams.

Localising at the idempotent $\xi_V(n,r)$ has the effect of cutting Schur algebras down to symmetric groups. When Γ is infinite affine, localising $\Pi^-_\Gamma(n,r)$ at this idempotent, we obtain the algebra,

$$\left(\Pi^-_\Gamma \otimes k[\lambda]/(\lambda^2 = \sum_i 1^{\otimes i} \otimes a \otimes 1^{\otimes r-i-1})\right) \rtimes \Sigma_r,$$

The Koszul dual of this algebra is isomorphic to its quadratic dual, the deformation $\mathcal{U}_r(\zeta) \rtimes \Sigma_r$ of $ZZ^-_\Gamma \wr \Sigma_r$. This completes the proof of the theorem. $\square$

Base change.

In this section, we consider a special situation, in which algebras are defined over an modular system (K,R,k), and possess similar structural properties over both fields K and k. Under certain circumstances, such a situation appears to occur naturally in representation theory. For example, close relations are expected between representations of quantum groups at roots of unity over K, and representations of algebraic groups over k.

Let l be a prime number, and let (K,R,k) be an l-modular system. Let A be an R-free R-algebra, of finite rank. We write $KA = K \otimes_R A$, $kA = k \otimes_R A$.

Let N be an R-free A-module. We write $KN = K \otimes_R N$, $kN = k \otimes_R N$. Let $Rad(KN)$ (respectively $Rad(kN)$) denote the Jacobson radical of KN (respectively kN). We define $Rad(N)$ to be the intersection $Rad(KN) \cap N$, an R-pure submodule of N. We have $KRad(N) = Rad(KN)$, and $kRad(N) \subseteq Rad(kN)$.

Any KA-module $\widehat{N}$ lifts (nonuniquely) to an R-free A-module N, such that $KN = \widehat{N}$. Any projective indecomposable kA-module lifts to a projective indecomposable A-module. A map $P \to N$ of A-modules is a minimal projective cover if, and only if $kP \to kN$ is a minimal projective cover.

Proposition 37 *Suppose that KP is indecomposable, for any projective indecomposable A-module P. Then,* *1. If KN is an irreducible KA-module, then kN is an irreducible kA-module.*

2. The irreducible KA-modules are in bijection with the irreducible kA-modules.

3. $kRad(A) = Rad(kA)$.

4. If KN is semisimple, then kN is semisimple.

Proof:

1. The Cartan numbers for kA are equal to Cartan numbers for KA, both being described by the R-rank of e_iRAe_j, where $\{RAe_i\}_{i\in I}$ is a collection of principal indecomposable modules for RA. Furthermore, RAe_i inherits a filtration $Rad^i(KAe_i) \cap RA$ from the radical filtration of KAe_i. Since the Cartan matrices of KA and kA are equal, the subquotients in this filtration cannot have any more composition factors over k than they do over K. The top of KAe_i is the first subquotient in the radical filtration, and therefore remains simple modulo l.

2. There is a natural map,

$$\{projective\ indecomposable\ kA-modules\} \rightarrow$$

$$\{projective\ indecomposable\ KA-modules\},$$

obtained by first lifting over R, and then tensoring over K. This map is surjective, since every projective module is a summand of the free module. The map is injective, since an isomorphism $KP \cong KQ$ of projective indecomposable KA-modules implies an isomorphism $KP/KRadP \cong KQ/KRadQ$ which implies, in turn, isomorphisms $kP/kRadP \cong kQ/kRadQ$, and $kP \cong kQ$.

3. Let us write $A = \bigoplus_i Ae_i$, where $\{e_i\}$ denotes a complete set of primitive orthogonal idempotents in A. Then $Rad(A) = \bigoplus \mathcal{M}_i$, where $\mathcal{M}_i = Rad(KAe_i) \cap A$. By 1, we see that $k\mathcal{M}_i = Rad(kAe_i)$, and so $kRad(A) = Rad(kA)$.

4. If KN is semisimple, then KN is a $KA/KRadA$-module. Therefore, RN is an $RA/RadA$-module, and kN is a $kA/kRad(A) = kA/Rad(kA)$-module, that is to say, semisimple. □

We have a similar result to proposition 37 for base change, over a splitting field.

Proposition 38 *Let FA be a finitely generated F-algebra, over a splitting field F. Let K be a field containing F. Then,*

1. Every projective indecomposable FA-module is indecomposable over K.

2. The irreducible FA-modules are in bijection with the irreducible KA-modules.

3. An FA-module FN is semisimple if, and only if KN is semisimple.

4. If FN is an FA-module, and $FP \to FN$ a minimal projective cover, then $KP \to KN$ is a minimal projective cover.

5. If FN is an FA-module, then FN is projective if, and only if, KN is projective.

6. Let FA be a positively graded algebra, whose graded pieces are finite dimensional. Then KA is Koszul if, and only if, FA is Koszul.

7. KA is quasi-hereditary if, and only if, FA is quasi-hereditary.$\square$

Lemma 39 *Suppose that A is a graded algebra, that N is a graded A-module, and that L is a graded submodule of KN. Then $\tilde{L} = L \cap KN$ is an R-pure graded submodule of N.* $\square$

Lemma 40 *Suppose that A is a graded R-algebra, that N is a graded A-module, and that M is a quotient of KN as a graded KA-module. Suppose that*

$$grHom_{KA}(KN, M) \cong K.$$

Let $\tilde{M} = N/KN \cap ker\phi$, where ϕ is some non-zero graded homomorphism between KN and M. Then $\tilde{M}$ is R-free, graded, and independent of the chosen ϕ, thus defining a canonical R-form for M.

Proof:

The intersection of an R-form for a K-vector space with a vector subspace is always R-pure. Therefore, $\tilde{M}$ is R-free. Since ϕ is defined uniquely up to a scalar, $ker\phi$ is uniquely defined, and $\tilde{M}$ is canonically defined. The grading on $\tilde{M}$ comes by applying lemma 39, with $L = ker\phi$. $\square$

Modular reduction of blocks of category $\mathcal{O}$.

Let l be a prime number, and let (K, R, k) be an l-modular system. Let $\mathfrak{g}$ be a complex semisimple Lie algebra over $\mathbb{C}$.

We define algebraic analogues $A(\mathfrak{g})$, and $B(\mathfrak{g})$, of principal blocks of category $\mathcal{O}(\mathfrak{g})$, over the discrete valuation ring R. Over fields of characteristic zero, these algebras were defined by W. Soergel, and they are known to be Koszul dual (and even isomorphic). Over R, we give a more convoluted definition, generalising Soergel's. The algebras we define are in Koszul duality over the field k. My justification for this construction is a desire to work over fields of arbitrary characteristic, whenever it is possible to do so. This is due to an interest in modular representations of finite groups, the exploration of which motivated this paper. In modular representation theory, one cares most about fields of positive characteristic.

In the last section of this article, we reveal extensions of tensor powers of Zigzag algebras by the algebras $B(\mathfrak{g})$.

Let G be a complex semisimple Lie group over $\mathbb{C}$, with Borel subgroup B, and Weyl group W. Let V denote the reflection representation of W. Let $C = S(V)/S_+(V)^W$ be the algebra of coinvariants associated to W. Then C is isomorphic to the cohomology ring of G/B, and Poincaré duality gives C the structure of a symmetric algebra. Over a field of characteristic zero, C is isomorphic, as a W-module, to $\mathbb{C}W$. We think of C is a graded algebra, generated by V in degree 2.

Given a simple reflection $s \in W$, let $C^s = S(V)^s/S_+(V)^W$ denote the subalgebra generated by s-fixed points. Then C^s is isomorphic to the cohomology ring of G/B_s, where B_s is the parabolic subgroup generated by B and s. Thus C^s also attains the structure of a symmetric algebra. Since s generates a subgroup of W of order 2, the dimension of C^s is half the dimension of C. As graded modules, we have ${}_{C^s}C \cong C^s \oplus C^s < 2 >$, where $< i >$ denotes the i^{th} degree shift.

The bimodule ${}_{C^s}C_C$ is projective on both sides, and its dual is isomorphic to ${}_CC_{C^s}$. The functors

$$Ind_{C^s}^C = C \otimes_{C^s} - : C^s - mod \to C - mod,$$

$$Res_{C^s}^C = C \otimes_C - : C - mod \to C^s - mod,$$

are therefore exact, and left and right adjoint to each other. In the graded module category, the left adjoint of $Res_{C^s}^C$ is $Ind_{C^s}^C$, whilst the right adjoint is the shifted functor, $Ind_{C^s}^C < -2 >$.

For $w = s_1...s_m$ a reduced expression in W, and a C-module M, let

$$\mathcal{F}^{s_1\cdots s_m}(M) = Ind_{C^{s_1}}^C Res_{C^{s_1}}^C ... Ind_{C^{s_m}}^C Res_{C^{s_m}}^C (M).$$

Then $\mathcal{F}^{s_1\cdots s_m}$ is an exact functor, whose left and right adjoint is $\mathcal{F}^{s_m\cdots s_1}$. In the graded category, the right adjoint of $\mathcal{F}^{s_1\cdots s_m}$ is $\mathcal{F}^{s_m\cdots s_1} < -2m >$

If M is a graded module with Hilbert polynomial $p(q)$, then $\mathcal{F}^{s_1\cdots s_m}(M)$ is a graded module over the coinvariant algebra with Hilbert polynomial $(1+q)^m.p(q)$. This module is dependent on the reduced expression chosen for w.

Theorem 41 *(Soergel [23]) There is a family* $\{\mathbb{C}M_w\}_{w\in W}$ *of mutually non-isomorphic, self-dual, graded* $\mathbb{C}C$*-modules, such that*

$$\mathcal{F}^{s_1\cdots s_m}(\mathbb{C}) \cong M_w \oplus \bigoplus_{v<w} (M_v)^{\oplus m_{vw}},$$

for some multiplicities $m_{vw} \in Z_+$. $\square$

Let $< i >$ denote the i^{th} graded shift. Let

$$\mathbb{C}A(\mathfrak{g}) = End(\bigoplus_{w \in W} M_w).$$

We have the following structural properties of $\mathbb{C}A(\mathfrak{g})$. The first is due to Soergel [23]. The second is due to E. Cline, B. Parshall and L. Scott ([8], 3.3c). The third was proved by Soergel, after a conjecture by A. Beilinson and V. Ginzburg ([23], see also [3]).

Theorem 42 *(i) The category $\mathbb{C}A(\mathfrak{g}) - mod$ is equivalent to the principal block of category $\mathcal{O}(\mathfrak{g})$.*

(ii) The algebra $\mathbb{C}A(\mathfrak{g})$ is quasi-hereditary. Its poset is W, with respect to the Bruhat order.

(iii) The algebra $\mathbb{C}A(\mathfrak{g})$ is Koszul, and Koszul self-dual. $\square$

The $\mathbb{Z}_+$-grading on $\mathbb{C}A(\mathfrak{g})$ comes from the grading on $\mathbb{C}C$, as described by the following formula:

$$\mathbb{C}A(\mathfrak{g}) = \bigoplus_{i \in \mathbb{Z}} grHom_{\mathbb{C}C}(M_v < l(v) >, M_w < l(w) + i >),$$

Lemma 43 *$grHom(\mathcal{F}^{s_1 \cdots s_m}(\mathbb{C}), \mathbb{C}M_w) \cong \mathbb{C}$.*

Proof:

The graded $\mathbb{C}C$-module $\mathcal{F}^{s_1 \cdots s_m}(\mathbb{C})$ is self-dual, and concentrated in degrees $0, ..., 2m$. Its degree zero part is isomorphic to $\mathbb{C}$, as is its degree $2w$ part. The component $\mathbb{C}M_w$ also has degree zero and $2w$ part isomorphic to $\mathbb{C}$. This implies that any graded component $\mathbb{C}M_{w'} < -2j >$ of $\mathcal{F}^{s_1 \cdots s_m}(\mathbb{C})$, for $w' \neq w$, satisfies the inequality

$$1 \leq j \leq l(w) - l(w') - 1.$$

If $grHom(\mathbb{C}M_{w'} < l(w') - d >, \mathbb{C}M_w < l(w) >)$ is non-zero, then we must have $d \geq l(w) - l(w')$, since $\mathbb{C}A(\mathfrak{g})$ is generated in degrees zero and one. Therefore, if

$$grHom(\mathbb{C}M_{w'} < -2j >, \mathbb{C}M_w) \cong grHom(\mathbb{C}M_{w'} < l(w) - 2j >, \mathbb{C}M_w < l(w) >)$$

is non-zero, we must have $l(w') - l(w) + 2j \geq l(w) - l(w')$, which implies

$$l(w) - l(w') \leq j.$$

It follows that $\mathbb{C}M_w$ is the only graded component of $\mathcal{F}^{s_1 \cdots s_m}(\mathbb{C})$ with any graded homomorphisms to $\mathbb{C}M_w$. Since $grEnd(\mathbb{C}M_w) \cong \mathbb{C}$, the proof of the lemma is complete. $\square$

Because $\mathbb{Q}$ is a splitting field for C, and $\mathbb{C}End(\mathcal{F}^{s_1\cdots s_m}(\mathbb{Q})) \cong End(\mathcal{F}^{s_1\cdots s_m}(\mathbb{C}))$, we may lift the decomposition of $\mathcal{F}^{s_1\cdots s_m}(\mathbb{C})$ over $\mathbb{Q}$, or indeed any field of characteristic zero. Thus, $\mathcal{F}^{s_1\cdots s_m}(K)$ decomposes as a direct sum of KM_w, and some KM_v's, for $v < w$.

Since $\mathcal{F}^{s_1\cdots s_m}(K)$ is graded, KM_w is a graded C-module. As a graded module, $\mathcal{F}^{s_1\cdots s_m}(K)$ is isomorphic to a direct sum of KM_w, and some modules $KM_v < i >$, where $w > v, i > 0$.

By lemma 43, up to a scalar, there is a unique graded homomorphism $\psi : \mathcal{F}^{s_1\cdots s_m}(K) \to KM_w$. These homomorphisms define a unique kernel $KL_{s_1\ldots s_m} \subset \mathcal{F}^{s_1\cdots s_m}(K)$. This subspace defines a unique R-pure sublattice

$$L_{s_1\ldots s_m} = KL_{s_1\ldots s_m} \cap \mathcal{F}^{s_1\cdots s_m}(R)$$

of $\mathcal{F}^{s_1\cdots s_m}(R)$. Let us define $M_{s_1\ldots s_m} = \mathcal{F}^{s_1\cdots s_m}(R)/L_{s_1\ldots s_m}$. By Lemma 40, $M_{s_1\ldots s_m}$ is a graded R-form for KM_w.

Proposition 44 *If $s_1\ldots s_m$ and $t_1\ldots t_m$ are two reduced expressions for w, then there is an isomorphism, $M_{s_1\ldots s_m} \cong M_{t_1\ldots t_m}$.*

Proof:

Since $M^0_{s_1\ldots s_m} \cong M^0_{t_1\ldots t_m} \cong R$, we have natural maps,

$$grHom(M_{s_1\ldots s_m}, M_{t_1\ldots t_m}) \to grHom(M_{s_1\ldots s_m}, M^0_{t_1\ldots t_m})$$
$$\cong grHom(M_{s_1\ldots s_m}, R) \cong Hom_R(R, R) \cong R.$$

Since the graded endomorphisms of KM_w are given by scalar multiplication, the map

$$\phi : grHom(M_{s_1\ldots s_m}, M_{t_1\ldots t_m}) \to grHom(M_{s_1\ldots s_m}, R)$$

is an injection from R to R. If ϕ is surjective, for any choice of s_i, t_i, then the natural composition map

$$grHom(M_{s_1\ldots s_m}, M_{t_1\ldots t_m}) \times grHom(M_{t_1\ldots t_m}, M_{s_1\ldots s_m}) \to$$
$$grHom(M_{s_1\ldots s_m}, M_{s_1\ldots s_m}),$$

is an isomorphism, and it follows that $M_{s_1\ldots s_m} \cong M_{t_1\ldots t_m}$. Therefore, to complete the proof of the proposition, we show that ϕ is surjective.

There is a commuting diagram of natural morphisms,

$$\begin{tikzcd}
grHom(M_{s_1\ldots s_m}, M_{t_1\ldots t_m}) \arrow[r, "\phi"] \arrow[d] & grHom(M_{s_1\ldots s_m}, R) \arrow[r, "\simeq"] \arrow[d] & R \arrow[d] \\
grHom(\mathcal{F}^{s_1\cdots s_m}(R), M_{t_1\ldots t_m}) \arrow[r] \arrow[d] & grHom(\mathcal{F}^{s_1\cdots s_m}(R), R) \arrow[r, "\simeq"] \arrow[d] & R \arrow[d] \\
grHom(R, \mathcal{F}^{s_m\cdots s_1}(M_{t_1\ldots t_m}) < -2m >) \arrow[r, "\psi"] & grHom(R, \mathcal{F}^{s_m\cdots s_1}(R) < -2m >) \arrow[r, "\simeq"] & R
\end{tikzcd}$$

The top left square is given by pulling back the map $\mathcal{F}^{s_1\cdots s_m}(R) \to M_{s_1\ldots s_m}$, and the map $M_{t_1\ldots t_m} \to R$. The bottom left square is obtained by adjunction. The downwards maps are all isomorphisms. It is therefore enough for us to show that ψ is surjective.

Pulling back the sequence of morphisms,

$$\mathcal{F}^{s_m\cdots s_1}\mathcal{F}^{t_1\cdots t_m}(R) \to \mathcal{F}^{s_m\cdots s_1}(M_{t_1\ldots t_m}) \to \mathcal{F}^{s_m\cdots s_1}(R),$$

we find it is enough for us to show that the map

$$grHom(R, \mathcal{F}^{s_m\cdots s_1}\mathcal{F}^{t_1\cdots t_m}(R) < -2m >) \to grHom(R, \mathcal{F}^{s_m\cdots s_1}(R) < -2m >)$$

is surjective. Forgetting the grading, we see it is enough for us to show that the map

$$Hom(R, \mathcal{F}^{s_m\cdots s_1}\mathcal{F}^{t_1\cdots t_m}(R)) \to Hom(R, \mathcal{F}^{s_m\cdots s_1}(R))$$

is surjective, or equivalently, by adjunction, the map

$$Hom(\mathcal{F}^{t_m\cdots t_1}\mathcal{F}^{s_1\cdots s_m}(R), R) \to Hom(\mathcal{F}^{s_1\cdots s_m}(R), R)$$

is surjective which, by Frobenius reciprocity, is to say the map

$$R \otimes_C \mathcal{F}^{t_m\cdots t_1}\mathcal{F}^{s_1\cdots s_m}(R) \to R \otimes_C \mathcal{F}^{s_1\cdots s_m}(R)$$

is surjective. In other words, it is sufficient for us to know that the natural map

$$R \otimes_C C \otimes_{C^{t_m}} C \otimes_{C^{t_{m-1}}} \ldots \otimes_{C^{t_1}} C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} \ldots \otimes_{C^{s_m}} C \otimes R \to$$

$$R \otimes_C C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} \ldots \otimes_{C^{s_m}} C \otimes R$$

is surjective, which is obvious. $\square$

Let $w = s_1 \ldots s_m$ be a reduced expression in W. We write M_w for $M_{s_1\ldots s_m}$, a uniquely defined R-form for KM^w. Let

$$A(\mathfrak{g}) = End_C(\bigoplus_W M_w).$$

Proposition 45 *The R-lattice $A(\mathfrak{g})$ is a $\mathbb{Z}_+$-graded algebra. The algebra $kA(\mathfrak{g})$ has the same Cartan-Hilbert matrix as $\mathbb{C}A(\mathfrak{g})$. All irreducible $kA(\mathfrak{g})$-modules are one dimensional.* $\square$

Extending powers of Zigzag algebras by principal blocks of category $\mathcal{O}$.

Levi's theorem states that any complex Lie algebra $\mathfrak{l}$ decomposes as a semidirect product,

$$\mathfrak{l} = \mathfrak{h} \rtimes \mathfrak{g},$$

of a solvable ideal $\mathfrak{h}$, and a semisimple subalgebra $\mathfrak{g}$. There are notable violations of this principle, when one restricts study to certain module categories of Lie algebras.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra. W. Soergel's investigations of Harish-Chandra bimodules reveal the existence of a graded algebra which is a non-split central extension $\mathbb{C}\widehat{A}(\mathfrak{g})$ of the finite dimensional algebra $\mathbb{C}A(\mathfrak{g})$ describing the principal block of $\mathcal{O}(\mathfrak{g})$, by $S(\mathfrak{h})$. Here, $\mathfrak{h}$ lies in degree two.

If the dimension of a vector space U is equal to the rank of $\mathfrak{g}$, we can identify $\mathfrak{h}$ with the diagonal entries of $\mathfrak{gl}(U) \cong U \otimes U^* \subset S(U + U^*)$, and form the tensor product

$$\mathbb{C}\widehat{A}(\mathfrak{g}) \bigotimes_{S(\mathfrak{h})} S(U + U^*).$$

The resulting extension of $A(\mathfrak{g})$ by $S(U + U^*)$ is generated in degrees zero, and one. It can even be seen to be Koszul, its Koszul dual being an extension of $\bigwedge(U + U^*)$ by $\mathbb{C}A(\mathfrak{g})$.

In this section, we describe a generalisation of this duality. We replace $S(U + U^*)$ with a tensor power of preprojective algebras, $\bigwedge(U + U^*)$ with a tensor power of Zigzag algebras. In this way, we release multiplicative extensions of tensor powers of Zigzag algebras by the algebras $B(\mathfrak{g})$ of the last section.

Consider the algebra $\widehat{C} = S(\mathfrak{h}) \bigotimes_{S(\mathfrak{h})^W} S(\mathfrak{h})$. Soergel has shown that on the stage of Harish-Chandra bimodules, this algebra plays a similar rôle to that of the coinvariant algebra, in the theatre of category $\mathcal{O}$.

Let the Weyl group W act on $\widehat{C}$, on the first component $(x \otimes y)^w = (x^w \otimes y)$. For $w = s_1 ... s_m$ a reduced expression in W, and a $\widehat{C}$-module M, let,

$$\widehat{IR}^{s_1 ... s_m}(M) = Ind_{\widehat{C}^{s_1}}^{\widehat{C}} Res_{\widehat{C}^{s_1}}^{\widehat{C}} ... Ind_{\widehat{C}^{s_m}}^{\widehat{C}} Res_{\widehat{C}^{s_m}}^{\widehat{C}}(M).$$

There exist $\widehat{C}$-modules $\mathbb{C}\widehat{M}^w$, for $w \in W$, such that $\widehat{IR}^{s_1,...,s_m}(S(\mathfrak{h}))$ is isomorphic to a direct sum, with one indecomposable summand $\mathbb{C}\widehat{M}^w$, and all other indecomposable summands isomorphic to $\mathbb{C}\widehat{M}^v$, for $v < w$ ([24], Theorem 2).

Let $S(\mathfrak{h})_1 = S(\mathfrak{h}) \bigotimes 1$ be the subalgebra of $\widehat{C}$ defined on the first component. This subalgebra is isomorphic to the symmetric algebra on $\mathfrak{h}$. Note that $\mathbb{C}\widehat{C}$, and $\mathbb{C}\widehat{M}^w$ are $S(\mathfrak{h})_1$-free, that

$$\mathbb{C} \bigotimes_{S(\mathfrak{h})_1} \widehat{C} \cong C,$$

as algebras, and,

$$\mathbb{C} \bigotimes_{S(\mathfrak{h})_1} \mathbb{C}\widehat{M}^w \cong \mathbb{C}M^w,$$

as C-modules. Thus, $\widehat{C}$ is truly a lift of C, and $\mathbb{C}\widehat{M}^w$ a lift of $\mathbb{C}M^w$.

Let us define $\widehat{A}(\mathfrak{g}) = End_{\widehat{C}}(\bigoplus_W \widehat{M}^w)$. Soergel's structure theorem for Harish-Chandra bimodules ([24], Theorem 3), states that,

Theorem 46 *The algebra $\mathbb{C}\widehat{A}(\mathfrak{g})$ is a graded algebra. It is a non-split central extension of $\mathbb{C}A(\mathfrak{g})$ by $S(\mathfrak{h})$. Thus, $S(\mathfrak{h})$ is a central subalgebra of $\widehat{A}(\mathfrak{g})$, and*

$$\mathbb{C} \bigotimes_{S(\mathfrak{h})} \mathbb{C}\widehat{A}(\mathfrak{g}) \cong \mathbb{C}A(\mathfrak{g}).$$

The category $\mathbb{C}\widehat{A}(\mathfrak{g}) - nil$ of modules on which the elements of positive degree act nilpotently is equivalent to the regular block of Harish-Chandra bimodules for $\mathfrak{g}$. $\square$

Let us assume that Γ is an affine Dynkin diagram, of type A.

There is a central embedding $k[z] \to \Pi_\Gamma^-$, taking z to the sum of terms $\sum_V x_v y_v$, where x_v, y_v are arrows corresponding to some edge attached to v. The tensor power of such maps defines an embedding

$$S(\mathfrak{h}) \to \Pi_\Gamma^{-\otimes r},$$

where r is the rank of $\mathfrak{g}$. Let us define,

$$\widehat{\Pi}_\Gamma^r = \Pi_\Gamma^{-\otimes r} \bigotimes_{S(\mathfrak{h})} \widehat{A}(\mathfrak{g}).$$

Then $\widehat{\Pi}_\Gamma^r$ is a multiplicative extension,

$$\begin{tikzcd} & \widehat{\Pi}_\Gamma^r \arrow[dr, two heads] & \\ \mathbb{C}^{\oplus W} \otimes \Pi_\Gamma^{\otimes r} \arrow[ur, hook] & & A(\mathfrak{g}) \otimes \mathbb{C}^{\oplus V^r}. \end{tikzcd}$$

By Theorem 26, we have the following.

Theorem 47 $\widehat{\Pi}_\Gamma^r$ *is a Koszul algebra. Its Koszul dual, $\widehat{ZZ}_\Gamma^r$, is a graded multiplicative extension,*

$$\begin{tikzcd} & \widehat{ZZ}_\Gamma^r \arrow[dr, two heads] & \\ \mathbb{C}^{\oplus V^r} \otimes A(\mathfrak{g}) \arrow[ur, hook] & & ZZ_\Gamma^{\otimes r} \otimes \mathbb{C}^{\oplus W}.\square \end{tikzcd}$$

Remark 48 For simplicity, we have only considered principal blocks of $\mathcal{O}$ in this paper. It should be possible to manipulate singular, and parabolic blocks as well, using fixed point subalgebras C^{W_λ} of C, where $W_\lambda < W$ is a parabolic subgroup.

References

[1] I. Ágoston, V. Dlab and E. Lukács, Standardly stratified extension algebras, *Communications in Algebra* **33** (2005), no. 5, 1357-1368.

[2] Interaction of Finite Dimensional Algebras with other areas of Mathematics, Banff International Research Station, September, 2004.

[3] A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory. *J. Amer. Math. Soc.* **9** (1996), no. 2, 473–527.

[4] J. Chuang, R. Kessar, Symmetric groups, wreath products, Morita equivalences, and Broué's abelian defect group conjecture. *Bull. London Math. Soc.* **34** (2002), no. 2, 174–184.

[5] J. Chuang, R. Rouquier, Derived equivalences for symmetric groups and sl_2-categorification, arXiv 0407205 (2004).

[6] J. Chuang, K.M. Tan, Representations of wreath products of algebras, Math. Proc. Cambridge Philos. Soc. **135** (2003), no. 3, 395–411.

[7] J. Chuang, W. Turner, Cubist algebras (2004),
http://www.math.uga.edu/archive/turner _ wb.html.

[8] E. Cline, B. Parshall, and L. Scott, Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.* **391** (1988), 85-99.

[9] S. Donkin, The q-Schur algebra. *London Mathematical Society Lecture Note Series* **253**. Cambridge University Press, Cambridge, 1998.

[10] P. Etingof, V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. *Invent. Math.* **147** (2002), no. 2, 243-348.

[11] I. Frenkel, Representation theory beyond affine Lie algebras: A perspective (2002).
http://www.msri.org/publications/ln/msri/2002/ssymmetry/ifrenkel/1/

[12] V. Ginzburg, N. Guay, E. Opdam, R. Rouquier, On the category $\mathcal{O}$ for rational Cherednik algebras. *Invent. Math.* **154** (2003), no. 3, 617–651.

[13] W.L. Gan, V. Ginzburg, Deformed preprojective algebras and symplectic reflection algebras for wreath products. *J. Algebra* **283** (2005), no. 1, 350–363.

[14] J. A. Green, Polynomial Representations of GL_n, *Lecture Notes in Mathematics* **830**, Springer-Verlag, 1980.

[15] R. S. Huerfano, M. Khovanov, A category for the adjoint representation. *J. Algebra* **246** (2001), no. 2, 514-542.

[16] B. Keller, A-infinity algebras in representation theory, Contribution to the Proceedings of ICRA IX, Beijing 2000.

[17] K. Lefèvre-Hasegawa, Sur les A-infini catégories,
http://www.math.jussieu.fr/ keller/lefevre/publ.html.

[18] R. Martinez-Villa, Application of Koszul algebras: the preprojective algebra, in *Representation theory of Algebras*, Canadian Mathematical Society Conference Proceedings, vol. 18, 1998, 487-504.

[19] V. Mazorchuk, Applications of the category of linear complexes of tilting modules associated with category $\mathcal{O}$, http://www.math.uu.se/ mazor/

[20] M. Peach, Phd. thesis, Bristol, 2004.

[21] C. Ringel, The preprojective algebra of a quiver. In: Algebras and Modules II. Can. Math. Soc. Conference Proceedings. Vol. 24 (1998), 467-480.

[22] R. Rouquier, Derived equivalences and sl_2-categorification, exposés au Workshop ICRA XI (août 2004),
http://www.math.jussieu.fr/ rouquier/preprints/preprints.html.

[23] W. Soergel, Kategorie $\mathcal{O}$, perverse Garben und Moduln uber den Koinvarianten zur Weylgruppe. *J. Amer. Math. Soc.* **3** (1990), no. 2, 421–445.

[24] W. Soergel, The combinatorics of Harish-Chandra bimodules. *J. Reine Angew. Math.* **429** (1992), 49–74.

[25] C. Stroppel, Category $\mathcal{O}$: Gradings and Translation Functors, *Journal of Algebra* **268**, 2003.

[26] W. Turner, Rock blocks (2004),
http://www.math.uga.edu/archive/turner _ wb.html.

[27] W. Turner, Tilting equivalences: from hereditary algebras to blocks of symmetric groups (2004),
http://www.math.uga.edu/archive/turner _ wb.html.

[28] C. Weibel, An introduction to homological algebra, *Cambridge Studies in Advanced Mathematics* **38**. Cambridge University Press, Cambridge, 1994.

[29] D. Woodcock, Cohen-Macaulay complexes and Koszul rings. *J. London Math. Soc.* (2) **57** (1998), no. 2, 398-410.