Bialgebras and Caterpillars.

W. Turner

March 27, 2006

Department of Mathematics, University of Oxford, Oxford, England.
Email: turnerw@maths.ox.ac.uk

Let p be a prime number. Broué's abelian defect conjecture [4] states that any p-block of a finite group is equivalent to its Brauer correspondent.

In [9], we introduced a double construction, which associates a symmetric associative algebra to certain bialgebras. We predicted that the derived category of a block b of a symmetric group is equivalent to the derived category of a block of some double. In fact, there are a number of different doubles which we expect to have components derived equivalent to b. There is the Rock double $\mathcal{D}$, which comes from a super-bialgebra, as well as a family of doubles $\mathcal{E}$, associated to quiver algebras of type A [11].

This note was formulated, with the intention of clarifying these conjectures. We first simplify our double construction, by giving a more symmetric formula, which holds for all bialgebras B. We prove that in degree one, the doubles associated to quiver algebras of type A have components which are isomorphic to Brauer caterpillar algebras, with exceptional multiplity one. More generally, we define bialgebras whose doubles have natural subalgebras which are isomorphic to Brauer caterpillar algebras, with exceptional multiplicity. Even in this simple situation, the bialgebras involved are a little subtle.

Bialgebras.

Let k be a field. The following result is a simplification of the double construction of [9].

Theorem 1 *Let B be a bialgebra over k, with dual B^*. Then the tensor product $D(B) = B \otimes B^*$ is a symmetric associative algebra, with product*

$$(a \otimes \alpha).(b \otimes \beta) = \sum a_{(2)} b_{(1)} \otimes \alpha_{(2)} \beta_{(1)} < a_{(1)}, \beta_{(2)} >< \alpha_{(1)}, b_{(2)} >,$$

and symmetric associative bilinear form,

$$< a \otimes \alpha, b \otimes \beta > = < a, \beta >< b, \alpha > .$$

Proof:

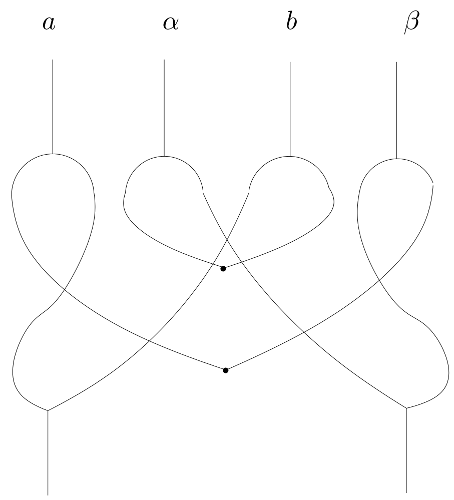
We prove $D(B)$ is associative.

$$((a \otimes \alpha).(b \otimes \beta)).(c \otimes \gamma) =$$
$$\left(\sum a_{(2)} b_{(1)} \otimes \alpha_{(2)} \beta_{(1)} < a_{(1)}, \beta_{(2)} >< \alpha_{(1)}, b_{(2)} >\right).(c \otimes \gamma) =$$
$$\sum (a_{(2)} b_{(1)})_{(2)} c_{(1)} \otimes (\alpha_{(2)} \beta_{(1)})_{(2)} \gamma_{(1)}$$
$$< a_{(1)}, \beta_{(2)} >< \alpha_{(1)}, b_{(2)} >< (a_{(2)} b_{(1)})_{(1)}, \gamma_{(2)} >< (\alpha_{(2)} \beta_{(1)})_{(1)}, c_{(2)} > =$$
$$\sum a_{(3)} b_{(2)} c_{(1)} \otimes \alpha_{(3)} \beta_{(2)} \gamma_{(1)}$$
$$< a_{(1)}, \beta_{(3)} >< \alpha_{(1)}, b_{(3)} >< a_{(2)}, \gamma_{(2)} >< b_{(1)}, \gamma_{(3)} >< \alpha_{(2)}, c_{(2)} >< \beta_{(1)}, c_{(3)} >,$$
$$(a \otimes \alpha).((b \otimes \beta).(c \otimes \gamma)) =$$
$$(a \otimes \alpha).\left(\sum b_{(2)} c_{(1)} \otimes \beta_{(2)} \gamma_{(1)} < b_{(1)}, \gamma_{(2)} >< \beta_{(1)}, c_{(2)} >\right) =$$
$$\sum a_{(2)} (b_{(2)} c_{(1)})_{(1)} \otimes \alpha_{(2)} (\beta_{(2)} \gamma_{(1)})_{(1)}$$
$$< b_{(1)}, \gamma_{(2)} >< \beta_{(1)}, c_{(2)} >< a_{(1)}, (\beta_{(2)} \gamma_{(1)})_{(2)} >< \alpha_{(1)}, (b_{(2)} c_{(1)})_{(2)} > =$$
$$\sum a_{(3)} b_{(2)} c_{(1)} \otimes \alpha_{(3)} \beta_{(2)} \gamma_{(1)}$$
$$< b_{(1)}, \gamma_{(3)} >< \beta_{(1)}, c_{(3)} >< a_{(1)}, \beta_{(3)} >< a_{(2)}, \gamma_{(2)} >< \alpha_{(1)}, b_{(3)} >< \alpha_{(2)}, c_{(2)} > .$$

The symmetry and non-degeneracy of the bilinear form on $D(B)$ are clear. Let us confirm its associativity.

$$< (a \otimes \alpha).(b \otimes \beta), c \otimes \gamma > =$$
$$\sum < a_{(2)} b_{(1)}, \gamma >< \alpha_{(2)} \beta_{(1)}, c >< a_{(1)}, \beta_{(2)} >< \alpha_{(1)}, b_{(2)} > =$$
$$\sum < a_{(2)}, \gamma_{(1)} >< b_{(1)}, \gamma_{(2)} >< \alpha_{(2)}, c_{(1)} >$$
$$< \beta_{(1)}, c_{(2)} >< a_{(1)}, \beta_{(2)} >< \alpha_{(1)}, b_{(2)} >,$$
$$< (a \otimes \alpha), (b \otimes \beta).(c \otimes \gamma) > =$$
$$\sum < a, \beta_{(2)} \gamma_{(1)} >< \alpha, b_{(2)} c_{(1)} >< b_{(1)}, \gamma_{(2)} >< \beta_{(1)}, c_{(2)} > =$$
$$\sum < a_{(1)}, \beta_{(2)} >< a_{(2)}, \gamma_{(1)} >< \alpha_{(1)}, b_{(2)} >$$
$$< \alpha_{(2)}, c_{(1)} >< b_{(1)}, \gamma_{(2)} >< \beta_{(1)}, c_{(2)} > . \square$$

A picture of the above construction can be drawn as follows:



Remark 2 The basic law of super-mathematics is: introduce a sign whenever two strings cross each other. Applying this to our situation, we find that given a super-bialgebra B, and a dual super-bialgebra B^*, we have a symmetric associative algebra $D(B) = B \otimes B^*$, with product

$$(a \otimes \alpha).(b \otimes \beta) = \sum (-1)^s a_{(2)} b_{(1)} \otimes \alpha_{(2)} \beta_{(1)} < a_{(1)}, \beta_{(2)} >< \alpha_{(1)}, b_{(2)} > .$$

Here,

$$s = |a_{(1)}|(|a_{(2)}| + |b_{(1)}|) + (|\alpha_{(2)}| + |\beta_{(1)}|)|\beta_{(2)}| +$$

$$|\alpha_{(1)}||b_{(1)}| + |\alpha_{(2)}||b_{(2)}| + |\alpha_{(2)}||b_{(1)}|,$$

whilst $|x|$ denotes the parity of a homogeneous element x.

let A be a unital associative k-algebra.

Lemma 3 *Let $B = B(A) = k \oplus A$. Then B is a bialgebra, with product*

$$(\lambda, a).(\lambda', a') = (\lambda\lambda', aa'),$$

and coproduct

$$\Delta(a) = 1_k \otimes a + a \otimes 1_k,$$

$$\Delta(1_k) = 1_k \otimes 1_k,$$

for $a, a' \in A$, and $\lambda, \lambda' \in k$. $\square$

Let M be a left A-module. Let d be a natural number.

Let $V = V(d)$ be a d-dimensional free left R-module, with basis $\mathcal{V} = \{v_1, ..., v_d\}$.

Let $X = X(d)$ be the set of right endomorphisms of V which are upper triangular with respect to $\mathcal{V}$. Thus, V is a right X-module, and X has a basis $\mathcal{X} = \{x_{i,j}, 1 \leq i \leq j \leq d\}$ of elementary upper triangular matrices. We write $x_{i,j} = 0$ for $i, j \in \mathbb{Z}$, if either $i \notin \{1, .., d\}$, or $i \notin \{1, .., d\}$, or $i > j$.

Let E be the tensor product of M, V over R, an A-X-bimodule. We write $M_i = M \otimes v_i \subset E$, and given $m \in M$, we write $m_i = m \otimes v_i \in M_i$.

Lemma 4 *Let $C = C(A, M, d) = A \oplus E \oplus X$. Then C is a bialgebra, with product*

$$(a, e, x).(a', e', x') = (aa', ae' + ex', xx'),$$

for $a, a' \in A, e, e' \in E, x, x' \in X$ and coproduct

$$\Delta(a) = (\sum_{l \in \mathbb{Z}} x_{ll}) \otimes a,$$

$$\Delta(m_i) = (\sum_{l \in \mathbb{Z}} x_{l,i+l}) \otimes m_i,$$

$$\Delta(x_{i,j}) = (\sum_{l \in \mathbb{Z}} x_{i+l,j+l}) \otimes x_{i,j},$$

for $a \in A, m \in M, 1 \leq i \leq j \leq d$. $\Box$

If B is a bialgebra, we define B_{assoc} to be the underlying associative algebra.

Lemma 5 *Let B_1, B_2 be bialgebras, with coproducts Δ_1, Δ_2. Suppose that $B = B_{1assoc} \cong B_{2assoc}$, and that*

$$\Delta_1(b).\Delta_2(b') = \Delta_2(b).\Delta_1(b') = 0,$$

for $b, b' \in B$.

Let $\Delta = \Delta_1 + \Delta_2$. Then B a bialgebra, with coproduct Δ. $\Box$

Definition 6 *Let*

$$\mathcal{B}_1 = \mathcal{B}_1(A, M, d) = k \oplus C(A, M, d),$$

$$\mathcal{B}_2 = \mathcal{B}_2(A, M, d) = B(C(A, M, d)_{assoc}).$$

Both $\mathcal{B}_1$ and $\mathcal{B}_2$ are bialgebras, and we have

$$\mathcal{B}_{1assoc} \cong \mathcal{B}_{2assoc} \cong k \oplus C(A, M, d)_{assoc}.$$

The coproducts Δ_1, Δ_2 of $\mathcal{B}_1, \mathcal{B}_2$ multiply to zero on $k \oplus C(A, M, d)_{assoc}$.

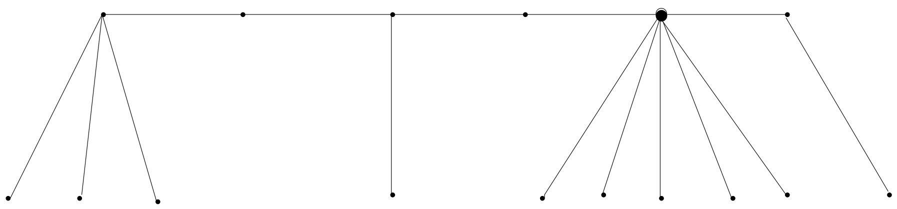
Definition 7 *Let $\mathcal{B} = \mathcal{B}(A, M, d)$ be the bialgebra, whose underlying algebra is $k \oplus C(A, M, d)_{assoc}$, and whose coproduct is $\Delta_1 + \Delta_2$.*

Brauer caterpillars.

Let C be a Brauer caterpillar. In other words, let C be a Brauer tree embedded in the plane, whose sets of edges E and vertices V may be divided into two parts, $E = \mathit{Abdomen} \amalg \mathit{Legs}$, $V = \mathit{Segments} \amalg \mathit{Suckers}$, so that,

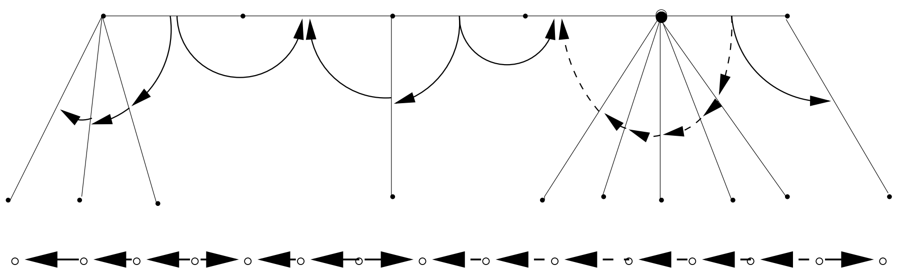
(i) A vertex is a sucker if, and only if, it is connected to precisely one edge.
(ii) An edge is a leg if, and only if, it is connected to a sucker.
(iii) Every abdominal edge is connected to precisely two segments.
(iv) The legs all rest beneath the abdomen.
(v) The exceptional vertex is a segment.

Here is an example of a Brauer caterpillar:



The exceptional vertex has multiplicity e.

To any Brauer caterpillar C, we may associate a quiver $Q = Q(C)$ of type A, whose vertices are in one-one correspondence with the edges of T, as illustrated by the following picture:



We thus start with the leftmost leg of C, and draw an arrow from this to its anticlockwise neighbour. We now draw an arrow between this edge, and its anticlockwise neighbour. We continue drawing arrows between edges and their anticlockwise neighbours, until we reach the rightmost leg of C. We do this in such a way that all arrows surrounding any given segment have the same orientation, whilst two arrows connected to the same abdominal edge should have opposite orientation. The arrows surrounding the exceptional vertex are drawn with dotted lines.

Let $A = kQ$ be the path algebra of Q. Thus A has a basis $\mathcal{P}$ of paths in Q. Let $\mathcal{P}^*$ be the dual basis to $\mathcal{P}$. If $p \in \mathcal{P}$, let p^* be the dual element in $\mathcal{P}^*$.

If v is a vertex of Q, let i_v be the path of length zero stationed at v. The element i_v is an idempotent in kQ.

Let t be the vertex of Q which is the terminus of the longest path in Q which can be obtained by composing dotted arrows. Thus, v is the unique vertex of Q which is the terminus of a dotted arrow, bit the source of no such arrow.

Let $M = kQi_t$. Thus, M is a projective kQ-module, with a basis given by paths of dotted arrows, terminating at v.

Let $d = e - 1$.

We write $\mathcal{V}^* = \{v_i^*\}$ for the basis of V^* dual to $\mathcal{V}$, and given $n \in M^*$, we write $n_i = n \otimes v_i^* \in M_i^*$.

Theorem 8 *Let $\mathcal{B} = \mathcal{B}(A, M, d)$. Let U be the Brauer tree algebra associated to C. Let*

$$T = T(A, M, d) = (A \otimes k^*) \oplus \left(\bigoplus_{i=1}^{e-1} M_i \otimes M_{e-i}^* \right) \oplus (k \otimes A^*).$$

Then T is a subalgebra of $D(\mathcal{B})$, and $T \cong U$.

Proof:

Let us first observe that T is indeed a subalgebra of $D(\mathcal{B})$...

The subalgebra A acts on the left and right of T in the natural way. The product of A^* with itself, and with $\left(\bigoplus_{i=1}^{e-1} M_i \otimes M_{e-i}^* \right)$ is zero. It remains for us to show that the product of elements $m_i \otimes n_{e-i}, m'_j \otimes n'_{e-j}$, lies in T, given $m, m' \in M, n, n' \in M^*$. We have

$$(m_i \otimes n_{e-i}).(m'_j \otimes n'_{e-j}) =$$

$$\sum m_{i(2)} m'_{j(1)} \otimes n_{e-i(2)} n'_{e-j(1)} < m_{i(1)}, n'_{e-j(2)} >< n_{e-i(1)}, m'_{j(2)} > .$$

Assuming this expression is non-zero, the following formulae are implied by our definition of bialgebra structure on $\mathcal{B}$:

$$m'_{j(2)} = m'_j, n_{e-i(1)} = n_j.$$

Tracing carefully through the definition of $\mathcal{B}$, we can now observe two possibilities. The first is,

$$m'_{j(1)} = x_{i,j}, m_{i(2)} = m_i,$$

$$n_{e-i(2)} = x^*_{j,e-i}, j \leq e-i,$$

$$n'_{e-j(1)} = n'_{e-i-j}, j < e-i,$$

$$n'_{e-j(2)} = x^*_{e-i-j,e-j}, m_{i(1)} = x_{e-i-j,e-j},$$

$$m_{i(2)} m'_{j(1)} = m_{i+j},$$

$$n_{e-i(2)} n'_{e-j(1)} = n'_{e-i-j}.$$

The second is,

$$m'_{j(1)} = 1_k, m_{i(2)} = 1_k,$$

$$m'_{j(2)} = m_j, m_{i(1)} = m_i,$$

$$n'_{e-j(2)} \in M^*_i, e-j = i, n'_{e-j(1)} \in A^*,$$

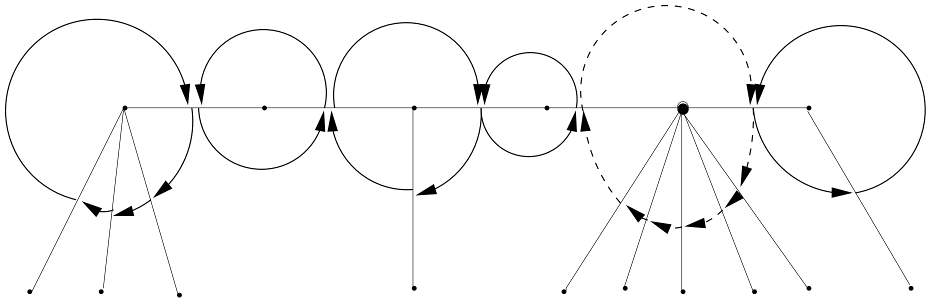
$$n_{e-i(1)} = n_{e-i}, n_{e-i(2)} = x^*_{e-i,e-i},$$

$$m_{i(2)} m'_{j(1)} = 1_k,$$

$$n_{e-i(2)} n'_{e-j(1)} \in A^*.$$

To clarify, we have shown that multiplication between $M_i \otimes M^*_{e-i}$ and $M_j \otimes M^*_{e-j}$ is just multiplication $End(M) \otimes End(M) \to End(M)$, when $i+j < m$. In case $i+j = m$, this multiplication is given by the map $M \otimes M^* \to A^*$, dual to the map $A \to End(M) \cong M \otimes M^*$ defined by the action of A on M. In case $i+j > m$, this multiplication is zero.

For a segment s, let q_s be an arrow which completes the path around the edges attached to s to a loop around s. The algebra U is generated by the path algebra of the quiver $Q' = Q \cup \{q_s, s \in Segments\}$:



Let $\mathcal{P}_{ord}$ be the set of paths around ordinary segments s in C, which are subpaths of a circuit c about s. Let $\mathcal{P}_{exc}$ be the set of paths around the exceptional segment s of C, which are subpaths of c^{d+1}. A basis for U is given by $\mathcal{P}_{ord} \amalg \mathcal{P}_{exc}$. We have,

$$dim(e_v U e_w) = dim(e_v T e_w) =$$

$$\begin{cases} d & \text{if } v = w \text{ is an exceptional abdominal edge,} \\ 2 & \text{if } v = w \text{ is not an exceptional abdominal edge,} \\ d-1 & \text{if } v \neq w \text{ are edges attached to the exceptional segment,} \\ 1 & \text{if } v \neq w \text{ are edges attached to a single ordinary segment,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, T and U have the same dimension.

We wish to identify T with U. We must introduce an extra generator a_s of T for every segment s of C. Let $p = p(s)$ be the longest path in Q around s. If s is not the exceptional segment, let $a_s = (1_k \otimes p^*)$, where $p \in A$. If s is the exceptional segment, then let $a_s = i_{t1} \otimes p_d^*$, where t is the terminal vertex of p. Note that $i_{t1} \in M_1, p_d^* \in M_d$.

Let $p = r.q$ be a decomposition of p as the product of two paths q, r. Let v be the tail of r, otherwise known as the source of q. If s is not the exceptional segment, then

$$(q \otimes 1_k^*).a_s.(r \otimes 1_k^*) = (q \otimes 1_k^*).(1_k \otimes p^*).(r \otimes 1_k^*) =$$

$$(q \otimes 1_k^*).(1_k \otimes q^*) = 1_k \otimes i_v^*,$$

in T. Otherwise, if s is the exceptional segment, then our previous computations of multiplication on T tell us,

$$((q \otimes 1_k^*).a_s.(r \otimes 1_k^*))^{d+1} = ((q \otimes 1_k^*).(i_{t1} \otimes p_d^*).(r \otimes 1_k^*))^{d+1} =$$

$$(q \otimes 1_k^*).(i_{t1} \otimes p_d^*).(p_1 \otimes p_d^*)^d.(r \otimes 1_k^*) =$$

$$(q \otimes 1_k^*).(i_{t1} \otimes p_d^*).(p_d \otimes p_1^*).(r \otimes 1_k^*) =$$

$$(q_1 \otimes p_1^*).(p_d \otimes q_d^*) = 1_k \otimes i_v^*,$$

in T.

An algebra homomorphism ϕ is now visible from U to T, which identifies arrows in Q, and takes q_s to a_s, for segments s. The computations above show that $c = 1_k \otimes e_v^*$, for circuits c about segments different from the exceptional vertex, whilst $c^{d+1} = 1_k \otimes e_v^*$, for circuits c around the exceptional segment. Such conditions imply that ϕ is injective. Since U and T have the same dimension, ϕ is an isomorphism. $\square$

Corollary 9 *Let B be a block of a finite group, of cyclic defect. Then*

$$D^b(B - mod) \cong D^b(T(A, M, d) - mod),$$

for some algebra A, module M, and dimension d. $\square$

Remark 10 It was my ambition, as I began to write this letter, to give a Schur algebra formulation of the above constructions, generalising the case $d = 0$ of other articles [9], [10], [11]. I would expect the double resulting from such a generalisation to describe blocks of finite general linear groups in nondescribing characteristic, up to derived equivalence. However, it is not clear to me how to make this extension.

References

[1] J. Alperin, Local Representation Theory, Cambridge Studies in Advanced Mathematics **11**, Cambridge Univ. Press, 1986.

[2] J. Alperin, Weights for finite groups, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), *Proc. Sympos. Pure Math.* **47**, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[3] M. Broué, Isométries de caractéres et équivalences de Morita ou dérivées, *Inst. Hautes Études Sci. Publ. Math.* **71** (1990), 45-63.

[4] M. Broué, Isométries parfaites, types de blocs, catégories dérivées, *Asterisque* **181-182** (1990), 61-92.

[5] M. Geck, Brauer trees of Hecke algebras, *Communications in Algebra* **20** (1992), 2937-2973.

[6] M. Linckelmann, Derived equivalences for cyclic blocks over a P-adic ring, *Mathematische Zeitschrift* **207** (1991), 293-304.

[7] J. Rickard, Derived categories and stable equivalences, *Journal of Pure and Applied Algebra* **61** (1989), 303-317.

[8] R. Rouquier, The derived category of blocks with cyclic defect groups, *Derived equivalences for group rings*, Springer Lecture notes in mathematics **1685** (1998), 199-220.

[9] W. Turner, Rock blocks (2004), *Representations and Cohomology archive*, http://www.maths.abdn.ac.uk/ bensondj/html/archive.html.

[10] W. Turner, On seven families of algebras (2005), *Representations and Cohomology archive* http://www.maths.abdn.ac.uk/ bensondj/html/archive.html.

[11] W. Turner, Tilting equivalences: from hereditary algebras to symmetric groups (2006), *Representations and Cohomology archive*, http://www.maths.abdn.ac.uk/ bensondj/html/archive.html.