CUBES AND COHOMOLOGY.

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Abstract. Via a species of noncommutative symplectic reduction, we associate algebras to parallelohedral tilings of Euclidean space. Such tilings arise as projections of cubical complexes in $\mathbb{Z}^r$ onto a linear space of dimension w. We study the combinatorics of such tilings, along with homological aspects of the corresponding algebras. In case $w = 2$, we obtain new theorems concerning planar tilings, such as the quasi-periodic rhombic tilings of R. Penrose.

We give an alternative definition, that is more general, in which algebras are associated to rhombohedral tilings of arbitrary Riemannian manifolds.

The Cubist algebras studied in the prequel to this paper correspond to the case $w = r-1$ [8]. In case $r = \frac{w(w+1)}{2}$, we expect there to be relations between Cubist algebras, and modular representations of symmetric groups, as we have previously found in case $w = 2$.

1. Introduction

Study of the intricate communion between algebra and geometry is an ancient ritual. According to modern orthodoxy, the mathematical approach to a space begins with the association of an algebra, such as a group of symmetries, a ring of functions, a group of homotopies, or a category of sheaves. Conversely, the authority of an algebraic object is often magnified when one associates to it some geometry, such as a space on which it acts, a space of which it is the fundamental group, or a category of representations.

Our paper is one more opus in this geometric-algebraic tradition. We couple certain associative algebras to rhombohedral tilings of a Riemannian manifold. These algebras are highly noncommutative, and their structure is reflected in the combinatorics of the corresponding tiling. We consider in detail the algebras associated to tilings of Euclidean space; much harmony is to be found in their nature, which is revealed homologically. We expect applications to modular representation theory, and to toric geometry.

I am most grateful to Joe Chuang, and to Raphaël Rouquier for their interest and insight.

2. Progression

We consider aspects of geometry and homological algebra associated to cubical tilings of Riemannian manifolds. We are especially interested in tilings of Euclidean space, where a detailed theory can be developed.

In chapter three, we examine geometric and combinatorial aspects of parallelohedral tilings of Euclidean space. We describe how such a parallelohedral tiling can be realised via linear projection of a subcomplex of an integral cubical tiling of some larger Euclidean space. We describe how such sets can arise, by taking a layer parallel to a linear subspace of Euclidean space. Examples arising this way include quasi-periodic Penrose tilings.

In chapter four, we consider bijections between facets and vertices of parallelohedral tilings, induced by vectors in the ambient Euclidean space. In chapter five, we consider related orderings on the collection of vertices. Orderings of a particular kind are necessary for our development of a homological theory associated to such a tiling.

In chapter six, we state our algebraic setup, which coincides with that of the prequel to this paper, "Cubist algebras" [8].

In chapter seven, we define the Cubist algebras, which are associated to parallelohedral tilings of Euclidean space. They possess a number of strong homological properties. In chapter eight, we observe that the Cubist algebras are Koszul, that they have highest weight module categories, that they obey a strict form of Serre duality, and that their derived categories admit a polarization [25]. We prove that the category of standardly filtered modules is a thick subcategory of the stable category.

In the prequel to this paper, we considered algebras associated to tilings of a special kind: those arising from projections of codimension one. Our proofs of Koszulity, and of the existence of highest weight structures resemble the proofs of the prequel, and therefore the passages in which the proofs are identical are only briefly outlined.

We describe derived equivalences between Cubist algebras in chapter nine. Such arise from local mutations on the relevant tiling. Similar results were also obtained in the prequel. However, here our exposition is quite different. We work less combinatorially, and more homologically. Whilst some of the combinatorial results of the prequel are missed by our new approach, further insight is gained, from a homological perspective. In particular, we see the derived equivalences are induced by a tilting bimodule. In chapter ten, we use this fact to show the derived equivalences are compatible with gradings in a particular way. We also observe the Cubist algebras obey an algebraic condition akin to integrability.

In the eleventh chapter, we consider deformations of Cubist algebras. Deformation parameters are in a natural correspondence with certain Cubist subquotients of the algebra, or combinatorially, with parallel strips in the tiling. These parallel strips are analogous to the nodes of a Dynkin diagram in classical Lie theory.

In chapter twelve, we generalize the definition of Cubist algebras to rhombohedral tilings of arbitrary Riemannian manifolds. Given a pair of tiled submanifolds of a Riemannian manifold tiled by rhombohedra, there is a functor between the representation categories of the corresponding Cubist algebras.

In chapter thirteen, we describe a conjectural application we had in mind when we began this work: an asymptotic description of modular representations of symmetric groups.

The combinatorial results of this paper are all exercises in discrete Euclidean geometry. Their proofs were figured out after drawing numerous rough pictures with a pen and paper. Many arguments and definitions, which might seem opaque in writing, become readily comprehensible after a couple of sketches. We have therefore included a number of figures. However, a large gallery would be required, if we were to exhibit all the doodles we have made, and we exhort the reader to make his own drawings as he makes his way through the article.

3. Cubist geometry

We consider here parallelohedral pavings of Euclidean space, and the associated combinatorics.

Let r be a natural number, or ∞, and let $\underline{r} = \{1, ..., r\}$. Let $E = \mathbb{R}^{\oplus \underline{r}}$ denote Euclidean space, of dimension r. Let ϵ_i denote the standard basis elements of E, for $i \in \underline{r}$.

Suppose $S \subset \underline{r}$. Let $F_S = F_1 \times ... \times F_r \subset E$, where $F_i = [0, 1]$, if $i \in S$, and $F_i = \{0\}$, if $i \notin S$.

Let $\mathcal{Z} = \mathcal{Z}_r$ denote the polytopal complex, homeomorphic to E, whose i-dimensional cells are i-cubes in E of the form $x + F_S, x \in \mathbb{Z}^r, |S| = i$. Let $\mathcal{Z}_r^{(j)}$ denote the j-skeleton of $\mathcal{Z}_r$, for $0 \leq j \leq r$. In other words, $\mathcal{Z}_r^{(j)}$ is the polytopal complex whose facets take the form $x + F_S, x \in \mathbb{Z}^r, |S| \leq j$.

Let w be a finite natural number between 0 and r. Let $c = r - w$. Let H be a w-dimensional vector space. Let $p : E \twoheadrightarrow H$ be a surjective linear map.

If E is finite dimensional, then we can identify H with the subspace $ker(p)^\perp$, and p with the orthogonal projection of E onto H.

Definition 1. *A polytopal subcomplex $\mathcal{C} \subset \mathcal{Z}$ is* Cubist *relative to H, if the projection $p : \mathcal{C} \to H$ is a homeomorphism.*

If $\mathcal{C}$ is a Cubist complex, we denote by $\mathcal{F}_i$ the set of i-dimensional cells in $\mathcal{C}$. We write $\mathcal{X} = \mathcal{F}_0$, and $\mathcal{F} = \mathcal{F}_w$.

Remark 2 Let $i : T(w) \to T(r)$ be an embedding of a torus of dimension w in a torus of dimension r. Applying $Hom(-, k^*)$, we obtain a map $\mathbb{Z}^r \to \mathbb{Z}^w$. Tensoring over $\mathbb{R}$, we obtain a linear map $E \to H$. Whilst not all the projections we consider are realised in this way, it can be conceptually helpful to think of our projection p as coming from a toric embedding.

Example 3 Let E be the permutation representation of the symmetric group Σ_r, defined by the action of Σ_r on $\underline{r}$. Let H be the $r-1$-dimensional irreducible component of E. Let $\mathcal{X}^-$ be a nonempty proper ideal of the partially ordered set $\mathbb{Z}^r$. Let $\mathcal{X} = \mathcal{X}^- \backslash (\mathcal{X}^- - (1, ..., 1))$. Let $\mathcal{C}$ be the subcomplex of $\mathcal{Z}$ whose cells are all cubes of the form $x + F_S$, for $x \in \mathcal{X}$, $S \subset \underline{r}$, such that $x + F_S$ is a subset of $\mathcal{X}$. Then $\mathcal{C}$ is a Cubist complex in E of dimension $r - 1$ (see [8], 2.2).

We now describe a special collection of Cubist sets, which are obtained by taking layers close to affine subspaces of Euclidean space.

Suppose $r < \infty$. Let $H \subset E$ be a subspace of dimension w. Let p denote the orthogonal projection of E onto H. Let $\pi : E \to H^\perp$ denote the orthogonal projection of E onto the orthogonal complement of H. Thus $H^\perp$ is a c dimensional space.

Identified as a subset of $\mathbb{R}^r$, the topological space $\mathcal{Z}_r^{(j)}$ is the set of vectors with at least $r - j$ integral coordinates.

Suppose that $H^\perp = \pi(\mathcal{Z}_r^{(c)})$. Let $x \in H^\perp \backslash \pi(\mathcal{Z}_r^{(c-1)})$. Let $\mathcal{C}(H, x)$ denote the complex whose cells are those cells of $\mathcal{Z}_r$ which are contained in a cube of the form $x + h + [0, 1]^r$, for some $h \in H$.

Theorem 4. *$\mathcal{C}(H, x)$ is a Cubist complex, relative to H.*

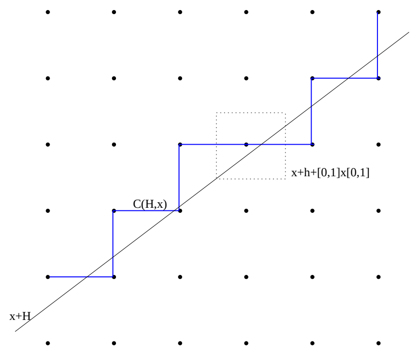

FIGURE 1. A Cubist set $\mathcal{C}(H, x)$, in case $w = 1$, and $r = 2$.

Before proving Theorem 4, let us make some preliminary comments on duality for polytopal complexes.

Recall that two w-dimensional polytopal complexes $\mathcal{A}$ and $\mathcal{A}'$ are said to be *dual* if the i-cells of $\mathcal{A}$ are in one-one correspondence with the $w - i$-cells of $\mathcal{A}'$, so that inclusion of faces in $\mathcal{A}$ corresponds to containment of faces in $\mathcal{A}'$, and inclusion of faces in $\mathcal{A}'$ corresponds to containment of faces in $\mathcal{A}$.

Lemma 5. *Suppose $\mathcal{A}$ is a polytopal complex, homeomorphic to $\mathbb{R}^w$. Then there exists a polytopal complex $\mathcal{A}'$, dual to $\mathcal{A}$, and a homeomorphism from $\mathcal{A}'$ to $\mathbb{R}^w$.*

Proof. Any polytope P in $\mathbb{R}^n$ containing 0 has a dual polytope P' (see [15]), which can be defined to be

$$P' = \{x \in \mathbb{R}^n | \langle x, P' \rangle \leq 1\}.$$

Let us define a set $P(\mathcal{A})$, by stereographically projecting $\mathcal{A} \cong \mathbb{R}^w$ onto a w-dimensional sphere in $\mathbb{R}^{w+1}$, and then taking the convex hull in $\mathbb{R}^{w+1}$ of the resulting vertices, all of which lie on the sphere. Note that $P(\mathcal{A})$ is not in fact a polytope, since it has an accumulation of faces at infinity, but we can define its dual $P(\mathcal{A})'$ nonetheless. We then define $\mathcal{A}'$ to be the polytopal complex obtained by linearly projecting the w-skeleton of $P(\mathcal{A})'$ onto the w-dimensional sphere from 0, before applying the inverse of stereographic projection, onto $\mathbb{R}^w$. Then $\mathcal{A}'$ is a polytopal complex, dual to $\mathcal{A}$, and homeomorphic to $\mathbb{R}^w$, as required. $\square$

Proof of Theorem 4: If C is an r-dimensional cell in $\mathcal{Z}_r$ which is contained in a cube of the form $x + h + [0,1]^r$, for some $x \in H^\perp, h \in H$, then $C = x + h + [0,1]^r$, and so $x + h \in \mathbb{Z}^r = \mathcal{Z}_r^{(0)}$. Therefore $x \in \pi(\mathcal{Z}_r^{(0)})$.

Now suppose $0 \leq i \leq c-1$. More generally, if C is an $r-i$-dimensional cell in $\mathcal{Z}_r$ which is contained in a cube of the form $x+h+[0,1]^r$, for some $x \in H^\perp, h \in H$, then $x + h$ has at least $r - i$ integral components. Therefore $x + h \in \mathcal{Z}_r^{(i)}$, and we conclude $x \in \pi(\mathcal{Z}_r^{(i)})$.

We have assumed that $x \notin \pi(\mathcal{Z}_r^{(c-1)})$. Therefore, every cell of $\mathcal{Z}_r$ which is contained in a cube of the form $x + h + [0,1]^r$, for some $h \in H$, has dimension smaller than $r - (c-1)$. In other words, every cell of $\mathcal{C} = \mathcal{C}(H,x)$ has dimension less than, or equal to, w.

We have also assumed that $x \in H^\perp = \pi(\mathcal{Z}_r^{(c)})$, and therefore $x + h \in \mathcal{Z}_r^{(c)}$, for some $h \in H$. For such an element h, the cube $x + h + [0,1]^r$ contains a w-cell of $\mathcal{Z}_r^{(c)}$. It follows that $\mathcal{C}$ contains some w-cell. In fact, each w-cell C in $\mathcal{C}$ corresponds to a unique point $h_C \in H$, such that $C \subset x + h_C + [0,1]^r$. More generally, each $w - i$-cell C in $\mathcal{C}$ corresponds to a $w - i$-dimensional set

$$h_C = \{h \in H \mid C \subset x + h + [0,1]^r\} \subset H.$$

Since all our spaces are linear in nature, h_C is enclosed by a collection of hyperplanes in a $w - i$-dimensional affine space. Therefore, h_C is a compact convex polytope.

We have thus shown that H is naturally homeomorphic to a polytopal complex $\mathcal{C}'$, whose i-cells are in correspondence with $w - i$-cells of $\mathcal{C}$. Inclusions of faces in $\mathcal{C}'$ correspond to reverse inclusions of faces in $\mathcal{C}$. In other words, the complex $\mathcal{C}'$ is dual to $\mathcal{C}$.

By Lemma 5, $\mathcal{C}$ is homeomorphic to H. It remains for us to prove that the projection map p defines a homeomorphism from $\mathcal{C}$ to H. To see this, some more delicate analysis is required.

Let x be a vertex of $\mathcal{C}$, and let X_x be the w-cell of $\mathcal{C}'$ which is dual to x. If y is a 1-cell of $\mathcal{C}$ containing x, then the line in H normal to the corresponding $w-1$-cell in X_x is parallel in H to the line $p(y)$. More generally, the affine subspace normal to a $w - i$-cell in X_x is parallel to the projection via p of the corresponding i-cell of $\mathcal{C}$ containing x. Let M_x denote the union of cells of $\mathcal{C}$ containing x. Let $N_x = p(M_x)$, a submanifold of H. By this parallel property, there is a polyhedron P_x contained in N_x, centred at $p(x)$, whose vertices lie on the lines $p(y)$, which is a polytopal dual polyhedron of X_x. Since P_x is the dual of the dual of the local configuration about x, in fact p restricts to a homeomorphism inside $p^{-1}(P_x)$. Furthermore, M_x is a piecewise linear extension of $p^{-1}(P_x)$, and we conclude p restricts to a homeomorphism from M_x to N_x.

Any face of $\mathcal{C}$ containing x takes the form $F = x + \sum_{s \in S}[0,1]\sigma_s\epsilon_s$, where $S \subset \underline{r}$ is a set of w elements, and $\sigma_s = \pm 1$. For such a face, let us define $F_x^{\frac{1}{2}} = x + \sum_{s\in S}[0, \frac{1}{2}]\sigma_s\epsilon_s$. Let $M_x^{\frac{1}{2}} = \bigcup_{x\in F} F_x^{\frac{1}{2}}$, a subset of M_x, whose volume is $\frac{1}{2^w}$ of the volume of M_x. Let $N_2^{\frac{1}{2}} = p(M_x^{\frac{1}{2}})$. The restriction of p to $M_x^{\frac{1}{2}}$ is a homeomorphism between $M_x^{\frac{1}{2}}$ and $N_x^{\frac{1}{2}}$. Furthermore, the vertices of X_x are in bijection with centres of faces F containing x, all of which are vertices of $N_x^{\frac{1}{2}}$. The polytope X_x can be homeomorphically deformed into $N_x^{\frac{1}{2}}$, in such a way as to conform with this bijection.

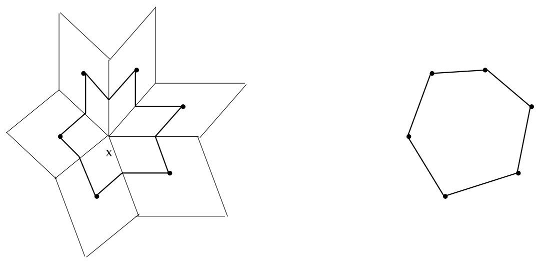

FIGURE 2. The sets $N_x^{\frac{1}{2}}$ and X_x.

We can write $\mathcal{C}$ as a union $\bigcup_{x\in\mathcal{X}} M_x^{\frac{1}{2}}$, since for every element x of a cube there is a vertex v, whose coordinates' lengths differ from those of x by at most one half. We have an entirely analogous decomposition of $\mathcal{C}'$ as a union $\bigcup_{x\in\mathcal{X}} X_x$. We can homeomorphically deform the polyhedra X_x, to obtain a decomposition of H as a union $\bigcup_{x\in\mathcal{X}} N_x^{\frac{1}{2}}$. We thus have a collection of commuting diagrams

$$\begin{tikzcd} M_x^{\frac{1}{2}} \arrow[rr, "p"] \arrow[dr, "\sim"'] & & N_x^{\frac{1}{2}} \arrow[dl, "\sim"] \\ & X & \end{tikzcd}$$

which unite to give a commutative diagram

$$\begin{tikzcd} \mathcal{C} = \bigcup_{x\in\mathcal{X}} M_x^{\frac{1}{2}} \arrow[rr, "p"] \arrow[dr, "\sim"'] & & \bigcup_{x\in\mathcal{X}} N_x^{\frac{1}{2}} \arrow[dl, "\sim"] \\ & H = \bigcup_{x\in\mathcal{X}} X_x & \end{tikzcd}$$

Since the downwards pointing arrows in this diagram are homeomorphisms, we conclude the projection p from $\mathcal{C}$ to H is also a homeomorphism, as required. $\square$

Example 6 Let $r = 5$. Let $w - 2$. Let C_5 be the cyclic group of order 5, with generator g. Let E be the regular representation of C_5 defined over $\mathbb{R}$, with the standard inner product defined by $\langle g^a, g^b\rangle = \delta_{ab}$, for $a, b \in \mathbb{Z}/5$. Then E decomposes as an orthogonal direct sum of irreducible representations, $E = \mathbb{R} \oplus E_1 \oplus E_2$, where E_i is a two dimensional irreducible representation, on which g acts as rotations by $2\pi i/5$.

Let $H = E_i$, for some i. Let $x \in H^{\perp} \backslash \pi(\mathcal{Z}_5^{(2)})$. Then $\mathcal{C}(H,x)$ is a Cubist subset of E of dimension 2, and the projection of $\mathcal{C}(H,x)$ onto H gives a tiling of the plane by rhombi, whose angles are a multiple of $\frac{\pi}{5}$. Our planar tiling is in fact a quasi-periodic Penrose tiling [21], as has been observed by de Bruijn [11]. Indeed, H is irrationally sloped, and so the local relation of the integer lattice to H at distinct lattice points can be arbitrarily close, but never identical. It follows that the tiling has no translational symmetry, and yet a copy any finite region of $\mathcal{C}(H,x)$ is to be found elsewhere in $\mathcal{C}(H,x)$, in infinitely many places.

Aperiodic planar tilings similar to those discovered by Penrose have been found in Islamic shrines, going back 500 years [18]. Beautiful pictures of Penrose tilings, and other rhombic tilings of the plane, can be found on the website of Jos Leys [17].

As the above example illustrates, there is a close intimacy between Cubist sets, and tilings of Euclidean space. Let us make this intuitive idea formal.

Definition 7. *A* parallelohedral tiling *of Euclidean space T is a cell complex $\mathcal{C}$, and a homeomorphism ϕ from that cell complex to T, such that*

(i) The image under ϕ of every i-cell in $\mathcal{C}$ is a parallelohedron of dimension i.

(ii) If two cells C_1, C_2 of $\mathcal{C}$ have a non-empty intersection, then the image $\phi(C_1) \cap \phi(C_2)$ of that intersection, is a face of the parallelohedron $\phi(C_i)$, for $i = 1, 2$.

When we discuss a tiling T, we call the images of cells in $\mathcal{C}$ under ϕ, cells of T. We call the cells of T with the same dimension as T, the *tiles* of T.

Now suppose C is a 1-cell in a tiling T of Euclidean space by parallelohedra. Let us write $C = \eta + [0,1]\zeta$.

Definition 8. *A* strip *of T, parallel to C, is a cell subcomplex Υ of T, minimal such that*

(i) Υ contains at least one tile.

(ii) If t_1, t_2 are tiles of T, if C is a face of the parallelohedron $x + t_1 \cap t_2$, for some $x \in T$, and if t_1 is a tile of T, then t_2 is a tile of Υ.

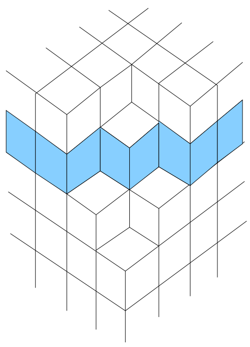

FIGURE 3. A strip

Lemma 9. *Suppose T is a tiling of Euclidean space by parallelohedra of dimension w. Any strip Υ of T, parallel to C, is homeomorphic to $\mathbb{R}^{w-1} \times C$. The complement $T \backslash \Upsilon$ is a union of two connected components,*

$$T\backslash\Upsilon = (T\backslash\Upsilon)^+ \amalg (T\backslash\Upsilon)^-,$$

where $(T\backslash\Upsilon)^+ = \Upsilon + \mathbb{R}_{>1}\zeta$, and $(T\backslash\Upsilon)^- = \Upsilon + \mathbb{R}_{<-1}\zeta$.

Proof. Consider the projection π of a strip Υ onto $C^\perp$. The image is nonempty, by assumption (i). Let t be a tile of Υ. By the minimality of Υ, the strip Υ is a union of tiles $t = t_1, t_2, t_3, ...$, where $t_i \cap t_{i+1}$ has a face $x_i + C$, for some x_i. Since the fibres of π, restricted to t_i are all isomorphic to C, we conclude that $\Upsilon \cong \pi(\Upsilon) \times C$. Note however, that π is surjective, by assumption (ii). Thus, Υ is homeomorphic to $\mathbb{R}^{w-1} \times C$, as required.

The fibres f of π are all isomorphic to $\mathbb{R}$. Such fibres can be written $f = f^- \amalg f_0 \amalg f^+$, where $f_0 = f \cup \Upsilon \cong C$, and where $f^- < f^0 < f^+$. For this reason, we have a decomposition of T, as written. □

For a fixed 1-cell C of T, let $\mathcal{S}_C$ denote the set of strips of T, parallel to C. As an immediate corollary of lemma 9, we can index such strips by an interval:

Lemma 10. *There is an isomorphism γ between $\mathcal{S}_C$ and an interval in $\mathbb{Z}$, such that $\gamma(\Upsilon) \leq \gamma(\Upsilon')$ if, and only if, $\Upsilon \subset \Upsilon' + \mathbb{R}_{\leq 0}\zeta$.*

Lemma 11. *The projection of a Cubist complex onto H gives a parallelohedral tiling of H.*

Conversely, if T is a tiling of w dimensional Euclidean space by parallelohedra, then T is the projection of some Cubist complex in Euclidean space onto a subspace of dimension w.

Proof. A Cubist complex $\mathcal{C}$ is tiled by w-dimensional cubes. Since the image of a cube under a linear map is a parallelohedron, such a complex $\mathcal{C}$ defines a tiling of H by parallelohedra.

Now suppose T is a tiling of $H = \mathbb{R}^w$ by parallelohedra. Let us translate all 1-cells of T inside H so that they form a collection of vectors $\Xi = \{\xi_\alpha\}_{\alpha \in A}$ be the set of all vectors in H obtained by translating 1-cells to the origin. We make choices as we define Ξ, so that 1-cells which are parallel of the same length are identified in Ξ. Therefore, $\xi_\alpha \in \Xi$ implies that $-\xi_\alpha \notin \Xi$. However, parallel 1-cells of different lengths, we do not identify in Ξ.

Let $\alpha \in A$. The collection of slices parallel to ξ_α can be identified with an interval $I_\alpha \subset \mathbb{Z}$. For each α, let us fix such an identification γ_α, so that I_α contains zero. The strip associated to $i \in I_\alpha$, we denote $\Upsilon_\alpha(i)$. We write

$$T_\alpha(i) = (T\backslash\Upsilon_\alpha(i+1))^- \cap (T\backslash\Upsilon_\alpha(i))^+,$$

for the region lying between $\Upsilon_\alpha(i)$ and $\Upsilon_\alpha(i+1)$.

If t lies on strip $\Upsilon_\alpha(i)$, we write $t_\alpha = i + x$, where $x \in [0,1]$ is the v_α coordinate of t in $\Upsilon_\alpha(i)$. If t lies on $T_\alpha(i)$, we write $t_\alpha = i$. The function $t \mapsto t_\alpha$ is a continuous map from T to $\mathbb{R}$.

Let us choose an identification of A with $\underline{r}$, where $r = |A|$. Let $p : E \to H$ denote the linear map taking ϵ_α to v_α. We have a map from T to $\mathcal{Z}$, taking t to $\sum t_\alpha \epsilon_\alpha$. This provides a splitting of the projection $p : \mathcal{Z} \to H$, and realises T as a Cubical subcomplex of $\mathcal{Z}$, as required. □

Example 12 Cubist sets of dimension w, and codimension 1, as in Example 3, are in one-one correspondence with tilings of Euclidean space of dimension w, whose vertices form the weight lattice of SL_{w+1}.

Remark 13 A *strip* in a Cubist set $\mathcal{C}$ is the intersection of $\mathcal{C}$ with $\mathbb{R}^{l-1} \times [z, z+1] \times \mathbb{R}^{r-l}$, for some $l \in \underline{r}$, and some integer z.

Strips of Cubist sets are naturally identified with strips of parallelohedral tilings, under the correspondence of Lemma 11.

Let $Cub(r, w)$ denote the collection of Cubist subsets of dimension w, in Euclidean space of dimension r.

For $l \in \underline{r}$, let q_l denote the projection of $E = \mathbb{R}^r$ onto the l^{th} component.

Lemma 14. *Let $\mathcal{C} \in Cub(r, w)$, such that $[z, z+1] \subset q_l(\mathcal{C})$. Then there exists a Cubist set $\mathcal{C}_{l,z} \in Cub(r-1, w-1)$, such that*

$$\mathfrak{l} = \mathfrak{l}_{l,z} = \{x \in \mathcal{C} | q_l(x) = z, x + \epsilon_l \in \mathcal{C}\} \cong \mathcal{C}_{l,z}.$$

Proof. Let C be a w-dimensional cube in $\mathcal{C}$, such that $q_l(C) = [z, z+1]$. If F is a face of C of dimension $w-1$, such that $q_l(F) = [z, z+1]$, then there exists a w-dimensional cube C' in $\mathcal{C}$, distinct from C, which contains F. Then $q_l(C') = [z, z+1]$. Let us define a polytopal complex C_1 by adjoining C' to C.

Continuing to extend in this manner, in all directions, we may define a polytopal complex which, upon projection to $H_l = H \cap \epsilon_l^\perp$, covers a large region of H_l. Extending infinitely, we obtain a polytopal complex which projects onto the $w-1$-dimensional space H_l. The fibres take the form $x + [0,1]\epsilon_l$, where $q_l(x) = z$. The collection of elements x appearing in such a fibre form a Cubist complex $\mathcal{C}_{l,z}$ of dimension $w-1$. These elements are precisely those which lie on the boundary of $\mathcal{C} \cap q_l^{-1}((-\infty, z])$, which is homeomorphic to H_l. But the boundary of $\mathcal{C} \cap q_l^{-1}((-\infty, z])$ consists precisely of those elements, such that $q_l(x) = z, x + \epsilon_l \in \mathcal{C}$. This completes the proof of the lemma. □

Definition 15. *Given $\mathcal{C} \in Cub(r, w)$, such that $[z, z+1] \subset q_l(\mathcal{C})$, let*

$$\mathcal{C}^{S_\mathfrak{l}} = S_{z,l}(\mathcal{C}) = \left(q_l^{-1}((-\infty, z] \cap \mathcal{C})\right) \cup \left(q_l^{-1}([z+1, \infty) \cap \mathcal{C}) - \epsilon_l\right)$$

be the Cubist set, obtained by slicing *the strip $\mathfrak{l}_{l,z} + [z, z+1]\epsilon_l$ from $\mathcal{C}$.*

Let $l \in \underline{r}$. Let $E^l = \epsilon_l^\perp$, a subspace of E of dimension $r-1$. Let $H^l = p(\epsilon_l)^\perp$, a subspace of H of dimension $w-1$.

For $z \in \mathbb{Z}$, let $\mathcal{Z}^z = \mathcal{Z} \cap q_l^{-1}(z)$, a polytopal complex homeomorphic to E^l.

The inverse procedure to slicing from a Cubist set is parting it:

Lemma 16. *Let $\mathcal{C} \in Cub(r,w)$, Suppose that $\mathfrak{l}$ is a polytopal subcomplex of $\mathcal{C} \cap \mathcal{Z}^z$, such that $\mathfrak{l}$ is a Cubist subset relative to H^l. Then there exists a Cubist set $\mathcal{C}^{P_\mathfrak{l}} \in Cub(r,w)$, obtained by* parting *$\mathcal{C}$ along $\mathfrak{l}$ in direction ϵ_l. We have*

$$(\mathcal{C}^{P_\mathfrak{l}})^{S_\mathfrak{l}} = \mathcal{C}.$$

Given $\mathcal{C} \in Cub(r,w)$ such that $[z, z+1] \subset q_l(\mathcal{C})$, and $\mathfrak{l} \cong \mathcal{C}_{l,z}$, we have

$$(\mathcal{C}^{S_\mathfrak{l}})^{P_\mathfrak{l}} = \mathcal{C}.$$

Proof. Note that $H \backslash H^l$ has two connected components, E^+, E^-. Without loss of generality, we may assume that $q_l(E^-) \leq z$, and $q_l(E^+) \geq z$. Since $\mathfrak{l}$ is homeomorphic to H^l, $\mathcal{C} \backslash \mathfrak{l}$ also has two connected components, $\mathcal{C}^+, \mathcal{C}^-$. We define

$$\mathcal{C}^{P_\mathfrak{l}} = \mathcal{C}^- \amalg (\mathfrak{l} + [0,1]\epsilon_l) \amalg (\mathcal{C}^+ + \epsilon_l),$$

a Cubist subcomplex of $\mathcal{Z}$ relative to H, which satisfies the hypotheses of the lemma. $\square$

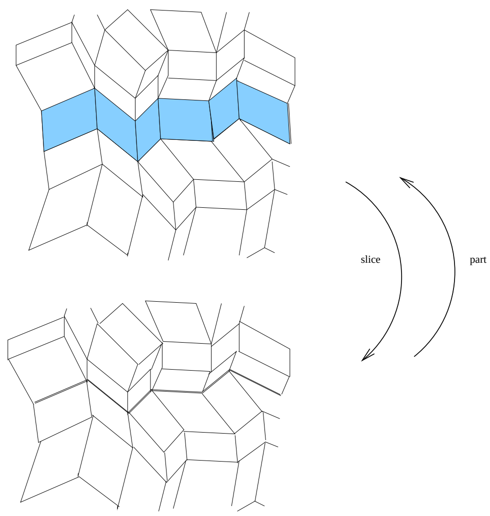

FIGURE 4. Slicing and parting a Cubist set

Let $Cub(r,w)$ denote the collection of Cubist complexes of dimension w in Euclidean space of dimension r.

Let $Cub_b(r,w)$ denote the set of convex Cubist complexes with boundary. By definition, these are w-dimensional polytopal subcomplexes $\mathcal{C}$ of $\mathcal{Z}$, such that $p : \mathcal{C} \to H$ is an embedding, whose image is a convex subset of H.

Lemma 17. *Any element of $Cub_b(r,w)$ can be extended to an element of $Cub(r,w)$.*

Proof. Let $\mathcal{C} \in Cub_b(r, w)$. Suppose that $\mathcal{C}$ is not an element of $Cub(r, w)$.

Since $p(\mathcal{C})$ is convex, we may find $i \in \underline{r}$, $\sigma \in \{\pm 1\}$, and a $w-1$-dimensional face F on the boundary of $\mathcal{C}$, such that $p((F + \sigma t \epsilon_i) \cap \mathcal{C}) = \emptyset$, for all $t \in \mathbb{R}_{>0}$. Upon fixing such an i, σ, we define Θ to be the collection of all $w-1$-dimensional faces F of $\mathcal{C}$, such that $p((F + \sigma t \epsilon_i) \cap \mathcal{C}) = \emptyset$, for all $t \in \mathbb{R}_{>0}$.

Let $\mathcal{C}'$ be the smallest polytopal subcomplex of $\mathcal{Z}$ containing $\mathcal{C}$, as well as the w-dimensional cubes $F + \sigma[t, t+1]\epsilon_i$, for $t \in \mathbb{R}_{>0}$.

Note that $\mathcal{C}'$ is strictly larger than $\mathcal{C}$, and embeds in H under the map p. We claim further, that it is convex, and therefore an element of $Cub_b(r, w)$. Indeed, $p(\mathcal{C}')$ can be thought of as the shadow cast by the body $p(\mathcal{C})$ from a light source at in infinite distance in the direction $-p(\sigma\epsilon_i)$. The convex body, along with the silhouette cast by it, together form a convex region.

So passing from $\mathcal{C}$ to $\mathcal{C}'$, we obtain a larger convex Cubical complex with boundary. Iterating this procedure, we may ensure that the resulting convex Cubist complex envelops a large finite region in H, upon the application of p. A number of iterations ensures the whole of H is covered, so we have an extension of $\mathcal{C}$ belonging to $Cub(r, w)$. $\square$

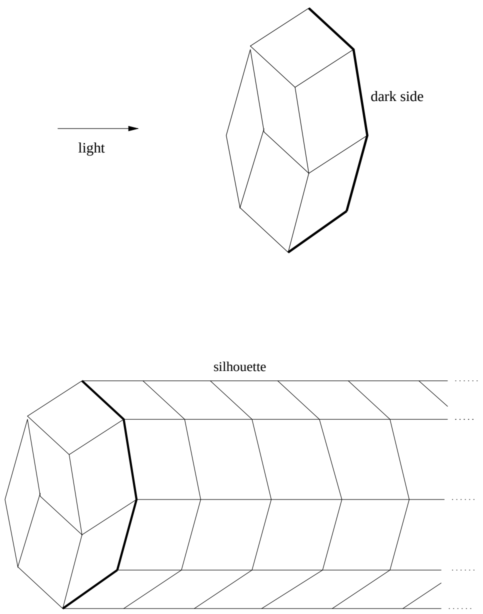

Figure 5. Casting a shadow

For $l \in \underline{r}$, let Φ_l denote the orthogonal projection of E onto $\mathbb{R}^{l-1} \times 0 \times \mathbb{R}^{r-l}$.

Lemma 18. *Any Cubist complex $\mathcal{C} \in Cub(r, w)$ can be extended to a Cubist complex $\tilde{\mathcal{C}} \in Cub(r, w+1)$.*

Proof. Note that $\Phi_l(H)$ is an affine subspace of $\mathbb{R}^{r-1}$ of dimension w, for some l. Therefore, the space $\Phi_l(\mathcal{C})$ is an element of $Cub_b(r, w+1)$. By Lemma 17, the convex Cubist set with boundary $\Phi_l(\mathcal{C})$ can be extended to an element $\mathcal{C}'$ of $Cub(r, w+1)$. Let $\tilde{\mathcal{C}} = \mathcal{C}' \times \mathbb{R}_l$. Then $\tilde{\mathcal{C}}$ is in $Cub(r, w+1)$, and contains $\mathcal{C}$, as required. $\square$

We have seen that we can remove slices from Cubist sets, and part Cubist sets at suitable subsets. The next lemma demonstrates that we can also truncate Cubist sets. Note that the truncated Cubist set $\mathcal{C}^T$ defined here is not unique.

Lemma 19. *Let $\mathcal{C}$ be a Cubist set, such that $z \in q_l(\mathcal{C})$. Then there exists a truncated Cubist set $\mathcal{C}^T$, containing $\mathcal{C} \cap q_l^{-1}[z, \infty)$, such that $q_l(\mathcal{C}^T) \subset [z, \infty)$.*

Proof. If $q_l(\mathcal{C}) \subset [z, \infty)$, then let $\mathcal{C}^T = \mathcal{C}$.

Otherwise, to obtain such a $\mathcal{C}^T$, we first form $q_l^{-1}([z, \infty)) \cap \mathcal{C}$, whose boundary $\partial = \mathcal{C}_{z-1}$ lies in $Cub(r-1, c)$, by Lemma 14. We then extend ∂ to an element of $Cub(r-1, c-1)$, by Lemma 18. Taking a product with $\{z\}$ in component l, we obtain $\tilde{\partial} \in Cub(r, c-1)$, such that $q_l(\tilde{\partial}) = z$.

We now divide $\mathcal{C}$ into two portions $\mathcal{C} = \mathcal{C}_1 \amalg \mathcal{C}_2$, where $\mathcal{C}_1 = q_l^{-1}([z, \infty))$, and $\mathcal{C}_1 = q_l^{-1}((\infty, z))$. We decompose $\tilde{\partial} = \tilde{\partial}_1 \amalg \tilde{\partial}_2$ into two portions, so that the images under p of $\mathcal{C}_1$ and ∂_1 are identified, and the images under p of $\mathcal{C}_2$ and ∂_2 are identified. We finally define $\mathcal{C}^T$ to be the union of $\mathcal{C}_2$, and $\tilde{\partial}_2$. By definition, $\mathcal{C}^T$ satisfies the conditions proposed in the statement of the lemma. $\square$

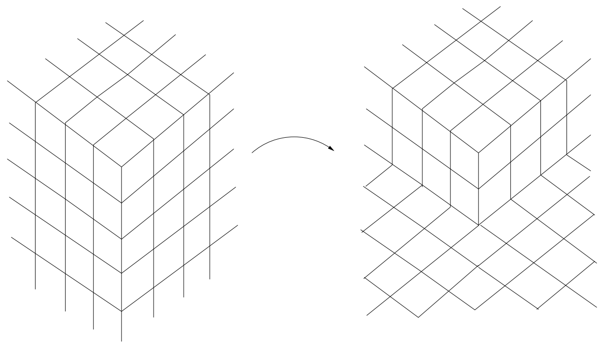

FIGURE 6. Truncation

4. Bijections on Cubist sets

We consider bijections between vertices and facets of Cubist sets, and related combinatorics.

Let $\mathcal{C}$ be a Cubist complex. Let

$$\mathbb{R}_S = \{(x_1, ..., x_r) \in E, \quad x_i = 0, i \notin S\},$$

for $S \subset \underline{r}$. Let $v \in H$ be a vector, such that $\mathbb{R}v \cap p(\mathbb{R}_S) = 0$, for $|S| = w-1$.

Definition 20. *Let $\lambda = \lambda_v : \mathcal{X} \to \mathcal{F}$ be the map which takes a vertex x to the unique w-dimensional facet containing $x + \epsilon v$, for small $\epsilon > 0$.*

Lemma 21. *λ is a bijection between $\mathcal{X}$ and $\mathcal{F}$.*

Proof. Let $F \in \mathcal{F}$. Let f be the centre of F. The parallelohedron F divides up into 2^w similar parallelohedra, after one divides its edges in half. Each vertex x of F is contained in precisely one such parallelohedron pa_x, and the line $f - \epsilon v$ is contained in one such parallelohedron, $pa_{x(f)}$, for small ϵ. An inverse to λ is given by the map which takes F to $x(f)$. $\square$

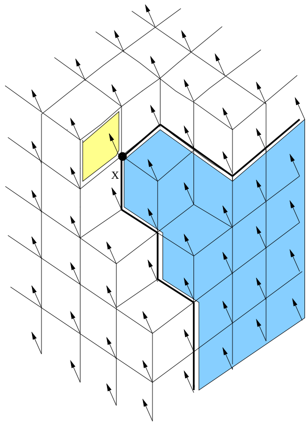

FIGURE 7. Vertex facet bijection; λx and μx.

Given $x \in \mathcal{X}$, let us write $\lambda x = x + \sum_{i \in S_x} [0,1]\sigma_i \epsilon_i$, where $S_x \subset \underline{r}$ contains w elements, and where $\sigma_i \in \{\pm 1\}$. Let

$$\mu x = \left(x + (\sum_{s \in S_x} \mathbb{R}_{\leq 0} \sigma_i \epsilon_i) + (\sum_{s \in \underline{r} \setminus S_x} \mathbb{Z}\epsilon_i) \right) \cap \mathcal{C}.$$

Then λx is the tile of $\mathcal{C}$ containing x into which v is directed. The set μx is the region of $\mathcal{C}$ containing x, cut out by the collection of parallel strips passing through λx.

For $x, y \in \mathbb{Z}^r$, let $d(x,y) = \sum_{i=1}^r |q_i(x) - q_i(y)|$.

Let $D_{U_\mathcal{C}}(q)$ denote the $\mathcal{X} \times \mathcal{X}$ matrix whose xy entry is $q^{d(x,y)}$, if $y \in \lambda x$, and zero otherwise.

Let $D_{V_\mathcal{C}}(q)$ denote the $\mathcal{X} \times \mathcal{X}$ matrix whose xy entry is $q^{d(x,y)}$, if $y \in \mu x$, and zero otherwise.

Theorem 22. *Let $\mathcal{C} \in Cub(r,w)$. Then $D_{U_\mathcal{C}}(q).D_{V_\mathcal{C}}(-q)^T = 1$.*

Proof. Let $x, y \in \mathcal{X}$. The xy entry in $D_{U_\mathcal{C}}(q).D_{V_\mathcal{C}}(-q)^T$ is given by

$$\sum_{z \in \lambda x \cap \mu y} (-1)^{d(y,z)} q^{d(x,z)+d(y,z)}.$$

Note that $\lambda x \cap \mu y$ is a cube of dimension $\leq w$, and that $\sum_{z \in C} (-1)^{d(y,z)} q^{d(x,z)+d(y,z)} = 0$, for all cubes of dimension greater than zero. Furthermore, $\lambda y \cap \mu y = \{y\}$, so it suffices to show that

$$|F \cap \mu y| = 1 \text{ implies } F = \lambda y, \text{ for all } F \in \mathcal{F}. \quad (*)$$

We work by induction on w. The case $w = 0$ is trivial. Therefore, assume that statement $(*)$ is true for all smaller w. We proceed to divide $\mathcal{C}$ up into three subcomplexes $\mathcal{C}^-, \mathcal{C}^0, \mathcal{C}^+$, and establish $(*)$ for all $F \in \mathcal{F}$ contained in each of the three subcomplexes.

Let

$$\mathcal{C}^- = \mu y = \left(y + (\sum_{i \in S_y} \mathbb{R}_{\leq 0} \sigma_i \epsilon_i) + (\sum_{i \in \underline{r} \setminus S_y} \mathbb{R}\epsilon_i) \right) \cap \mathcal{C},$$

$$\mathcal{C}^1 = \left(y + (\sum_{i \in S_y} \mathbb{R}_{\leq 1} \sigma_i \epsilon_i) + (\sum_{i \in \underline{r} \setminus S_y} \mathbb{R}\epsilon_i) \right) \cap \mathcal{C},$$

Let $\mathcal{C}^0$ denote the closure in $\mathcal{C}$ of $\mathcal{C}^1 \backslash \mathcal{C}^-$. Let $\mathcal{C}^+$ denote the closure in $\mathcal{C}$ of $\mathcal{C} \backslash \mathcal{C}^1$. By definition, we have

$$\mathcal{C} = \mathcal{C}^- \cup \mathcal{C}^0 \cup \mathcal{C}^+.$$

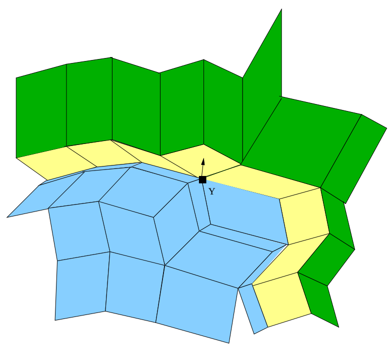

FIGURE 8. $\mathcal{C} = \mathcal{C}^- \cup \mathcal{C}^0 \cup \mathcal{C}^+$.

Note that $\mathcal{C}^- \cap \mathcal{C}^+ = \emptyset$, since $q_i(\xi) \leq 0$, for all $i \in S_y$, for all elements $\xi \in \mathcal{C}^-$, whilst $q_i(\eta) \geq 1$, for some $i \in S_y$, for all $\eta \in \mathcal{C}^+$. Therefore, for every cube $F \in \mathcal{F}$ contained in $\mathcal{C}^+$, we have $|F \cap \mu y| = 0$.

It remains to consider the case $F \subset \mathcal{C}^- \cup \mathcal{C}^0$. Let $N(F)$ denote the cardinality of the set $\{i \in S_y \quad | \quad (y_i + \sigma_i) \in q_i(F)\}$.

We claim that $|F \cap \mu y| = 2^{w-N(F)}$. To see this, let us begin with a face F which lies in $\mathcal{C}^- = \mu y$. Note that here, $N(F) = 0$, for $F \subset \mu y$, and $|F \cap \mu y| = 2^w$. Therefore the formula is true for all faces contained in $\mathcal{C}^-$. Let us now walk around from face to face inside $\mathcal{C}^- \cup \mathcal{C}^0$, at each step moving to an adjacent face, by crossing a $w-1$-cell. As we move out of $\mathcal{C}^-$ into $\mathcal{C}^0$, in direction $\sigma_i \epsilon_i$, we inevitably find our chosen face intersects the hyperplane $x \in \mathbb{R}^r, q_i(x) = (y_i + \sigma_i)$. At the same time, $N(F)$ increases to one, and the number of vertices in $|F \cap \mu y|$ is halved. More generally, weaving between faces, one step at a time, we find that each time $N(F)$ increases by one, $|F \cap \mu y|$ halves, and each time $N(F)$ decreases by one, $|F \cap \mu y|$ doubles. If $N(F)$ is not altered by a step, then $|F \cap \mu y|$ is not altered. Since the formula $|F \cap \mu y| = 2^{w-N(F)}$ holds inside $\mathcal{C}^-$, we find the same formula holds inside $\mathcal{C}^- \cup \mathcal{C}^0$, by induction.

Now suppose $|F \cap \mu y| = 1$. We conclude $N(F) = w$, and so $(y_i + \sigma_i) \in q_i(F)$, for all $i \in S_y$. For $F \subset \mathcal{C}^- \cup \mathcal{C}^0$, we have $q_i(F) \subset (y_i + \mathbb{R}_{\leq 1}\sigma_i)$, for all $i \in S_y$, and therefore $F = \lambda y$, as required.

This completes the proof of the proposition. □

5. Orderings on Cubist sets

We describe special orderings on the collection of vertices of Cubist set.

Definition 23. *A Cubist set $\mathcal{C}$ is* ν-ordered*, if there exists a bijection $\nu : \mathcal{X} \to \mathcal{F}$, and a partial order $\succeq$ on $\mathcal{X}$, generated by the relations $x \succeq y$, for $y \in \nu x$.*

Let $v \in H$, such that $\mathbb{R}v \cap p(\mathbb{R}_S) = 0$, for $S \subset \underline{r}$, $|S| = w - 1$. The dominant result of this chapter is the following:

Theorem 24. *Let $\mathcal{C}$ be a Cubist set in E. Then $\mathcal{C}$ is λ_v-ordered.*

In the sequel, we write $\succeq_v$ for the ordering induced by λ_v, or else we drop the v and write $\succeq$.

Theorem 24 is obvious, in case $w = 0$, or $w = 1$. It can be proved in case $w = 2$, or $c = 1$, fairly easily, by induction on w, c, as in the prequel to this paper [8]. The case $w \geq 3$ is more subtle. In this chapter, we give a proof by induction on w.

We define a *bounded Cubist set* to be a bounded subcomplex $\mathcal{C}$ of $\mathcal{Z}$, such that the projection p restricted to $\mathcal{C}$ is an embedding.

Let us make some technical comments. The following lemma is easy to prove.

Lemma 25. *Suppose $\mathcal{C}$ is a bounded Cubist set. The map λ_v^{-1} defines an injection from the facets of $\mathcal{C}$ to the vertices of $\mathcal{C}$.*

We define strips of bounded Cubist sets, and the operations of slicing and parting of bounded Cubist sets, just as we did for Cubist sets.

Definition 26. *A Cubist complex $\mathcal{C}$ is* v-convex *if $x, x + tv \in p(\mathcal{C})$ implies $x + [0,1]tv \in p(\mathcal{C})$.*

Definition 27. *A* source tile *in a bounded Cubist set $\mathcal{C}$ is a facet f, such that $\lambda_v^{-1}(f)$ lies on the light side of $\mathcal{C}$, and such that every $w-1$-dimensional face on the light side of $\lambda_v^{-1}(f)$ lies on the light side of $\mathcal{C}$.*

Let $\mathcal{L}$ be the collection of all $w-1$-dimensional faces of $\mathcal{C}$, which are illuminated in by light cast from in direction v. Projection in H orthogonal to v identifies $\mathcal{L}$ with a bounded Cubist set in $v^\perp$.

Let $\mathcal{L}_1 = \mathcal{L}$, and $v_1 = v$. We inductively define $\mathcal{L}_i$ to be the collection of all $w-i-1$-dimensional faces of $\mathcal{L}_{i-1}$, which are illuminated in by light cast from in direction v_i, where $v_i \in v_{i-1}^\perp$. Projection in $v_{i-1}^\perp$ orthogonal to v_i identifies $\mathcal{L}_i$ with a bounded Cubist set in $v_i^\perp$.

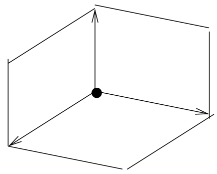

FIGURE 9. A source facet, and its corresponding vertex, viewed from ∞, in case $w = 3$.

Proposition 28. *Suppose $\mathcal{C}$ is a bounded v-convex Cubist set, such that every i-dimensional face is contained in some $w-1$-dimensional face, for $i \leq w-1$. Suppose that $\mathcal{L}_i$ is a v_i-convex Cubist set, with respect to some $v_i \in v^{\perp}$, for every i. Then either $\mathcal{L}$ contains a v_2-source, which is not a face of any tile of $\mathcal{C}$, or else $\mathcal{C}$ contains a v-source in $\mathcal{L}$. Furthermore, $\mathcal{C}$ is λ_v-ordered.*

Proof. For the length of this proof, when we write the "light side", or "dark side", we mean, as seen illuminated by a light in direction v, unless stated otherwise.

We work by induction on w, and on the total number of faces in $\mathcal{C}$. When $\mathcal{C}$ has only one $w-1$-dimensional face, the lemma is obvious. Therefore assume $\mathcal{C}$ has more than one such face.

Note that $\mathcal{L}$ contains no i-dimensional faces which are not contained in some $w-1$-dimensional facet of $\mathcal{L}$. Indeed, such a face could not be contained in any $w-1$-dimensional face of $\mathcal{C}$, since otherwise there would be some $w-1$-dimensional face of $\mathcal{C}$, not contained in $\mathcal{L}$, yet containing an i-dimensional face on the light side, isolated from the facets of $\mathcal{L}$. This is impossible, because v does not run perpendicular to any face in $\mathcal{C}$. So the light side of $\mathcal{L}$, with respect to v_2 cannot contain any facet, which is not a facet of $\mathcal{L}$. Therefore, by induction on w, the $w-1$-dimensional bounded Cubist set $\mathcal{L}$ contains a source, with respect to λ_{v_2}.

Let f' denote such a source tile in $\mathcal{L}$, relative to v_2. There are now two possibilities to consider. Either f' is a $w-1$-dimensional face of some w-dimensional tile in $\mathcal{C}$, or it is not. If it is not, then we have overturned a v_2-source in $\mathcal{L}$, which is not a face of any tile of $\mathcal{C}$, as desired.

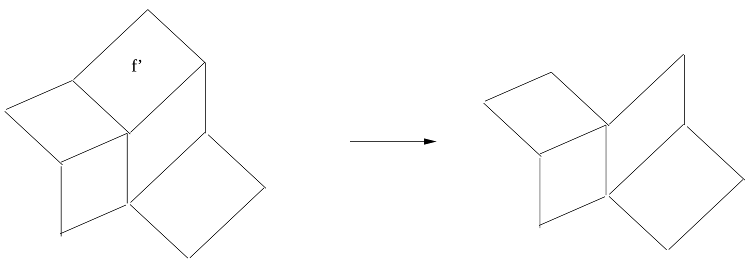

FIGURE 10. Removing a $w-1$-dimensional tile f', in case $w = 3$.

Otherwise, assume that f' is a $w-1$-dimensional face of some w-dimensional tile f'' of $\mathcal{C}$. Then f'' lies on the boundary of $\mathcal{C}$. If we remove f'' from $\mathcal{C}$, we obtain a smaller Cubist set $\mathcal{C}'$.

By induction, we conclude that $\mathcal{C}'$ contains a source tile f'''.

There are now two possibilities. Either f'' shades some face of f''', or it does not. If not, then f''' is a source of $\mathcal{C}$, as well as of $\mathcal{C}'$, and our lust for a source of $\mathcal{C}$ is sated. Otherwise, f'' shades some light side face of f'''. We prove that in such circumstances, f'' is a source of $\mathcal{C}$.

Suppose first that f'' shades some $w-1$-dimensional face of f'''. Thus, f'' and f''' are adjacent in $\mathcal{C}$, and their intersection is a $w-1$-dimensional face W, which is on the light side of f''', and the dark side of f''. We have a bijection between the light-side faces of f''', and the light side faces of f'', which takes W to f', and any other light side face F to that face of f''' which intersects F in a $w-2$-dimensional face of W. We claim this map is a bijection between the light side faces of f'', and the light side faces of f''' in $\mathcal{C}\backslash f''$. Since f''' is a source in $\mathcal{C}\backslash f''$, it follows that f'' is a source of $\mathcal{C}$. Indeed, we know that f' is not shaded in $\mathcal{C}$. If any other light side face of f'' is cast in the shade by a facet of $\mathcal{C}\backslash f''$, then it follows that the intersection with f''' is cast in the shade by the same facet, which is a contradiction, since f''' is a source in $\mathcal{C}\backslash f''$. Therefore, $\mathcal{C}$ contains a source, as required.

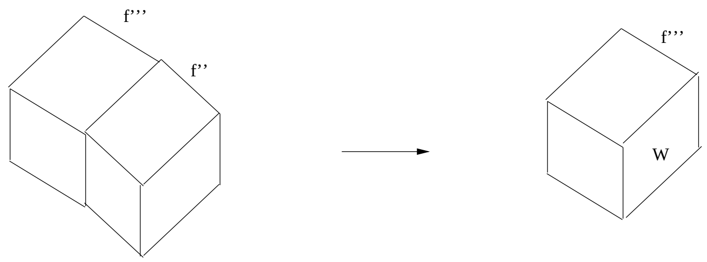

FIGURE 11. Removing f'', when it shades a $w-1$-dimensional face of f'''.

The other case to consider is when f'' does not shade a $w-1$-dimensional face of f''', but does cover some face ξ of dimension $< w-1$. In this case, f'' and f''' are joined only along ξ, This contradicts the v-convexity of $\mathcal{C}$ (see Figure 12)

Let us now prove that $\mathcal{C}$ is ordered, under the hypotheses of the proposition. We again proceed by induction on the total number of faces of $\mathcal{C}$. There are two cases to consider. Either $\mathcal{L}$ contains a v_2-source, which is not a face of any tile of $\mathcal{C}$, or else $\mathcal{C}$ contains a v-source in $\mathcal{L}$.

Suppose $\mathcal{C}$ contains a v-source in $\mathcal{L}$. If $f = x + \sum_{s\in S} \sigma_s[0,1]\epsilon_s$ is that source facet, for some light side vertex x in $\mathcal{C}$, and $\sigma_s = \pm 1$, $S \subset \underline{r}$, such that $|S| = w$, let us write $\hat{f} = x + \sum_{s\in S} \sigma_s[0,1)\epsilon_s$. Then $\mathcal{C}\backslash\hat{f}$ is a bounded Cubist set satisfying the hypotheses of the Lemma, with a smaller number of tiles. Therefore $\mathcal{C}\backslash\hat{f}$ is λ_v-ordered. Since f is a source, $\mathcal{C}$ is also λ_v-ordered. Note that removing such

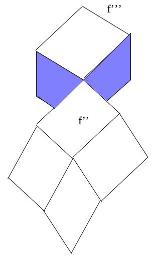

FIGURE 12. When f'' does not shade a $w-1$-dimensional face of f'''.

a source has little effect on the topology of $\mathcal{L}_i$, since it corresponds to a Cubist mutation (cf [8], and chapter 9):

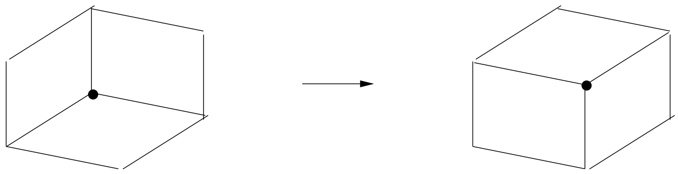

FIGURE 13. A Cubist mutation: removing a source.

Otherwise, if $\mathcal{L}$ contains a v_2-source, which is not a face of any tile of $\mathcal{C}$, we can remove it, and obtain a smaller set satisfying the hypotheses of the proposition. This set is λ_v-ordered by induction, and since removing a $w-1$-dimensional face has no effect on the ordering whatsoever, we conclude that $\mathcal{C}$ itself is λ_v-ordered. This completes the proof of the Proposition. $\square$

Corollary 29. *Suppose $\mathcal{C}$ is a bounded convex Cubist set. Then $\mathcal{C}$ is λ_v-ordered.*

Proof. Bounded Cubist sets satisfy the hypotheses of Proposition 28 $\square$

Before proceeding further towards a the proof of Theorem 24, let us give a number of technical definitions.

Definition 30. *We say two tiles t, t' of a bounded Cubist set $\mathcal{C}$ are* strongly connected, *if there exists a sequence $t = t_1, t_2 ..., t_n = t'$ of tiles, such that $t_i \cap t_{i+1}$ is a $w-1$-dimensional face, for $1 \leq i \leq n-1$.*

We say a bounded Cubist set $\mathcal{C}$ is strongly connected *if any two tiles in $\mathcal{C}$ are strongly connected.*

Definition 31. *A polytopal subcomplex $\mathfrak{l}$ of a bounded Cubist set $\mathcal{C}$ is* submerged *if $\mathfrak{l}$ is a bounded Cubist set, relative to $v^{\perp}$, if every cell is the face of some tile of $\mathfrak{l}$, and if $\mathfrak{l}$ is strongly connected.*

Definition 32. *A submerged subcomplex $\mathfrak{l}$ of $\mathcal{C}$ is* restricted, *if no tile of $\mathfrak{l}$ which lies in the light side of $\mathcal{C}$ is a face of a tile of $\mathcal{C}$.*

Definition 33. *A polytopal subcomplex* $\mathfrak{l}$ *of a bounded Cubist set* $\mathcal{C}$ *is a* part set *if it is submerged, and there exist connected subcomplexes* $\mathcal{C}_+, \mathcal{C}_-$ *of* $\mathcal{C}$*, such that* $\mathcal{C} \cap (\mathfrak{l} + \mathbb{R}_+ v) \subset \mathcal{C}_+$*, such that* $\mathcal{C} \cap (\mathfrak{l} + \mathbb{R}_- v) \subset \mathcal{C}_-$ *and such that* $\mathcal{C}_+ \cap \mathcal{C}_- = \mathfrak{l}$*.*

The reason for our interest in part sets is that *slicing* a strip from a bounded region of a Cubist set leaves a part set along its remnant boundary. Furthermore, the operation of *parting* may be performed along a part set $\mathfrak{l}$ inside a bounded region of a Cubist set.

Lemma 34. *Let* $\mathcal{C}$ *be a connected bounded Cubist set, satisfying the hypotheses of proposition 28. Suppose that* $\mathfrak{l}$ *is a restricted submerged subcomplex of* $\mathcal{C}$*.*

Then there exists a restricted part set $\tilde{\mathfrak{l}}$ *in* $\mathcal{C}$*, containing* $\mathfrak{l}$*.*

Proof. We work by induction on the number of tiles in $\mathcal{C}$.

We apply proposition 29, to conclude that $\mathcal{C}$ either $\mathcal{L}$ contains a v_2-source, which is not a face of any tile of $\mathcal{C}$, for some v_2, or else $\mathcal{C}$ contains a source tile.

Let us consider the first possibility. We denote the pertinent v_2-source by f. Either f is contained in $\mathfrak{l}$, or it is not.

Suppose it is not. Then we may remove it, apply induction to the resulting smaller bounded Cubist set $\mathcal{C}'$, to obtain a restricted part set $\mathfrak{l}'$ in $\mathcal{C}'$. If f is not connected to $\mathcal{C}_-$, then we put f in $\mathcal{C}_+$, and set $\tilde{\mathfrak{l}} = \mathfrak{l}'$ - done.

If f is not connected to $\mathcal{C}_+$, then we put f in $\mathcal{C}_-$, and set $\tilde{\mathfrak{l}} = \mathfrak{l}'$ - done.

Otherwise, if f is connected to both $\mathcal{C}_+$ and $\mathcal{C}_-$, in which case we set $\tilde{\mathfrak{l}} = \mathfrak{l}' \cup f$ - done.

Now suppose f is contained in $\mathfrak{l}$. Then we remove f from both $\mathcal{C}$ and $\mathfrak{l}$, apply induction, and glue f onto the resulting restricted part set - done

We now consider the case when $\mathcal{C}$ contains a source tile s. This we can remove, to obtain a smaller Cubist set $\mathcal{C}'$, in which $\mathfrak{l}$ is contained, since it is restricted. We can apply induction, to extend to a part set $\mathfrak{l}'$ submerged in $\mathcal{C}'$. If the shadow of s in $\mathcal{C}'$ lies in $\mathcal{C}_-$, then we put s in $\mathcal{C}_-$, and set $\tilde{\mathfrak{l}} = \mathfrak{l}'$- done.

If the shadow of s in $\mathcal{C}'$ lies in $\mathcal{C}_+$, then we put s in $\mathcal{C}_+$, and set $\tilde{\mathfrak{l}} = \mathfrak{l}'$- done. Otherwise, half the shadow of s in $\mathcal{C}'$ lies in $\mathcal{C}_-$, and the other half in $\mathcal{C}_+$. Let us call these halves ss_- and ss_+ respectively. We put s in $\mathcal{C}_-$, and set $\tilde{\mathfrak{l}} = \mathfrak{l}' \cup ss_+$- done.

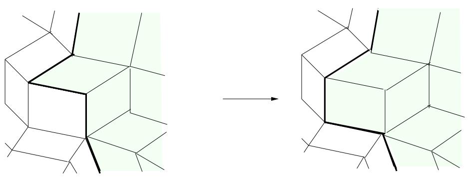

FIGURE 14. $\mathfrak{l}$ and $\tilde{\mathfrak{l}}$, in case half the shadow of s in $\mathcal{C}'$ lies in $\mathcal{C}_-$, and the other half in $\mathcal{C}_+$.

□

Corollary 35. *Suppose $\mathcal{C}$ is a bounded convex Cubist set. Suppose that $\mathfrak{l}$ is a restricted submerged subcomplex of $\mathcal{C}$.*

Then there exists a restricted part set $\tilde{\mathfrak{l}}$ in $\mathcal{C}$, containing $\mathfrak{l}$.

Proposition 36. *Every Cubist set can be approximated, in an arbitrarily large finite region, by a bounded convex Cubist set.*

Proof. (See Figure 15) We prove this by induction on the number of tiles in the finite region F, which we assume, without loss of generality, to be contractible. The case when F has only one tile is trivial.

We extend F in seven steps. Let S be a strip in F, parallel to $p(\epsilon_l)$.

• If necessary, we tilt the vector $v = p(\epsilon_l)$ by a miniscule amount, within the strip, so that S is the unique strip in F parallel to v, and so that $\mathbb{R}v \cap p(\mathbb{R}_S) = 0$, for $S \subset \underline{r}\backslash\{l\}$, $|S| = w - 1$. Note this does not alter the combinatorics of F in any way.

• We slice F, removing the strip S. Inside the resulting finite region F^1, we leave a part set $\mathfrak{l}$.

• We extend F^1 to a large bounded convex Cubist set F^2, by induction.

• We extend $\mathfrak{l}$ within F^2 to a part set $\tilde{\mathfrak{l}}$ containing $\mathfrak{l}$, by Corollary 35.

• We part F^2 in direction v along $\tilde{\mathfrak{l}}$, creating a strip $\tilde{S}$. The resulting Cubist complex F^3 contains F.

• We cast shadows in direction v, to create an unbounded Cubist complex F^4 containing F^3.

• We extend $\tilde{S}$ to a strip $\hat{S}$ of F^4. We put $F^5 = F^3 \cup \hat{S}$, a bounded Cubist complex containing F.

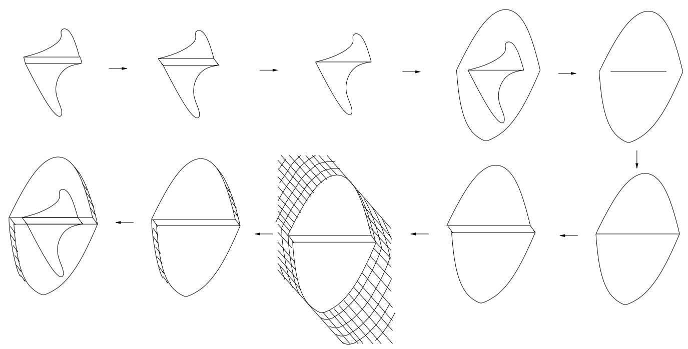

FIGURE 15. Approximation with a convex Cubist set.

To complete the proof of the Proposition, we need to observe that F^5 is a convex Cubist set. Note that $q_l(F^5)$ is an interval $[z, z+1]$ in $\mathbb{R}$, since S was the unique strip in F parallel to v. It is not difficult to see that F^5 is in fact the convex hull in H of the projection under p of the union $q_l^{-1}(z) \cup q_l^{-1}(z+1)$.

□

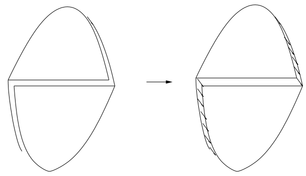

FIGURE 16. F_5 as a convex hull.

Proof of theorem 24: Every finite region of a Cubist set can be approximated by a bounded convex Cubist set, by Proposition 36. Every bounded convex Cubist set is λ_v-ordered, by Proposition 29. Since any circuit $x = x_1 \succ x_2 \succ ... \succ x_n = x$ in the equivalence relation induced by λ_v must occur in some finite region, in fact any Cubist set is λ_v-ordered. □

We have now proved that $\mathcal{C}$ is λ_v-ordered. We wish to show further that $\mathcal{C}$ can be approximated by a Cubist set, in which every interval is finite.

Definition 37. *A Cubist set $\mathcal{C}$ possesses the* finite interval property*, with respect to $\succeq_v$, if the interval $[x, y]$ in the partially ordered set $(\mathcal{X}, \succeq_v)$ is finite, for all $x, y \in \mathcal{X}$.*

Suppose that $r < \infty$, and $\mathcal{C}^0 \in Cub_b(r, w)$ is compact. Let $v_1, ..., v_w$ be a sequence of vectors in H, of the same length as v, and extremely close to v. Let $\theta_i \in [0, 2\pi]$ denote the angle between v_i and v, for $1 \leq i \leq w$. Suppose that $\frac{\theta_{i+1}}{\theta_i} << 1$, for $1 \leq i \leq w - 1$. In other words, we assume that $v_1, ..., v_w$ gets successively closer to v.

Suppose that $\mathcal{C}^w$ is the Cubist set, obtained by casting shadows from $\mathcal{C}^0$, in directions $\pm v_1, \pm v_2, ..., \pm v_w$ successively.

Lemma 38. *The Cubist set $\mathcal{C}^w$ possesses the finite interval property, with respect to $\succeq_v$.*

Proof. Obviously, within $\mathcal{C}^0$, there can only be finitely many strictly descending sequences $x = x^0 \succ x^1 \succ ... \succ x^N = y$ between $x, y \in \mathcal{X}^0$. Casting shadows in directions $\pm v_1$, we obtain $\mathcal{C}^1$, a convex Cubist set with boundary.

Let $x, y \in \mathcal{X}^1$. Any strictly decreasing sequence of elements $x = x^0 \succ x^1 \succ ... \succ x^N = y$ in $\mathcal{X}^1$ must divide up into subsequences, $g_1 = (x^i)_{i=1}^{n_1}$, $g_2 = (x^i)_{i=n_1+1}^{n_2}$, ..., $g_3 = (x^i)_{i=n_{N-1}}^{n_N}$, where the v_1 components of elements of g_i are all equal, and greater than the v_1 components of g_{i+1}. Furthermore, since $\mathcal{C}^1$ is obtained from $\mathcal{C}^0$ by casting a shadow in direction v_1, these subsequences g_i can be identified with strictly decreasing sequences of elements of $\mathcal{C}^0$. There are only finitely many such sequences, and $y_r - x_r$ is finite. Therefore, there are only finitely many descending sequences from x to y. Therefore, $\mathcal{C}^1$ obeys the finite interval property. Note that every w-dimensional cube on the boundary of $\mathcal{C}^1$ has an edge parallel to v_1.

Shining a light in direction v_2, the dark side of $\mathcal{C}^1$ is isomorphic to $\mathcal{D}^1 \times \mathcal{Z}_1 v_1$, where $\mathcal{D}^1$ is a subset of $\mathcal{C}^0$. Casting a shadow in direction v_2, and then in direction $-v_2$, we obtain a Cubist set $\mathcal{C}^2$ with boundary. Descending sequences in $\mathcal{X}^2$ can be thought of as compositions of finitely many subsequences g_i, whose v_1 and v_2-components are all equal. As before, the sequences g_i can be identified with descending sequences in $\mathcal{C}^0$, and so our convex Cubist set with boundary $\mathcal{C}^2$ possesses the finite interval property.

The dark side of $\mathcal{C}^2$ in direction v_3, is isomorphic to $\mathcal{D}^2 \times \mathcal{Z}_1\epsilon_{r-1} \times \mathcal{Z}_1\epsilon_r$, where $\mathcal{D}^2$ is a subset of $\mathcal{C}^0$. Casting a shadow in directions $\pm v_3$, we obtain a Cubist set with boundary, possessing the finite interval property. Iterating this procedure, we eventually cover the entire space with a Cubist set $\mathcal{C}^w$, possessing the finite interval property.

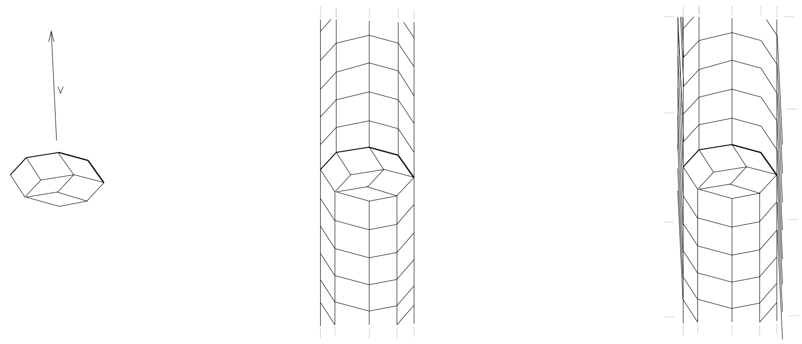

FIGURE 17. An example of $\mathcal{C}^0$, $\mathcal{C}^1$, $\mathcal{C}^2$, in case $w = 2$.

$\square$

Corollary 39. *Any Cubist set can be approximated in an arbitrarily large finite region by some Cubist set which possesses the finite interval property.*

Proof. Let $\mathcal{C}$ be a Cubist set, and F a finite region. We can extend F to a convex Cubist set F', by Proposition 36. Casting shadows from F', as in Lemma 38 we obtain a Cubist set containing F, which possesses the finite interval property. $\square$

6. Algebraic preliminaries

We state our general algebraic setup, which coincides with that of the prequel to this article [8]. Further technicalities can be found in that paper.

Let k be a field. We shall be working with associative k-algebras A graded over the integers. So $A = \oplus_{i\in\mathbb{Z}} A_i$ and $A_i A_j \subset A_{i+j}$. While not assuming the existence of a unit, we require A to be equipped with a set of mutually orthogonal idempotents $\{e_s \mid s \in \mathcal{S}\} \subset A_0$ such that $A = \oplus_{s,s'\in\mathcal{S}} e_s A e_{s'}$. Unless stated otherwise, all A-modules M are assumed to be *graded* left modules, so that $M = \oplus_{i\in\mathbb{Z}} M_i$ and $A_i M_j \subset M_{i+j}$, and to be quasi-unital, i.e., $M = \oplus_{s\in\mathcal{S}} e_s M$. Given $n \in \mathbb{Z}$, we let

$M\langle n\rangle$ be the A-module obtained by shifting the grading by n, so that $M\langle n\rangle_i = M\langle n-i\rangle$.

Now suppose that A is positively graded, i.e. $A_i = 0$ for $i < 0$, and that $\{e_s \mid s \in \mathcal{R}\}$ is a basis for A_0. Let us also impose the finiteness condition $\dim e_s A_i < \infty$ for all $s \in$ and $i \in \mathbb{Z}$. Let A be the category of all graded A-modules, where the space of morphisms between graded modules M and N, which we denote $\mathrm{Hom}_A(M,N)$, consists of A-module homomophisms preserving degree. We denote by A-mod the full subcategory consisting of modules M such that $\dim e_s M_i < \infty$ for all $s \in$ and $i \in \mathbb{Z}$, and that $M_i = 0$ for $i << 0$.

We define A-nod to be the category of (not necessarily graded) finite dimensional A-modules.

7. Definitions

Here we define the Cubist algebras, via a species of noncommutative symplectic reduction.

We wish to consider algebras defined over a field k, such as $\mathbb{F}_p$. Cubist complexes are geometric objects, defined over $\mathbb{R}$. We therefore require an algebraic setup which is flexible enough for change base from $\mathbb{R}$ to $\mathbb{F}_p$.

Our general conditions are the following: R is a subring of $\mathbb{R}$; k is a field, and we have a ring homomorphism $R \to k$. For example, we might take $R = \mathbb{Z}$, and $k = \mathbb{F}_p$.

Let $E_R = R^{\oplus \underline{r}}$.

We assume the existence of a free R-module H_R, and epimorphism of R-modules, $p_R : E_R \twoheadrightarrow H_R$, such that $\mathbb{R} \otimes_R H_R \cong H$, and $\mathbb{R} \otimes_R p_R \cong p$. We write

$$E_k = k \otimes_R E_R, \quad H_k = k \otimes_R H_R, \quad p_k = k \otimes_R p_R.$$

Thus, $p_k : E_k \twoheadrightarrow H_k$.

Let Q be the quiver with vertex set $\mathbb{Z}^r$, and set of arrows

$$\{a_{x,i}, b_{x,i} | x \in \mathbb{Z}^r, 1 \leq i \leq r\}.$$

The arrow $a_{x,i}$ is directed from x to $x + \epsilon_i$. The arrow $b_{x,i}$ is directed from x to $x - \epsilon_i$. Let f_x be the primitive idempotent in kQ corresponding to $x \in \mathbb{Z}^r$

Let Π be the path algebra kQ, modulo *commutation relations*

$$a_{x,i}a_{x+\epsilon_i,j} = a_{x,j}a_{x+\epsilon_j,i},$$

$$b_{x,i}b_{x-\epsilon_i,j} = b_{x,j}b_{x-\epsilon_j,i},$$

$$a_{x,i}b_{x+\epsilon_i,j} = b_{x,j}a_{x-\epsilon_j,i},$$

for $1 \leq i, j \leq r$.

Let $\Gamma = k[a_1, ..., b_r, b_1, ..., b_r]$ be a polynomial ring in $2r$ variables. The algebra Π is a Γ-Γ-bimodule, where a_i acts on the right of Πf_x as $a_{x,i}$ and on the left of $f_x\Pi$ as $a_{x-\epsilon_i,i}$; where b_i acts on the right of Πf_x as $b_{x,i}$ and on the left of $f_x\Pi$ as $a_{x+\epsilon_i,i}$.

We have an embedding $\phi : E_k \hookrightarrow \Gamma^2$, taking $\epsilon_i \in E_k$ to the quadratic element $a_i b_i$ of Γ. By the commutation relations, the left action of E_k on Π is identical to the right action of E_k on Π. The subalgebra $S(H_k)$ of Γ therefore acts centrally on Π.

Let $V = \Pi \bigotimes_{S(ker(p_k))} k$, the quotient of Π by the ideal generated by H_k. As a graded algebra, V is quadratic. We define U be the quadratic dual of V. We denote the generating arrows of U, dual to $a_{x,i}, b_{x,i}$, by $\alpha_{x,i}, \beta_{x,i}$, for $x \in \mathbb{Z}^r, i \in \underline{r}$.

Definition 40. *Given a Cubist complex $\mathcal{C}$, let*

$$U_{\mathcal{C}} = U / \sum_{x \in \mathbb{Z}^r \backslash \mathcal{X}} U f_x U,$$

$$V_{\mathcal{C}} = \bigoplus_{x,y \in \mathcal{X}} f_x V f_y.$$

The algebras $U_{\mathcal{C}}, V_{\mathcal{C}}$ are the *Cubist algebras* associated to $\mathcal{C}$.

Remark 41 Thinking of Π as the ring of regular functions on a smooth noncommutative variety $\mathcal{P}$, of dimension $2r$, the Cubist algebras $V_{\mathcal{C}}$ can be thought of as representing a symplectic quotient

$$\mathcal{P}//T$$

of $\mathcal{P}$ by a c-dimensional torus T [19]. Thus, V represents the zero fibre $\mathcal{V}$ of a moment map $\mu : \mathcal{P} \to Lie(T)^*$, whilst cutting by idempotents $f_x, x \in \mathcal{X}$ corresponds to factoring out further the action of T on $\mathcal{V}$.

We will see that the algebra $V_{\mathcal{C}}$ has finite global dimension, equal to $2(r-c)$. In the language of noncommutative geometry, our symplectic quotient is a smooth variety, of dimension $2(r-c)$. By comparison, note that $\mathcal{V}$ is singular.

The prima materia for classical symplectic reduction is a Hamiltonian group action on a symplectic manifold. After transmutation, one obtains a second symplectic manifold. In our situation, strong properties of the algebra Π descend to strong properties of the algebra $V_{\mathcal{C}}$. Indeed, both $\Pi, V_{\mathcal{C}}$ are Koszul algebras, with highest weight module categories, respecting a form of Serre duality.

There are numerous different possible Cubist complexes $\mathcal{C}$ tiling H, and therefore numerous different algebras $V_{\mathcal{C}}$. However, there exist derived equivalences relating many of these algebras, representing their common origin.

Remark 42 Any lattice in Euclidean space defines a collection of parallelohedral tilings: namely, those tilings whose vertices lie on the lattice. There are algebras in Lie theory corresponding to lattices; for example, there is the vertex operator algebra construction of Borcherds [5]. It would be interesting if there were some relation with the corresponding Cubist algebras.

8. Homological algebras

The Cubist algebras possess many strong homological properties, such as Koszulity, quasi-heredity, and self-injectivity. In this section, we prove these facts, generalising the case $c = 1$ treated in the prequel to this paper. Many of the arguments required to prove the results of this section are identical to arguments given in the prequel to this paper [8]. We do not repeat these here, but choose instead to outline proofs, providing details only where the logic of our previous article seems insufficient.

We prove that the category of standardly filtered modules is a thick subcategory of the stable module category, for a self-injective algebra with highest weight module category.

We prove that the derived categories of Cubist algebras admit a polarization ([25]).

Following the convention of the prequel, given a positively graded algebra A, we denote by $C_A(q)$ the Cartan-Hilbert matrix, which records the graded composition series of principal indecomposable modules of A. In case the degree zero part of A is a field, $C_A(q)$ is the Hilbert polynomial of A.

Lemma 43. *Γ is free over $S(E_k)$. The algebra $\Lambda = \Gamma \bigotimes_{S(ker(p_k))} k$ is Koszul, with Hilbert polynomial $(1-q^2)^c/(1-q)^{2r}$.*

Proof. It is easy to check that Γ is free over $S(E_k)$. Furthermore, $S(E_k)$ is free over $S(ker(p_k))$, so Γ is free over $S(ker(p_k))$. Therefore, Λ has Hilbert polynomial

$$C(\Gamma)/C(S(ker(p_k))) = (1-q^2)^c/(1-q)^{2r}.$$

Furthermore, Λ is a quadratic complete intersection. Quadratic complete intersections are always Koszul [13]. □

Corollary 44. *Λ acts freely on the left and right of V. V is a Koszul algebra, and its Koszul dual is isomorphic to U.*

Proof. Π is a Γ-Γ-bimodule. Tensoring over k with $S(ker(p_k))$, we find that V has the structure of a Λ-Λ-bimodule.

The Koszul complex for Λ is a linear resolution of the trivial Λ-module. Inducing up to V, we obtain a linear resolution of $V^0 = V \otimes_V k$, the degree zero part of V. Therefore, V is Koszul. The Koszul dual of V is the quadratic dual of V, namely U. □

We wish to prove that $V_{\mathcal{C}}$-mod is a highest weight category. To this end, we define a collection of Λ-modules, a subset of which, after inducing up to V, and projecting down to $V_{\mathcal{C}}$, will define a collection of standard $V_{\mathcal{C}}$-modules. These modules possess natural linear resolutions which allow us to prove the Koszulity of $V_{\mathcal{C}}$, as well.

Let $\tilde{r} = \{a_i, b_i | i \in \underline{r}\}$. Let γ be the projection of $\tilde{r}$ onto $\underline{r}$, which takes α_i and β_i to i.

Let θ be a subset of $\tilde{r}$ of order w, such that the restriction $\gamma : \theta \hookrightarrow \underline{r}$ of γ to theta is an embedding. Let $\Omega_\theta = k[\theta]$ be a polynomial ring in w variables. We have a natural algebra homomorphism from Ω_θ to Λ.

Lemma 45. *The algebra Λ is free over Ω_θ, with basis*

$$\mathcal{B}_\theta = \bigcup_{\phi \in \tilde{r}, \theta \subset \phi, \gamma : \phi \cong \underline{r}.} \{ \text{ monomials in } \phi \}.$$

Proof. We first show that Ω_θ acts freely. Indeed, let F_θ be the subspace of E_k spanned by $\{\epsilon_i, i \in \gamma(\theta)\}$. Note that $E_k = F_\theta \oplus ker(p_k)$, by assumption. Furthermore, Γ is free over $S(E_k)$. Therefore, $\Lambda = \Gamma \otimes_{S(ker(p_k))} k$ is free over $S(F_\theta)$. Moreover, $\Lambda \otimes_{S(F_\theta)} k \cong \Gamma \otimes_{S(E_k)} k$ is free over $\Omega_\theta \otimes_{S(F_\theta)} k$. It follows that Λ is free over Ω_θ.

We now show that $\mathcal{B}_\theta$ forms a basis. Note that the quadratic relations $ker(p_k)$ in Λ allow us to reduce any monomial in $\tilde{r}$ to a sum of monomials of the form $m_1 m_2$, where $m_1 \in \mathcal{B}_\theta$, and where m_2 is a monomial in θ. Therefore, $\Lambda = \mathcal{B}_\theta.\Omega_\theta$. This implies that $C(\mathcal{B}_\theta).C(\Omega_\theta) \geq C(\Lambda)$, with equality if, and only if, $\mathcal{B}_\theta$ forms a basis.

By Lemma 43, we have

$$\begin{aligned} C(\Lambda) = (1-q^2)^c/(1-q)^{2r} &= (1-q)^c(1+q)^c/(1-q)^c(1-q)^{w+r} \\ &= (1+q)^c/(1-q)^{w+r}. \end{aligned}$$

However, by definition, $C(\Omega_\theta) = 1/(1-q)^w$. Furthermore, we can decompose $\mathcal{B}_\theta$ as a disjoint union

$$\mathcal{B}_\theta = \coprod_{\psi \subset \tilde{r}, \gamma : \psi \cong \underline{r} \backslash \gamma(\theta)} \left(\prod_{\beta_i \in \psi} \beta_i.\{ \text{ monomials in } \theta \cup \psi \} \right),$$

so that

$$\begin{aligned} C(\mathcal{B}_\theta) &= \sum_{\psi \subset \tilde{r}, \gamma : \psi \cong \underline{r} \backslash \gamma(\theta)} q^{|\{\beta_i \in \psi\}|} 1/(1-q)^r \\ &= (1+q)^c/(1-q)^r. \end{aligned}$$

Combining our three formulas, we see that $C(\Lambda) = C(\mathcal{B}_\theta).C(\Omega_\theta)$. □

Suppose $\theta \subset \tilde{r}$ has order w. Let P_θ denote the subalgebra of V generated by f_x, and $f_x\theta$, for $x \in \mathbb{Z}^r$. Let $\Delta_\theta(x) = V \otimes_{P_\theta} kx$, for $x \in \mathbb{Z}^r$.

Given $x \in \mathcal{X}$, we have $\lambda(x) = x + \sum_{i=1}^{w} [0,1] \sigma_i \epsilon_{x_i}$, for some increasing sequence $x_i \in \underline{r}$, and some $\sigma_i = \pm 1$. Let $\theta_x \subset \tilde{r}$ be the subset of $\tilde{r}$ of order w, which contains α_{x_i}, whenever $\sigma_i = 1$, and β_{x_i}, whenever $\sigma_i = -1$, for $i = 1, ..., w$.

Let $\Delta_{V_\mathcal{C}}(x)$ denote the standard $V_\mathcal{C}$-module, for $x \in \mathcal{X}$.

Lemma 46. *Let $x \in \mathcal{X}$. We have an isomorphism of $V_\mathcal{C}$-modules*

$$\Delta_{V_\mathcal{C}}(x) \cong \oplus_{x \in \mathcal{X}} Hom_V(V f_\mathcal{X}, \Delta_{\theta_x}(x)).$$

We have a linear projective resolution,

$$\ldots \to \bigoplus_{\substack{y \in \lambda x \\ d(x,y)=2}} V_\mathcal{C} f_y \langle 2 \rangle \to \bigoplus_{\substack{y \in \lambda x \\ d(x,y)=1}} V_\mathcal{C} f_y \langle 1 \rangle \to V_\mathcal{C} f_x \to \Delta_{V_\mathcal{C}}(x).$$

Proof. (see [8], Lemma 39) We induce a Koszul resolution of Ω_{θ_x} up to V, and then project down to $V_{\mathcal{C}}$ via $\oplus_{x\in\mathcal{X}} Hom_V(Vf_{\mathcal{X}}, -)$. Taking a component, we obtain a linear resolution of the standard $V_{\mathcal{C}}$-module, as required. □

Theorem 47. *$U_{\mathcal{C}}$ and $V_{\mathcal{C}}$ are Koszul dual algebras.*

$U_{\mathcal{C}}$-mod is a highest weight category, with respect to $\succeq$. $V_{\mathcal{C}}$-mod is a highest weight category, with respect to $\succeq^{op}$.

The graded decomposition matrix of $U_{\mathcal{C}}$ is $D_{U_{\mathcal{C}}}(q)$. The graded decomposition matrix of $V_{\mathcal{C}}$ is $D_{V_{\mathcal{C}}}(q)$.

Proof. The highest weight structure on $V_{\mathcal{C}}$-mod follows by a numerical criterion, from Proposition 22 and Lemma 46 (see [8], Theorem 41).

To prove the Koszulity of $V_{\mathcal{C}}$, one then applies a result of Ágoston, Dlab, and Lukács [1], which states that a finite dimensional algebra with a highest weight module category, whose standard modules have linear resolutions, is Koszul, and its Koszul dual has a highest weight module category. In order to apply this result to our situation, it must be possible to approximate a Cubist set in an arbitrarily large region by a Cubist set satisfing the finite interval property. This is possible by Lemma 38 (see [8], Theorem 47).

We know that U and V are Koszul dual algebras. From a recollement of E. Cline, B. Parshall and L. Scott, it then follows that the Koszul dual of $V_{\mathcal{X}}$ is in fact $U_{\mathcal{X}}$ (see [8], Theorem 53). We then conclude that $U_{\mathcal{X}}$-mod is a highest weight category (see [8], Corollary 56). □

The statements above imply a homological duality between $U_{\mathcal{C}}$ and $V_{\mathcal{C}}$ [3]:

$$K_{\mathcal{C}} : D^b(U_{\mathcal{C}}\text{-mod}) \cong D^b(V_{\mathcal{C}}\text{-mod}),$$

as well as an algebraic stratification of the derived categories of these algebras. A second set of results describes the homological self-duality of $U_{\mathcal{C}}$ and $V_{\mathcal{C}}$:

Theorem 48. *There exist graded algebra automorphisms η_U, η_V of $U_{\mathcal{C}}$, $V_{\mathcal{C}}$, acting trivially on the degree zero parts of these algebras, such that*

$$Hom(X,Y)^{\eta_U} \cong Hom(Y,X)^*,$$

naturally in $X \in U_{\mathcal{C}}\text{-perf}, Y \in D^b(U_{\mathcal{C}}\text{-mod})$, such that

$$Hom(X,Y)^{\eta_V} \cong Hom(Y,X[2w])^*,$$

naturally in $X \in V_{\mathcal{C}}\text{-perf}, Y \in D^b(V_{\mathcal{C}}\text{-mod})$, and such that the diagram

$$\begin{tikzcd} D^b(U_{\mathcal{C}}\text{-mod}) \arrow[r, "K_{\mathcal{C}}"] \arrow[d, "\eta_U"] & D^b(V_{\mathcal{C}}\text{-mod}) \arrow[d, "\eta_V"] \\ D^b(U_{\mathcal{C}}\text{-mod}) \arrow[r, "K_{\mathcal{C}}"] & D^b(V_{\mathcal{C}}\text{-mod}) \end{tikzcd}$$

commutes.

Proof. The algebra $U_{\mathcal{C}}$ is a graded Frobenius algebra (for a proof, see [8], Theorem 69). Therefore, $U_{\mathcal{C}}$ possesses a graded twist η_U, known as the Nakayama automorphism, such that $U_{\mathcal{C}}^{\eta_U} \cong \bigoplus_{x,y\in\mathcal{X}} Hom(f_x U_{\mathcal{C}} f_y, k)$, as $U_{\mathcal{C}}$-$U_{\mathcal{C}}$-bimodules. It follows that the derived category of $U_{\mathcal{C}}$ satisfies a duality of the form stated in the theorem. Since the injective hull of kx is isomorphic to the projective cover of kx, for $x \in \mathcal{X}$, twisting by η_U fixes simple modules. Therefore, η_U acts trivially on the semisimple degree zero part of $U_{\mathcal{C}}$.

Passing through the Koszul duality functor $K_{\mathcal{C}}$, we find that there is a graded Morita self-equivalence η_V of $V_{\mathcal{C}}$, twisting by which induces a commutative diagram, and a natural isomorphism as stated. The Morita self-equivalence is induced by an algebra automorphism, since $V_{\mathcal{C}}$ is basic. $\square$

Remark 49 In case $c = 1$, the automorphisms η_U, η_V can be taken to be the identity (see [8]). In this case, the Cubist algebras are Calabi-Yau. It would be interesting to know whether this is true, for $c > 1$.

Remark 50 We have combinatorial formulas ([8], Theorem 90),

$$C_{V_{\mathcal{C}}}(q)_{xy} = q^{d(x,y)}(1-q^2)^{-w}, \qquad C_{V_{\mathcal{C}}}(q) = D_{V_{\mathcal{C}}}(q)^T D_{V_{\mathcal{C}}}(q),$$

$$C_{U_{\mathcal{C}}(q)} C_{V_{\mathcal{C}}(q)} = 1, \quad C_{U_{\mathcal{C}}(q)} = D_{U_{\mathcal{C}}}(q)^T D_{U_{\mathcal{C}}}(q), \quad C_{U_{\mathcal{C}}}(q^{-1}) = q^{-2w} C_{V_{\mathcal{C}}}(q).$$

We have a recollement

$$D^b(V\text{-mod})_{\mathcal{C}} \rightleftarrows D^b(V\text{-mod}) \rightleftarrows D^b(V_{\mathcal{C}}\text{-mod})$$

where $D^b(V\text{-mod})_{\mathcal{C}}$ denotes the subcategory of $D^b(V\text{-mod})$ of complexes of modules, whose homology is given by simple modules outside $\mathcal{C}$ ([8], Corollary 52).

If T is the tiling of Euclidean space corresponding to the Cubist set $\mathcal{C}$, we have algebra isomorphisms $U_{\mathcal{C}} \cong U_T$, and $V_{\mathcal{C}} \cong V_T$ ([8], Proposition 58, Theorem 53).

Remark 51 Since the collection of Cubist sets is so rich, in case $w > 1$, we obtain a large collection of Koszul Frobenius algebras of Loewy length $2w + 1$, whose Nakayama automorphism fixes simple modules, which possess infinitely many highest weight structures. This contrasts markedly with Loewy length three, in which case there is a unique self-injective algebra whose Nakayama automorphism fixes simple modules, in possession of a highest weight structure: the Brauer tree algebra on an infinite line.

The category of self-injective algebras with highest weight module category is a pleasing one. For example, the subcategory of Δ-filtered modules is not abelian in a general highest weight category, and is therefore homologically somewhat unsatisfactory. However, if the algebra is also self-injective, the stable category of Δ-filtered modules admits a triangulated structure:

Theorem 52. *Let A be a self-injective algebra, with highest weight module category, which is not semisimple. Then A is Ringel self-dual, and infinite dimensional. The* *category $\mathcal{F}(\Delta)$ of Δ-filtered modules is a thick subcategory of the stable module category, A-$\underline{\text{nod}}$.*

Proof. Recall the stable module category A-$\underline{\text{nod}}$ of a self-injective algebra is triangulated [16].

A finite dimensional algebra with highest weight module category has finite global dimension. A non-semisimple self-injective algebra has infinite global dimension, since the Heller translation functor is invertible. Therefore, A is infinite dimensional.

By definition, all projective A-modules are filtered by standard modules.

Costandard A-modules are dual to standard A^{op}-modules. Since A is self-injective, projective A-modules are dual to projective A^{op}-modules, Therefore, projective A-modules are filtered by costandard modules.

Therefore, the projective indecomposable A-modules are precisely the indecomposable tilting modules for A. Consequently, $A = End_{A\text{-nod}}(A)$ is Ringel self-dual (see [12], A4). Thus, ${}_AA_A$ is a tilting bimodule. Since Ringel duality exchanges standard and costandard modules, the standard A-modules can also be thought of as costandard modules, with respect to the opposite partial order.

We can thus write $\mathcal{F}(\Delta) \cong \mathcal{F}^{op}(\nabla)$. Since the category $\mathcal{F}(\Delta)$ is precisely the collection of modules left $Ext^{>0}$-orthogonal to the corresponding costandard modules, whilst the category $\mathcal{F}^{op}(\nabla)$ is precisely the collection of modules right $Ext^{>0}$-orthogonal to the corresponding standard modules ([12], A2, proposition 1(v)) , we conclude that if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence in A-mod, and any two of M_1, M_2, M_3 lie in $\mathcal{F}(\Delta)$, then the third also lies in $\mathcal{F}(\Delta)$.

Projective A-modules are the same as injective A-modules. These lie in $\mathcal{F}(\Delta)$, so Heller translation preserves $\mathcal{F}(\Delta)$. Also, we know that taking a pushout of maps between objects of $\mathcal{F}(\Delta)$, one of which is injective, we obtain an object of $\mathcal{F}(\Delta)$. It follows that any map $M_1 \to M_2$ in the stable category, with $M_1, M_2 \in \mathcal{F}(\Delta)$ can be extended to an exact triangle

$$M_1 \to M_2 \to M_3 \rightsquigarrow,$$

with $M_3 \in \mathcal{F}(\Delta)$, since exact trangles in the stable category are formed by taking such pushouts [16]. We have thus proven that the category of Δ-filtered modules is a triangulated subcategory of the stable module category of A.

In the module category, direct summands of Δ-filtered modules are also Δ-filtered, as can be seen by their characterisation as the modules which are $Ext^{>0}$ orthogonal to costandard modules. Modulo projective indecomposable summands, modules which are isomorphic in the stable category, are isomorphic in the module category. Therefore, modulo projective indecomposable summands, direct summands in the module category coincide with direct summands in the stable category. Therefore, $\mathcal{F}(\Delta)$ is a thick subcategory of the stable module category of A. $\square$

Thanks to the above theorem, for a self-injective quasi-hereditary algebra, the standard hierarchy of triangulated categories has a natural extension,

$$K^b(A\text{-nod}) \to D^b(A\text{-nod}) \to A\text{-}\underline{\text{nod}} \to A\text{-}\underline{\text{nod}}_\Delta\,.$$

Here, we write $A\text{-}\underline{\text{nod}}_\Delta$ for the quotient of the stable category by the thick subcategory $\mathcal{F}(\Delta)$.

Corollary 53. *The category $\mathcal{F}(\Delta)$ of Δ-filtered modules is a thick subcategory of the stable module category, $U_{\mathcal{C}}\text{-}\underline{\text{nod}}$.*

We have introduced the following homological restriction:

Definition 54. *(see* [25]*) A* polarization *of a triangulated category $\mathcal{T}$ is a quadruple (ϕ, X^+, X^-, w), where X^+ and X^- are generating collections of objects of $\mathcal{T}$, where*

$$Hom(X^+, X^-[i]) = Hom(X^-, X^+[i]) = 0, \text{ for } i \neq w,$$

and where $\phi : \mathcal{T} \to \mathcal{T}^{op}$ is an equivalence, such that

$$\phi(X^+) = X^-, \quad \phi(X^-) = X^+, \quad \phi^\vee.\phi = 1.$$

The most basic example of a triangulated category admitting a polarization is the derived category $D^b(k[W]\text{-mod})$, where W is a symplectic vector space of dimension $2w$. We have proved the following theorem.

Theorem 55. *(see* [25]*) Let A, B be Koszul dual algebras. Suppose that B is self-injective, and that B is standard Koszul, with respect to partial orderings $\preceq_i$ of a set Λ, for $i \in \mathbb{Z}/2$. Suppose that the standard modules, taken with respect to $\preceq_i$, are also standard modules for a graded cellular structure on B, for $i \in \mathbb{Z}/2$. Suppose that all standard B-modules have Loewy length $w+1$, and there exists a bijection $\theta : \Lambda \to \Lambda$, such that*

$$\Delta_i(\lambda) \cong \nabla_{i+1}(\lambda^\theta),$$

$$Hom(\Delta_i(\lambda), \nabla_i(\mu)) = 0,$$

for $\lambda \neq \mu \in \Lambda$, $i \in \mathbb{Z}/2$. Then $D^b(A)$ admits a polarization $(\phi, \Delta_0, \Delta_1, w)$.

Theorem 56. *The derived category $D^b(V_{\mathcal{C}}\text{-mod})$ admits a polarization.*

Proof. We observe that $U_{\mathcal{C}}$ satisfies the hypotheses of Theorem 55, from which it follows that $D^b(V_{\mathcal{C}}\text{-mod})$ admits a polarization.

Upon choosing a non-zero vector $v \in H$ which does not run parallel to any edges in $\mathcal{T}$, we obtain a pair of orderings $\preceq_v$ and $\preceq_{-v}$ with respect to which the standard Koszulity, and graded cellularity of $U_{\mathcal{C}}$ are known (Theorem 47, see also [8], theorem 60).

We show that there is a bijection $\theta : \mathcal{X} \to \mathcal{X}$, such that

$$\Delta^{\preceq_{sv}}_{U_{\mathcal{C}}}(x) \cong \nabla^{\preceq_{-sv}}_{U_{\mathcal{C}}}(x^{\theta}),$$

for $x \in \mathcal{X}$, and $s \in \pm 1$. Indeed, we already have a bijection λ_v between the vertices and faces of $\mathcal{T}$. We define θ to be the map which takes a vertex x to the vertex opposite x in the parallelogram $\lambda_v(x)$. The inverse to this map, θ^{-1}, takes a vertex y to the vertex opposite y in the parallelogram $\lambda_{-v}(x)$. The standard module $\Delta^{\prec_{sv}}_{U_{\mathcal{C}}}(x)$ is a facetious module, whose dual is the facetious module $\Delta^{\prec_{-sv}}_{U_{\mathcal{C}}}$.

It only remains for us to show that

$$Hom(\Delta^{\preceq_{sv}}_{U_{\mathcal{C}}}(x), \nabla^{\preceq_{sv}}_{U_{\mathcal{C}}}(y)) = 0,$$

for $x \neq y \in \mathcal{X}$, and $s \in \pm 1$.

We claim that in the graded module category, $Hom(\Delta^{\preceq_v}_{U_{\mathcal{C}}}(x), \Delta^{\preceq_{-v}}_{U_{\mathcal{C}}}(y)\langle i \rangle)$ is isomorphic to the field k, if x, y are opposite in $\lambda_v(x)$ and $i = w$, and zero otherwise. Indeed, since the facetious module $\Delta^{\preceq_v}_{U_{\mathcal{C}}}(x)$ has simple top kx and simple socle kx^{θ}, this formula is clear in case x, y are opposite in $\lambda_v(x)$. However, if there is a non-zero homomorphism from $\Delta^{\preceq_v}_{U_{\mathcal{C}}}(x)$ to $\Delta^{\preceq_{-v}}_{U_{\mathcal{C}}}(y)$, then $x \in \lambda_{-v}(y) = \lambda_v(y^{\theta^{-1}})$. This implies $x \preceq_v y^{\theta^{-1}}$. Furthermore, by duality there is a non-zero homomorphism from $\Delta^{\preceq_v}_{U_{\mathcal{C}}}(y^{\theta^{-1}})$ to $\Delta^{\preceq_{-v}}_{U_{\mathcal{C}}}(x^{\theta})$ which implies $y^{\theta^{-1}} \in \lambda_v(x)$. This implies $y^{\theta^{-1}} \preceq_v x$. Therefore, $x = y^{\theta^{-1}}$, which implies that x, y are in fact opposite in $\lambda_v(x) = \lambda_{-v}(y)$.

We have thus proven that $Hom(\Delta^{\preceq_v}_{U_{\mathcal{C}}}(x), \nabla^{\preceq_v}_{U_{\mathcal{C}}}(y)\langle i \rangle)$ is isomorphic to the field k, if $x = y$ and $i = w$, and zero otherwise.

Similarly, we find that $Hom(\Delta^{\preceq_{-v}}_{U_{\mathcal{C}}}(x), \nabla^{\preceq_{-v}}_{U_{\mathcal{C}}}(y)\langle i \rangle)$ is isomorphic to the field k, if $x = y$ and $i = w$, and zero otherwise. This completes the proof of the Theorem. □

9. Derived equivalences

In this section, we define derived equivalences between Cubist algebras whose Cubist sets are related by local mutations. The proof given here is different from that given in case $c = 1$ in the prequel to this paper. In our previous paper, the proof of the derived equivalences were tied to an interesting combinatorial formula for the Cartan matrix of $U_{\mathcal{C}}$, of which we do not have an analogue here. However, our new perspective allows us to describe equivalences of a form not considered there, and provides further theoretical insight. For example, it allows us to observe the compatibility of the derived equivalences with various gradings (section 10), and limiting procedures [23].

Our method is to derive naturally defined functors between Cubist algebras $V_{\mathcal{C}}$. It is convenient to perform homological calculations on the Koszul dual algebra $U_{\mathcal{C}}$.

Let us begin with a lemma. For the notation $D^{\uparrow}, D^{\downarrow}$, and the Koszul duality equivalences, we refer to the article of Beilinson, Ginzburg, and Soergel [3].

Lemma 57. *Suppose U, V are Koszul dual algebras, that e is an idempotent in $U^0 \cong V^0$, and that $U_e = U/U(1-e)U$ and $V_e = eVe$ are also Koszul dual algebras.*

Then we have commutative diagrams,

$$\begin{tikzcd} D^{\uparrow}(U\text{-mod}) \arrow[r, "\sim", "K"'] \arrow[d, "RHom_U(U_e,-)"] & D^{\downarrow}(V\text{-mod}) \arrow[d, "Hom_V(Ve,-)"] \\ D^{\uparrow}(U_e\text{-mod}) \arrow[r, "\sim", "K_e"'] & D^{\downarrow}(V_e\text{-mod}), \end{tikzcd}$$

$$\begin{tikzcd} D^{\downarrow}(U\text{-mod}) \arrow[d, "U_e\otimes^L_U -"] & D^{\uparrow}(V\text{-mod}) \arrow[l, "K"', "\sim"] \arrow[d, "eV\otimes_V -"] \\ D^{\downarrow}(U_e\text{-mod}) & D^{\uparrow}(V_e\text{-mod}), \arrow[l, "K_e"', "\sim"] \end{tikzcd}$$

where K and K_e denote the Koszul duality functors. We have a full embedding of bounded derived categories

$$D^b(U_e\text{-mod}) \to D^b(U\text{-mod}).$$

Proof. The U-U_e-bimodule U_e corresponds, under Koszul duality on the right, to the U-V_e-bimodule U^0e. The V-V_e-bimodule Ve corresponds, under Koszul duality on the left, to the same U-V_e-bimodule U^0e. Therefore, the U-U_e-bimodule U_e corresponds, under Koszul duality, to the V-V_e-bimodule Ve. We thus have a commutative diagram,

$$\begin{tikzcd} D^{\uparrow}(U\text{-mod}) \arrow[r, "\sim", "K"'] \arrow[d, "RHom_U(U_e,-)"] & D^{\downarrow}(V\text{-mod}) \arrow[d, "Hom_V(Ve,-)"] \\ D^{\uparrow}(U_e\text{-mod}) \arrow[r, "\sim", "K_e"'] & D^{\downarrow}(V_e\text{-mod}), \end{tikzcd}$$

We similarly have a commutative diagram

$$\begin{tikzcd} D^{\downarrow}(U\text{-mod}) \arrow[d, "U_e\otimes^L_U -"] & D^{\uparrow}(V\text{-mod}) \arrow[l, "K"', "\sim"] \arrow[d, "eV\otimes_V -"] \\ D^{\downarrow}(U_e\text{-mod}) & D^{\uparrow}(V_e\text{-mod}). \arrow[l, "K_e"', "\sim"] \end{tikzcd}$$

Now suppose that S, T are irreducible U_e-modules, and e_S, e_T corresponding degree zero idempotents in U. Let F denote the exact functor

$$U_e \otimes_{U_e} - : D^b(U_e\text{-mod}) \to D^b(U\text{-mod}).$$

satisfies the condition $Hom(FS, FT\langle i\rangle) \cong e_S V_i e_T = e_S V_e e_T \cong Hom(S, T\langle i\rangle)$. Therefore F is a full embedding on the collection of irreducible U_e-modules. Since the collection of irreducible U_e-modules generates , F is a full embedding on $D^b(U_e\text{-mod})$. □

Let $\mathcal{C}$ and $\mathcal{D}$ be Cubist subcomplexes of $\mathcal{Z}$ of dimension w, whose sets of vertices are denoted $\mathcal{X}$ and $\mathcal{Y}$.

Definition 58. *Suppose there exist $x \in \mathcal{X}$, $S \subset \underline{r}$, such that $|S| \geq c$, and*

$$\mathcal{X} \cup \mathcal{Y} \supset x + \mathbb{Z}^{\underline{r}\backslash S} + \sum_{s\in S}\{0,1\}\epsilon_s,$$

$$\mathcal{X}\backslash\mathcal{Y} = x + \mathbb{Z}^{\underline{r}\backslash S}, \quad \mathcal{Y}\backslash\mathcal{X} = x + \mathbb{Z}^{\underline{r}\backslash S} + \sum_{s\in S}\epsilon_s.$$

Then we say $\mathcal{C},\mathcal{D}$ are related by a Cubist mutation.

Remark 59 Let S be fixed. If the above definition is satisfied for some $x \in \mathcal{X}\backslash\mathcal{Y}$, then it is satisfied for every $x \in \mathcal{X}\backslash\mathcal{Y}$.

A picture of a mutation, in case $w = 2$, and$|S| = 3$, is given in Figure 13.

Lemma 60. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are related by a Cubist mutation. Then*

$$\mathcal{X} \cap (z + \sum_{s\in S}\mathbb{Z}_{\geq 0}\epsilon_s + \sum_{t\in \underline{r}\backslash S}\mathbb{Z}\epsilon_t) = \emptyset,$$

for $z \in \mathcal{Y}\backslash\mathcal{X}$, and

$$\mathcal{Y} \cap (z + \sum_{s\in S}\mathbb{Z}_{\leq 0}\epsilon_s + \sum_{t\in \underline{r}\backslash S}\mathbb{Z}\epsilon_t) = \emptyset,$$

for $z \in \mathcal{X}\backslash\mathcal{Y}$.

Suppose $\mathcal{X},\mathcal{Y}$ are Cubist sets related by a mutation. Let $z \in \mathcal{Y}\backslash\mathcal{X}$.

Let V^S denote the subalgebra of V generated by the arrows $a_{z,i}$, for $i \in S$. As a vector space, $V^S f_z$ is isomorphic to the symmetric algebra $S(kS)$, for $z \in \mathbb{Z}^r$.

Let U^S denote the quotient of U by the arrows $\beta_{z,i}$, for $i \in \underline{r}\backslash S$, and $\alpha_{z,i}$, for $i \in \underline{r}$. As a vector space, $M_z = U^S e_z$ is isomorphic to the exterior algebra $\bigwedge(kS)$.

The algebras U^S and V^S are Koszul duals.

If M and N are modules over an algebra A, with fixed direct sum decompositions,

$$M = \oplus_{i\in I} M_i, N = \oplus_{j\in J} N_j,$$

where M_i and N_i are indecomposable objects of A-mod. Then we write

$$Hom^\star(M,N) = \bigoplus_{i\in I, j\in J, k,l\in\mathbb{Z}} Hom(M, N\langle k\rangle[l]), \quad End^\star(M) = Hom^\star(M,M).$$

If $\mathcal{A},\mathcal{B}$ are subsets of $\mathcal{Z}^0$, we write

$$V_{\mathcal{A},\mathcal{B}} = \sum_{a\in\mathcal{A}, b\in\mathcal{B}} f_a V f_b.$$

Then $V_{\mathcal{A},\mathcal{B}}$ is a $V_{\mathcal{A}}$-module, with a canonical decomposition $V_{\mathcal{A},\mathcal{B}} = \oplus_{b\in\mathcal{B}} V_{\mathcal{A},\{b\}}$, as a direct sum of indecomposables.

We have natural isomorphisms

$$V_{\mathcal{C}} \cong V_{\mathcal{X},\mathcal{X}} \cong End^\star_{V_{\mathcal{C}}}(V_{\mathcal{X},\mathcal{X}}).$$

Lemma 61. *We have*

$$Hom^\star_V(Vf_{\mathcal{X}}, K(M_z)) = Hom^\star_U(k\mathcal{X}, M_z) =$$
$$Hom^\star_U(M_z^*, k\mathcal{X}) = Hom^\star_V(K(M_z^*), Vf_{\mathcal{X}}) =$$
$$Hom^\star_V(K(M_z), Vf_{\mathcal{Y}}) = Hom^\star_U(M_z, k\mathcal{Y}) =$$
$$Hom^\star_U(k\mathcal{Y}, M_z^*) = Hom^\star_V(Vf_{\mathcal{Y}}, K(M_z^*)) = 0,$$

for $z \in \mathcal{Y}\backslash\mathcal{X}$.

Proof.

$$K(M_z) = V \otimes_{k\mathcal{Z}^0} M_z^* \cong V \otimes_{V^S} V^S \otimes_{k\mathcal{Z}^0} M_z^*$$

But U^S and V^S are in Koszul duality, and M_z^* is an injective U^S-module, and therefore

$$V^S \otimes_{k\mathcal{Z}^0} M_z^* \cong kz.$$

Consequently $K(M_z) \cong V \otimes_{V^S} kz$ The composition factors of $V \otimes_{V^S} kz$ lie in $z + \sum_{s\in S} \mathbb{Z}_{\geq 0}\epsilon_s + \sum_{t\in \underline{r}\backslash S} \mathbb{Z}\epsilon_t$, which does not intersect $\mathcal{X}$ by Lemma 60. Therefore

$$Hom_V^\star(Vf_{\mathcal{X}}, K(M_z)) \cong Hom_V^\star(Vf_{\mathcal{X}}, K(M_z)) \cong Hom_V^\star(Vf_{\mathcal{X}}, V \otimes_{V^S} kz) \cong 0$$

By Koszul duality, and k-duality, we also have

$$Hom_U^\star(k\mathcal{X}, M_z) = Hom_U^\star(M_z^*, k\mathcal{X}) = Hom_V^\star(K(M_z^*), Vf_{\mathcal{X}}) = 0.$$

So the first four spaces of homomorphisms listed in the statement of the theorem are zero. The four following spaces are zero, by the same argument, using $\mathcal{Y}$ instead of $\mathcal{X}$. □

We denote by L_z the kernel of the U-module homomorphism $M_z \twoheadrightarrow kz$.

Lemma 62. *L_z is a $U_{\mathcal{C}}$-module.*

Proof. The composition factors of M_z lie in $x + [0,1]\sum_{s\in S}\epsilon_s$, where $x = z - \sum_{s\in S}\epsilon_s \in \mathcal{X}\backslash\mathcal{Y}$. Since $\mathcal{C}$ and $\mathcal{D}$ are related by a Cubist mutation, the composition factors of L_z lie in $\mathcal{X}$. Therefore $f_w L_z = 0$, for all $w \in \mathbb{Z}^r\backslash\mathcal{C}$, and so L_z is a $U_{\mathcal{C}}$-module. □

Lemma 63. *$K_{\mathcal{C}}(L_z)[-1] \cong f_{\mathcal{X}} V f_z$.*

Proof. We have a short exact sequence

$$0 \to L_z \to M_z \to kz \to 0.$$

Applying the Koszul duality functor, and the exact functor $Hom_V^\star(Vf_{\mathcal{X}}, -)$, we obtain an exact triangle,

$$Hom_V^\star(Vf_{\mathcal{X}}, K(L_z)) \to Hom_V^\star(Vf_{\mathcal{X}}, K(M_z)) \to Hom_V^\star(Vf_{\mathcal{X}}, Vf_z) \rightsquigarrow,$$

which by Lemma 61 is an exact triangle,

$$K_{\mathcal{C}}(L_z) \to 0 \to f_{\mathcal{X}} V f_z \rightsquigarrow .$$

We therefore have an isomorphism $f_{\mathcal{X}} V f_z \cong K_{\mathcal{C}}(L_z)[-1]$, as required. □

Lemma 64. *We have*

$$Hom_{U_{\mathcal{C}}}(X, L_z[-1]) \cong Hom_U(X, kz),$$

$$Hom_{U_{\mathcal{C}}}(L_z[-1], X) \cong Hom_U(kz, X),$$

for $X \in D^b(U_{\mathcal{C}}\text{-mod})$.

Proof. We have an adjunction

$$D^-(U\text{-mod}) \underset{\psi}{\overset{\phi}{\rightleftarrows}} D^-(U_{\mathcal{C}}\text{-mod}),$$

where $\phi = RHom_U(U_e, -)$, and $\psi = U_{\mathcal{C}} \otimes_{U_{\mathcal{C}}} -$. There is an isomorphism $\phi(kz) \cong L_z[-1]$, by Lemmas 57 and 63. We therefore have isomorphisms

$$Hom_{U_{\mathcal{C}}}(X, L_z[-1]) \cong Hom_{U_{\mathcal{C}}}(X, \phi(kz)) \cong Hom_U(\psi X, kz),$$

for $X \in D^b(U_{\mathcal{C}}\text{-mod})$. However, ψ is the lift functor, and so

$$Hom_{U_{\mathcal{C}}}(X, L_z[-1]) \cong Hom_U(X, kz),$$

as required.

The second formula is proved analogously. □

Theorem 65. *Suppose that $\mathcal{C}, \mathcal{D}$ are Cubist complexes, related by a Cubist mutation. Then we have an equivalence of triangulated categories*

$$D^b(V_{\mathcal{D}}\text{-mod}) \to D^b(V_{\mathcal{C}}\text{-mod}).$$

Proof. There is a natural map $V_{\mathcal{D}} \to End^{\star}_{V_{\mathcal{C}}}(V_{\mathcal{X},\mathcal{Y}})$, whose image has homological degree zero. We prove this map is an isomorphism. Indeed,

$$Hom^{\star}_{V_{\mathcal{C}}}(V_{\mathcal{X},\mathcal{Y}}, V_{\mathcal{X},\mathcal{Y}}) \cong$$

$$Hom^{\star}_{V_{\mathcal{C}}}(V_{\mathcal{X},\mathcal{X}\cap\mathcal{Y}} \oplus V_{\mathcal{X},\mathcal{Y}\backslash\mathcal{X}}, V_{\mathcal{X},\mathcal{Y}}) \cong$$

$$Hom^{\star}_{V_{\mathcal{C}}}(V_{\mathcal{X},\mathcal{X}\cap\mathcal{Y}}, V_{\mathcal{X},\mathcal{Y}}) \oplus Hom^{\star}_{V_{\mathcal{C}}}(V_{\mathcal{X},\mathcal{Y}\backslash\mathcal{X}}, V_{\mathcal{X},\mathcal{Y}}) \cong$$

$$V_{\mathcal{X}\cap\mathcal{Y},\mathcal{Y}} \oplus Hom^{\star}_{U_{\mathcal{C}}}(K_{\mathcal{C}}^{-1}(V_{\mathcal{X},\mathcal{Y}\backslash\mathcal{X}}), K_{\mathcal{C}}^{-1}(V_{\mathcal{X},\mathcal{Y}})),$$

which, by Lemma 63 is isomorphic to

$$V_{\mathcal{X}\cap\mathcal{Y},\mathcal{Y}} \oplus Hom^{\star}_{U_{\mathcal{C}}}(L_{\mathcal{Y}\backslash\mathcal{X}}[-1], L_{\mathcal{Y}\backslash\mathcal{X}}[-1] \oplus k\mathcal{X}\cap\mathcal{Y}) \cong$$

$$V_{\mathcal{X}\cap\mathcal{Y},\mathcal{Y}} \oplus Hom^{\star}_{U_{\mathcal{C}}}(L_{\mathcal{Y}\backslash\mathcal{X}}[-1], L_{\mathcal{Y}\backslash\mathcal{X}}[-1] \oplus k\mathcal{X}\cap\mathcal{Y})$$

which, by Lemma 64 is isomorphic to

$$V_{\mathcal{X}\cap\mathcal{Y},\mathcal{Y}} \oplus Hom^{\star}_{U}(k\mathcal{Y}\backslash\mathcal{X}, L_{\mathcal{Y}\backslash X}[-1] \oplus k\mathcal{X}\cap\mathcal{Y})$$

which, by Lemma 61 is isomorphic to,

$$V_{\mathcal{X}\cap\mathcal{Y},\mathcal{Y}} \oplus Hom^{\star}_{U}(k\mathcal{Y}\backslash\mathcal{X}, k\mathcal{Y}\backslash X \oplus k\mathcal{X}\cap\mathcal{Y}) \cong$$

$$V_{\mathcal{X}\cap\mathcal{Y},\mathcal{Y}} \oplus Hom^{\star}_{U}(k\mathcal{Y}\backslash\mathcal{X}, k\mathcal{Y}) \cong$$

$$V_{\mathcal{X}\cap\mathcal{Y},\mathcal{Y}} \oplus V_{\mathcal{Y}\backslash\mathcal{X},\mathcal{Y}} \cong V_{\mathcal{Y},\mathcal{Y}} \cong V_{\mathcal{D}}.$$

Therefore,

$$End^{\star}_{V_{\mathcal{X}}}(V_{\mathcal{X},\mathcal{Y}}) \cong V_{\mathcal{Y}},$$

as required. By Rickard's tilting theory [22], we have a derived equivalence between $V_{\mathcal{X}}$ and $V_{\mathcal{Y}}$. □

Remark 66 We expect derived equivalences between Cubist algebras to pass to the subcategory of $D^b(U_{\mathcal{C}}\text{-mod})$ generated by standard modules to the subcategory of $D^b(U_{\mathcal{D}}\text{-mod})$ generated by standard modules. Therefore, the equivalences should also pass to an equivalence of quotients

$$U_{\mathcal{C}}\text{-}\underline{\mathrm{mod}}_{\Delta} \cong U_{\mathcal{D}}\text{-}\underline{\mathrm{mod}}_{\Delta}.$$

10. Gradings

Here, we consider the existence of certain gradings on the Cubist algebras. The degree zero parts of these gradings are Koszul algebras, whose derived categories are quotients of the graded derived category of the Cubist algebras.

These quotients are sometimes compatible with the derived equivalences, so that derived equivalences between Cubist algebras pass to derived equivalences between degree zero parts.

Geometrically, these degree zero parts correspond to closed Lagrangian subvarieties of the noncommutative symplectic variety.

Group theoretically, these gradings correspond, via derived equivalences, to gradings on the group algebra of the defect group of a block.

We describe Cubist algebras as multiplicative extensions, giving an algebraic structure with features resembling an integrable system.

Recall we have a natural 2-1 map from $\tilde{r}$ to $\underline{r}$, which takes a_i and b_i to $i \in \underline{r}$. Let $\xi : \underline{r} \to \tilde{r}$ be some section of this map.

We have a $\mathbb{Z}_+$-grading on Γ, where $\tilde{r}\backslash\xi(\underline{r})$ lies in degree zero, and where $\xi(\underline{r})$ lies in degree one. This grading extends to a $\mathbb{Z}_+$-grading on Π

The subspace $\phi(E_k)$ of Γ is homogeneous of degree one, with respect to this grading, Therefore the grading on Γ descends to a grading on Λ, and the grading on Π descends to a grading on V. Taking a component, we obtain a $\mathbb{Z}_+$-grading on $V_{\mathcal{C}}$. We denote by $\mathcal{V}_{\mathcal{C}}$ the degree zero part of $V_{\mathcal{C}}$, with respect to this grading. We denote by $D^b(V_{\mathcal{C}}\text{-grmod})$ the derived category of $V_{\mathcal{C}}$-modules which respects this grading. We denote by $\mathcal{J}_{\mathcal{C}}$ the subcategory of $D^b(V_{\mathcal{C}}\text{-grmod})$, whose terms in homology have composition factors which all lie in non-zero degree.

We have a $\mathbb{Z}_+$-grading on U which places the variables dual to $\xi(\underline{r})$ in degree one, and the variables dual to $\tilde{r}\backslash\xi(\underline{r})$ in degree zero. We denote by $\mathcal{U}_{\mathcal{C}}$ the degree zero part of $U_{\mathcal{C}}$, with respect to this grading. We denote by $D^b(U_{\mathcal{C}}\text{-grmod})$ the derived category of $U_{\mathcal{C}}$-modules which respects this grading. We denote by $\mathcal{I}_{\mathcal{C}}$ the subcategory of $D^b(U_{\mathcal{C}}\text{-grmod})$, whose terms in homology have composition factors which all lie in non-zero degree.

Note the quiver of $\mathcal{U}_{\mathcal{C}}$ is the opposite of the quiver of $\mathcal{V}_{\mathcal{C}}$.

Theorem 67. *$\mathcal{U}_{\mathcal{C}}$, and $\mathcal{V}_{\mathcal{C}}$ are Koszul dual algebras. $\mathcal{U}_{\mathcal{C}}$ is concentrated in degrees $0, 1, ..., w$, and $\mathcal{V}_{\mathcal{C}}$ has global dimension w.*

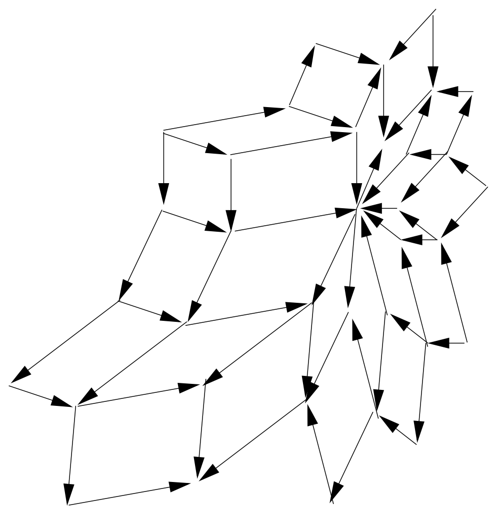

FIGURE 18. The quiver of $\mathcal{V}_{\mathcal{C}}$

We have recollements of triangulated categories, and a commuting diagram,

$$\begin{tikzcd} \mathcal{I}_{\mathcal{C}} \arrow[r, shift left] \arrow[d, "\wr"] & D^b(U_{\mathcal{C}}\text{-grmod}) \arrow[l, shift left] \arrow[r, shift left] \arrow[d, "\wr K_{\mathcal{C}}"] & D^b(\mathcal{U}_{\mathcal{C}}\text{-mod}) \arrow[l, shift left] \arrow[d, "\wr"] \\ \mathcal{J}_{\mathcal{C}} \arrow[r, shift left] & D^b(V_{\mathcal{C}}\text{-grmod}) \arrow[l, shift left] \arrow[r, shift left] & D^b(\mathcal{V}_{\mathcal{C}}\text{-mod}) \arrow[l, shift left] \end{tikzcd}$$

Proof. The Koszul complex for $V_{\mathcal{C}}$ defines a projective resolution of $k\mathcal{X}$,

$$V_{\mathcal{C}} \otimes_{k\mathcal{X}} U_{\mathcal{C}}^* \to k\mathcal{X},$$

linear with respect to the Koszul grading. The differentials in this complex thus have a degree one part, corresponding to arrows in $U_{\mathcal{C}}$ in $\xi(\underline{r})^*$, and a degree one part, corresponding to arrows in $U_{\mathcal{C}}$ outside $\xi(\underline{r})^*$. Restricting to degree zero, we thus obtain a linear projective resolution

$$\mathcal{V}_{\mathcal{C}} \otimes_{k\mathcal{X}} \mathcal{U}_{\mathcal{C}}{}^* \to k\mathcal{X}.$$

It follows that $\mathcal{V}_{\mathcal{C}}$ is Koszul, with Koszul dual $\mathcal{U}_{\mathcal{C}}$.

Note that paths in a cube in the quiver of $U_{\mathcal{C}}$ must have minimal length. If two paths have the same endpoints, are equal, up to a sign in $\mathcal{U}_{\mathcal{C}}$, by the supercommutation relations. Furthermore, paths in the quiver of $\mathcal{U}_{\mathcal{C}}$ which do not lie in a single w-dimensional cube either have a component of length two in some direction, or are equal, up to sign, to a path in the quiver of U which does not all lie in $\mathcal{C}$. In case the path has a component of length two, the path must vanish in $\mathcal{U}_{\mathcal{C}}$, by the square relations. Paths in the quiver of U which do not lie in $\mathcal{C}$ must vanish in $\mathcal{C}$, since vertex idempotents outside $\mathcal{C}$ are zero. In conclusion, paths in the quiver of $\mathcal{U}_{\mathcal{C}}$, which do not vanish, must all lie in a single w-dimensional cube, Therefore $\mathcal{U}_{\mathcal{C}}$ is concentrated in degrees $0, ..., w$. The Koszul dual algebra $V_{\mathcal{C}}$ has global dimension w.

The existence of the bottom recollement follows from the finite global dimension of $\mathcal{V}_{\mathcal{C}}$, by a theorem of Parshall and Scott ([20], Theorem 2.7(b)). The recollement lifts to $D^b(U_{\mathcal{C}}\text{-grmod})$, by Koszul duality. □

Remark 68 Suppose two Cubist sets $\mathcal{C}, \mathcal{D}$ are related by a mutation, with respect to some $S \subset \underline{r}$. Suppose that $\xi(S) \subset \{a_1, ..., a_r\}$. Then the bimodule $V_{\mathcal{C},\mathcal{D}}$ inducing a derived equivalence between $V_{\mathcal{C}}$, $V_{\mathcal{D}}$ is naturally compatible with the gradings on these algebras. It is not difficult to see that, in these circumstances, we have a commuting diagram of equivalences.

$$\begin{tikzcd}
\mathcal{J}_{\mathcal{C}} \arrow[r, shift left] \arrow[d, "\wr"] & D^b(V_{\mathcal{C}}\text{-grmod}) \arrow[l, shift left] \arrow[r, shift left] \arrow[d, "\wr"] & D^b(\mathcal{V}_{\mathcal{C}}\text{-mod}) \arrow[l, shift left] \arrow[d, "\wr"] \\
\mathcal{J}_{\mathcal{D}} \arrow[r, shift left] & D^b(V_{\mathcal{D}}\text{-grmod}) \arrow[l, shift left] \arrow[r, shift left] & D^b(\mathcal{V}_{\mathcal{D}}\text{-mod}). \arrow[l, shift left]
\end{tikzcd}$$

The quivers of the algebras $\mathcal{V}_{\mathcal{C}}$, $\mathcal{V}_{\mathcal{D}}$ induce the structure of a partially ordered set on $\mathcal{C}, \mathcal{D}$. Poset algebras, and derived equivalences between them, have been studied by S. Ladkani.

Remark 69 We think of the degree zero part $\mathcal{V}_{\mathcal{C}}^{\xi}$ as cutting out a Lagrangian subvariety L_{ξ} in a noncommutative symplectic variety $Y_{\mathcal{C}}$ defined by $V_{\mathcal{C}}$ (cf Example 72).

Let T be a w-dimensional torus, and $\mathfrak{t}$ its Lie algebra.

Theorem 70. *We have a multiplicative extension of algebras,*

$$\begin{tikzcd}
 & V_{\mathcal{C}} \arrow[dr, two heads] & \\
k\mathcal{X} \otimes k[\mathfrak{t}^*] \arrow[ur, hook] & & \bar{V_{\mathcal{C}}} \otimes k.
\end{tikzcd}$$

and a split embedding

$$\bar{V_{\mathcal{C}}} \to \prod_{\xi} \mathcal{V}_{\mathcal{C}}^{\xi}$$

of $\bar{V_{\mathcal{C}}}$ into an algebra of global dimension w.

Proof. Multiplicative extensions of algebras are analogues in the category of algebras, of group extensions in the category of groups (see [24]).

The algebra $k[\mathfrak{t}^*]$ acts centrally on the left and right of $V_{\mathcal{C}}$, with $\mathfrak{t}^*$ acting in degree two. This action should be thought of as a collection of commuting Hamiltonions on $Y_{\mathcal{C}}$, defining a map from the variety $Y_{\mathcal{C}}$ to affine space $\mathfrak{t}^*$. The zero fibre has a ring of functions $\bar{V_{\mathcal{C}}} = V_{\mathcal{C}} \otimes_{k[\mathfrak{t}^*]} k$ of infinite global dimension, and is therefore singular.

Let $x, y \in \mathcal{C}$. The xy entry in the Cartan-Hilbert matrix of $V_{\mathcal{C}}$ is $q^{d(x,y)}(1-q^2)^{-w}$, where $d(x, y)$ is the length of a shortest path in the one-skeleton of $\mathcal{C}$ between x and y. A basis for $f_x V_{\mathcal{C}} f_y$ is $p.k[\mathfrak{t}^*]$, where p is a path in the one-skeleton of

$\mathcal{C}$ of length $d(x, y)$. The algebra $k[\mathfrak{t}^*]$ acts centrally and freely, and so we have a multiplicative extension as drawn. The vector space $f_x \bar{V_\mathcal{C}} f_y$ is therefore one dimensional, generated by any shortest path in the one-skeleton of $\mathcal{C}$. Furthermore, every shortest path in $\mathcal{X}$ is also a shortest path p in the one-skeleton of $\mathcal{Z}$, and so every time it passes along an arrow parallel to a given 1-cell, it points in the same direction. Therefore, for some ξ, the path p defines a non-zero path in $\mathcal{V}_\mathcal{C}^\xi$. It follows that the natural map

$$\bar{V_\mathcal{C}} \to \prod_\xi \mathcal{V}_\mathcal{C}^\xi$$

is a split embedding. The algebras ${\mathcal{V}_\mathcal{C}}^\xi$ all have global dimension w, by Theorem 67. □

Remark 71 We visualise our zero fibre $\bar{Y_\mathcal{C}}$ as a union of smooth Lagrangian subvarieties, L_ξ in $Y_\mathcal{C}$. The Lagrangians L_ξ should be thought of as noncommutative toric varieties (cf. example 72).

A symplectic manifold of dimension $2w$, with w commuting Hamiltonions, whose fibres are tori is known as an integrable system. The variety $Y_\mathcal{C}$ thus resembles a noncommutative integrable system.

Example 72 Suppose $c = 1$. Then $r = w + 1$. Let H be the hyperplane $\epsilon_1 + ... + \epsilon_r = 0$ in $E = \mathbb{R}^r$. Let $\mathcal{C}$ denote the polytopal subcomplex of $\mathcal{Z}_r$, whose cells are entriely contained in the band $\{x \in E| - 1 \leq \sum x_i \leq 1\}$. Then $\mathcal{C}$ is Cubist, and $\{z \in \mathbb{Z}^r | \sum z_i = 0\} \cong \mathbb{Z}^w$ acts on the collection of vertices $\mathcal{X}$. Let $\xi : i \mapsto \alpha_i$. Then the algebra $\mathcal{V}_\mathcal{C}^\xi$ admits a $\mathbb{Z}^w$-action.

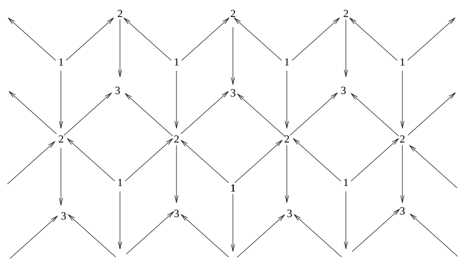

FIGURE 19. $k^{\times 2}$-equivariant $\mathbb{P}^2$

The algebra $\mathcal{V}_\mathcal{C}^\xi$ has no fixed points under the $\mathbb{Z}^w$-action. However, we can pass to a completion, which contains infinite sums $\sum_{x,y \in \mathcal{X}} v_{xy}$ of elements v_{xy}, of bounded degree. The resulting algebra now contains fixed points. Indeed, the algebra of $\mathbb{Z}^w$-fixed points is an algebra, given by a quiver with $w + 1$ vertices,

$$\bullet \underset{a_r}{\overset{a_1}{\underset{\cdots}{\overset{\cdots}{\rightrightarrows}}}} \bullet \underset{a_r}{\overset{a_1}{\underset{\cdots}{\overset{\cdots}{\rightrightarrows}}}} \bullet \quad \cdots \quad \bullet \underset{a_r}{\overset{a_1}{\underset{\cdots}{\overset{\cdots}{\rightrightarrows}}}} \bullet$$

and relations $a_i a_j - a_j a_i$, for $1 \leq i, j \leq w+1$. This algebra was proved by Beilinson to be derived equivalent to $\mathbb{P}^w$ [2].

Given that this is so, it makes sense to enquire as to the geometric meaning of the Cubist algebra in this case. We expect the derived category of modules over the Cubist algebra to be equivalent to the derived category of $k^{\times w}$-equivariant coherent sheaves over $T^*\mathbb{P}^w$.

We expect to study more general toric varieties than $\mathbb{P}^w$ from our perspective, in future work.

11. Strips

We discuss strips of Cubist sets, and their relevance for the Cubist algebras.

Recall a *strip* in a Cubist set $\mathcal{C}$ is the intersection of $\mathcal{C}$ with $\mathbb{R}^{l-1}\times[z,z+1]\times\mathbb{R}^{r-l}$, for some $l\in\underline{r}$, and some integer z.

We have already that strips have significance in Cubist combinatorics, when we defined the operations "slicing" and "parting" on Cubist sets. Indeed, slicing was the surgical operation of removing a strip from a Cubist set, whilst parting was the operation of enhancing a Cubist set by the insertion of a strip.

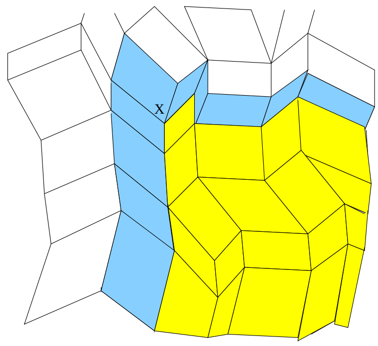

Figure 20. μx cut out by strips

The module $\Delta_{V_\mathcal{X}}(x)$ is correspondingly induced from a subalgebra P_θ. The region μx, for $x\in\mathcal{X}$, which describes the composition factors of the Cubist algebra $V_\mathcal{X}$ is cut out by a w-tuple of strips. By analogy with Lie theory, we think of the region μx as *Borel*, obtained as an intersection of parabolic subgroups. Continuing the analogy, we therefore think of the half-spaces $\mathfrak{p}=\mathcal{C}\cap(\mathbb{R}^{l-1}\times\mathbb{R}_{\geq z}\times\mathbb{R}^{r-l})$, corresponding to a strip, as *maximal parabolic*. We think of the Cubist subset $\mathfrak{l}\cong\mathcal{C}_{l,z}$ on the boundary of a strip, as *maximal Levi*. We do not have Cubist subalgebras corresponding to Levi subsets, in general. Nonetheless, there is a natural algebraic relation.

Proposition 73. *Let $\mathcal{C}$ be a Cubist set of dimension w. For a given strip in $\mathcal{C}$, let $\mathfrak{l}$ be the associated Levi subset of dimension $w-1$. Then $V_\mathfrak{l}$ is a subquotient of $V_\mathcal{C}$. There is a natural $V_\mathfrak{l}$- $V_\mathcal{C}$-bimodule, $M_{\mathfrak{l},\mathcal{C}}$.*

Proof. Let $V_{\mathfrak{p}} = V_{\mathcal{C}} / \sum_{b \in \mathfrak{l}} V_{\mathcal{C}} f_{b+\epsilon_l} V_{\mathcal{C}}$. Then we have an isomorphism

$$V_{\mathfrak{l}} \cong \bigoplus_{x,y \in \mathfrak{l}} f_x V_{\mathfrak{p}} f_y,$$

since by definition, $f_x a_l b_l = 0 \in V_{\mathfrak{p}}$, for $x \in \mathfrak{l}$. A $V_{\mathfrak{l}}$-$V_{\mathcal{C}}$-bimodule is given by

$$M_{\mathfrak{l},\mathcal{C}} = \bigoplus_{b \in \mathfrak{l}} f_b V_{\mathfrak{p}}.$$

□

Remark 74 The bimodules $M_{\mathfrak{l},\mathcal{C}}$ induce exact adjoint functors between derived categories of Cubist algebras $V_{\mathfrak{l}}$, $V_{\mathcal{C}}$. In the representation theory of symmetric groups, analogous functors are induction and restriction between blocks of different weights.

Let $\mathcal{S}$ denote the set of strips in $\mathcal{C}$. If we wish to think algebraically, by Lemma 10, we can identify a strip s with its parallel direction ϵ_l, and the minimal value of $q_l(s)$, an integer. We thus have a natural bijection,

$$\mathcal{S} \cong \coprod_{l \in \underline{r}} (q_l(\mathcal{X}) \backslash q_l(\mathcal{X})^{sup})$$

where $\mathcal{S}$ is identified with a collection of r intervals in $\mathbb{Z}$. Here, M^{sup} denotes the singleton set containing the supremum of M, if such exists, and denotes the empty set otherwise.

Proposition 75. *The Cubist algebra $V_{\mathcal{C}}$ has a natural $\mathcal{S}$-parameter deformation $V_{\mathcal{C},\mathcal{S}}$.*

Proof. In a Koszul algebra A, deformations are a natural consequence of the existence of quadratic central elements in the Koszul dual algebra [24]. Indeed, given such an element $z \in A^!$, the algebra $(A^! \otimes k[\tau]/(1 \otimes \tau^2 - z \otimes 1))^!$ is a deformation of A. Here, $\otimes$ denotes the graded tensor product, over k.

After choosing a strip

$$S = S(l, z) = \{x \in \mathcal{X} | n \leq x_i \leq n+1\}$$

inside $\mathcal{X}$, we define a quadratic element of $U_{\mathcal{C}}$ by $z_S = \sum_{s \in S} e_s \alpha_i \beta_i$. We should be a little careful, since Cubist sets are infinite. Strictly speaking, the element z_S does not lie in $U_{\mathcal{C}}$. However, it acts on the left and right of $U_{\mathcal{C}}$, and the left and right actions agree. This is sufficient to define a deformation.

Every such element gives rise to a deformation of $V_{\mathcal{C}}$. Assembling them together, we obtain a deformation $V_{\mathcal{C},\mathcal{S}}$, with parameters indexed by $\mathcal{S}$. □

Remark 76 Since Cubist sets are infinite, we should say precisely what we mean by an I-parameter deformation of A. We mean an algebra $\widehat{A}$, such that $Z_I = k[z_i | i \in I]$ acts on the left, and right of $\widehat{A}$, such that the left and right actions agree, and such that $\widehat{A} \otimes_{Z_I} k \cong A$.

Remark 77 If $\mathfrak{g}$ is a simple Lie algebra, and W its Weyl group, we have correspondences

$$\begin{tikzcd} \{\text{Nodes of the Dynkin diagram of } W\} \arrow[r] \arrow[d] & \{\text{ Deformation parameters for } H_W\}, \\ \{\text{Maximal Levi subalgebras of } \mathfrak{g}\} & \end{tikzcd}$$

where H_W denotes the rational Cherednik algebra associated to W [14]. Analogously, we have correspondences,

$$\begin{tikzcd} \mathcal{S} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \{\text{Deformation parameters for } V_{\mathcal{C},\mathcal{S}}\} \\ \{\text{Maximal Levi subquotients of } V_{\mathcal{C}}\} & \end{tikzcd}$$

The collection $\mathcal{S}$ of strips is therefore a Cubist analogue of the collection of nodes of a Dynkin diagram.

Remark 78 W. Gan, V. Ginzburg, E. Opdam and R. Rouquier have defined a category $\mathcal{O}$ for rational Cherednik algebras, and studied its properties [14]. We expect to be able to do something similar for the deformations $V_{\mathcal{C},\mathcal{S}}$. Let F be a field containing k. Let $f : \mathcal{S} \to F$. There is a natural algebra homomorphism from $Z_{\mathcal{S}}$ to F, taking z_s to $f(s)$. We should define a category $\mathcal{O}_{\mathcal{C},f}$ of modules for $V_{\mathcal{C},\mathcal{S}} \otimes_{Z_S} F$, which in case $F = k$, $f = 0$, realises the corresponding highest weight category of $V_{\mathcal{C}}$-modules. The category $\mathcal{O}_{\mathcal{C},f}$ should be a highest weight category. In case F is the field of fractions of $Z_{\mathcal{S}}$, $\mathcal{O}_{\mathcal{C},f}$ should be semisimple.

Example 79 Let $r = w = 1$. Thus $\mathcal{C} = \mathcal{Z}_1$. Then $V_{\mathcal{C}}$ is isomorphic to the preprojective algebra on an infinite line. Applying the above proposition, we obtain a deformation $V_{\mathcal{C},\mathcal{S}}$ of $V_{\mathcal{C}}$, with parameters indexed by the integers. The algebra $V_{\mathcal{C},\mathcal{S}}$ can be given by quiver and relations, as follows.

The quiver has vertices $v_i, i \in \mathbb{Z}$, arrows a_i from v_i to v_{i+1}, arrows b_i from v_{i+1} to v_i, and arrows t_{ij}, from v_i to v_i, for $i, j \in \mathbb{Z}$. The relations are

$$a_i b_i - b_{i-1} a_{i-1} + t_{ii}^2 - t_{ii-1}^2 = 0,$$

$$a_i t_{i+1j} - t_{ij} a_i = 0,$$

$$b_i t_{ij} - t_{i+1j} b_i = 0,$$

$$t_{ij} t_{ik} - t_{ik} t_{ij} = 0,$$

for $i, j, k \in \mathbb{Z}$.

The $k[t_j | j \in \mathbb{Z}]$ acts on the left and right of $V_{\mathcal{C}}$, with t_j acting as $\sum_i t_{ij}$. Specialising the deformation parameters t_j to elements of the field k, for $j \in \mathbb{Z}$, we recover the deformed preprojective algebras of W. Crawley-Boevey and M. Holland, in type A_∞^∞ [10].

Remark 80 The term *noncommutative geometry* is somewhat ambiguous. Roughly speaking, it is the study of geometries whose rings of functions are noncommutative. But what does this mean ? According to one interpretation, which we have assumed in this paper, the geometric object is taken to be the derived category of the noncommutative ring. Let us call this *triangulated noncommutative geometry*. Geometric aspects of a chimerical noncommutative variety are to be interpreted via homological aspects of the corresponding ring. For example, a smooth noncommutative variety of dimension d corresponds to an associative algebra of finite homological dimension d.

A different interpretation of noncommutative geometry, attributed to M. Kontsevich, has been advocated, in which the geometric object corresponding to a noncommutative algebra is not a triangulated category, but a genuine space, namely the space of representations of the algebra. Let us call this *noncommutative geometry of moduli*. Here, a noncommutative variety is thought to be a complete intersection, if the associated variety of representations is a complete intersection. A noncommutative variety is thought to be symplectic, if the corresponding variety of representations is symplectic.

There is by now an extensive body of writing, including work of W. Crawley-Boevey, M. Kontsevich, P. Etingof, V. Ginzburg, which explores the noncommutative symplectic geometry of moduli [9]. Our paper can be thought of as an excursion into triangulated noncommutative symplectic geometry (cf. [25]).

We wish to emphasise here that, from a naive perspective, the two approaches are genuinely different. For example, the cotangent space to the representations of a quiver can be formed by taking representations of the corresponding double quiver. So, in the geometry of moduli, the algebra associated to the cotangent space of the space associated to a quiver algebra is the path algebra of the corresponding double quiver. Performing Hamiltonian reduction on the representation variety, one obtains representations of the preprojective algebra, which if the quiver is not Dynkin, has homological dimension 2 [9].

However, both the path algebra of the quiver, and the path algebra of its double are hereditary, and therefore correspond to algebras of homological dimension 1. In triangulated noncommutative geometry, the cotangent space of the quiver algebra ought to give an algebra of homological dimension 2. Performing Hamiltonian reduction with respect to the action of a one dimensional group on a variety of dimension 2, in this setting, we obtain an algebra of homological dimension 0. That is not so interesting ! In triangulated noncommutative geometry, for non-Dynkin quivers, one rather thinks of the preprojective algebra as corresponding to the cotangent bundle of the path algebra of the quiver.

In the noncommutative geometry of moduli, the deformed preprojective algebra is the algebra obtained, after performing quantum Hamiltonian reduction on the cotangent space. In triangulated noncommutative geometry, the deformed preprojective algebra is the algebra obtained by performing deformation quantization on the cotangent space, and then specialising the deformation parameters. This is illustrated in example 79.

Let us note that a third interpretation of noncomutative geometry is thrust forward by the ideas of this paper, which we might call *discrete noncommutative geometry*. Here the geometry is a classical manifold. One forms a discrete model of that space, which is in our case a tiling by parallelohedra. One then associates a noncommutative algebra to that discrete model, and studies that noncommutative algebra (cf. chapter 12).

12. Riemannian manifolds

We now associate Cubist algebras, by quiver and relations, which makes sense for tilings of Riemannian manifolds. For an overview of Riemannian geometry, we refer to the book of Berger [4].

Definition 81. *Let M be a Riemannian manifold of dimension w. A* rhombohedron *in M is a homeomorphism ϕ from a $[0,1]^w$ to a subset of M, such that geodesics in $[0,1]^w$ map to geodesics in M, and the geodesics $c_1 \times ... \times c_{i-1} \times [0,1] \times c_{i+1} \times ... \times c_w$ map to geodesics of length 1, for $1 \leq i \leq w$, and $c_j \in \{0,1\}$.*

Definition 82. *A* Cubist manifold *T is a Riemannian manifold M, a cell complex $\mathcal{C}$, and a homeomorphism ϕ from $\mathcal{C}$ to M, such that*

(i) The image under ϕ of every i-cell in $\mathcal{C}$ is a rhombohedron of dimension i.

(ii) If two cells C_1, C_2 of $\mathcal{C}$ have a non-empty intersection, then the image $\phi(C_1) \cap \phi(C_2)$ of that intersection, is a face of the rhombohedron $\phi(C_i)$, for $i = 1, 2$.

Example 83 We can tile the hyperbolic plane with rhombi, by taking a tiling by equilateral triangles, and then deleting some edges. We thus obtain a hyperbolic Cubist manifold.

Projecting the surface of a cube in $\mathbb{R}^3$ centred at the origin onto a 2-sphere centred at the origin, we obtain a rhombic tiling of the 2-sphere. A more complicated rhombic tiling of the 2-sphere is obtained by projecting a triacontahedron onto it.

Let $T = (M, \mathcal{C}, \phi)$ be a Cubist manifold.

Let $\bar{T}$ denote the quiver, whose vertices are in one-one correspondence with vertices of T, and whose arrows are in two-one correspondence with vertices of T, with each edge of T corresponding to two opposing arrows, directed between the two vertices lying at the ends of the edge.

If x is a vertex of T, and e an edge of T with x as an endpoint, let $a_{x,e}$ denote the arrow in $\bar{T}$ along e away from x, and let $b_{x,e}$ denote the arrow along e into x. Let $\xi_{e,x}$ denote the vector of length one in the tangent space T_xM, which points in direction e.

If a is an arrow in $\bar{T}$, let $s(a)$ denote the source of an arrow a, and $t(a)$ the tail of a.

Let $\mathcal{X}$ denote the set of vertices of T, and E the set of edges of T. Let E_x denote the set of edges with vertex $x \in \mathcal{X}$. Let G_x denote the holonomy group at x, the subgroup of the linear isometry group of the tangent space to M at x, generated by the operators obtained by performing parallel transport around loops based at x.

Definition 84. *The* Cubist algebra V_T *associated to* T *is the* $\mathbb{R}$*-algebra given by generators:*

(G1) g*, for* $g \in G_x$*, and* $x \in \mathcal{X}$*;*

(G2) arrows a *in the quiver* $\bar{T}$*;*

modulo Group relations*:*

(R1) $h.g - (hg) = 0$*, for* $h, g \in G_x$*, and* $x \in \mathcal{X}$*;*

(R2) $h.g = 0$*, for* $h \in G_x, g \in G_y$*, and* $x \neq y \in \mathcal{X}$*;*

Quiver relations*:*

(R3) $1_{G_{s(\alpha)}}.\alpha - \alpha = 0, \quad \alpha.1_{G_{t(\alpha)}} - \alpha = 0$*, for all arrows* α *in* $\bar{T}$*;*

(R4) $g.\alpha = 0$*, for all arrows* α *in* $\bar{T}$*, and* $g \in G_x$*, such that* $x \neq s(\alpha) \in \mathcal{X}$*;*

(R4)' $\alpha.g = 0$*, for all arrows* α *in* $\bar{T}$*, and* $g \in G_x$*, such that* $x \neq t(\alpha) \in \mathcal{X}$*;*

(R5) $h.\alpha - \alpha.g = 0$*, for all arrows* α *in* $\bar{T}$*, and all operators* $h \in G_{s(\alpha)}$*,* $g \in G_{t(\alpha)}$ *such that the linear map from* $T_{s(\alpha)}M$ *to* $T_{t(\alpha)}M$ *given by multiplication by* h *followed by parallel transport along* α*, is equal to the linear map given by parallel transport along* α *followed by multiplication by* g*;*

Quadrilateral relations*:*

(R6) $p_1 - g.p_2 = 0$*, if* p_1 *and* p_2 *are two paths of length two between opposing vertices of a quadrilateral face in* T*, and* g *is the element of* G *obtained by performing parallel transport along the loop* $p_1 p_2^{-1}$ *in* M*;*

and Vertex relations*:*

(R7) $\sum_{e \in E_x} \lambda_e a_{e,x} b_{e,x} = 0$*, for all* $(\lambda_e) \in \mathbb{R}^{E_x}$*, such that* $\sum_{e \in E_x} \lambda_e \xi_{e,x} = 0$ *in* $T_x M$*,* $x \in \mathcal{X}$*.*

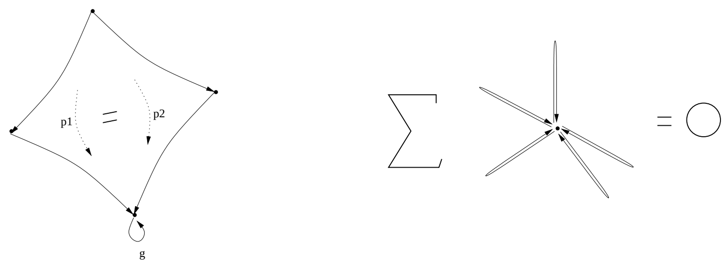

FIGURE 21. Quadrilateral and Vertex relations.

Example 85 Suppose $\mathcal{C}$ is a Cubist set, and T' is the corresponding parallelohedral tiling of Euclidean space. Rescaling the 1-cells to have length 1, we obtain a rhombohedral tiling T of Euclidean space H. The holonomy group of H is trivial, and the algebra V_T is the path algebra of $\bar{T}$, modulo relations

- $p_1 - p_2 = 0$, if p_1 and p_2 are two paths of length two between opposing vertices of a quadrilateral in T.
- $\sum_{e\in E_v} \lambda_e a_{e,v} b_{e,v} = 0$, for all $(\lambda_e) \in k^{E_v}$, such that $\sum_{e\in E_v} \lambda_e x_{e,v} = 0$ in $T_v M$.

We thus have a representation of the quadratic algebra V_T, given by the path algebra of $\bar{T}$, modulo these relations. We have isomorphisms of Cubist algebras $V_{\mathcal{C}} \cong V_{T'} \cong V_T$ (cf. Remark 50).

Remark 86 We can define Cubist manifolds with boundaries as well. If $T_1, T_2 \subset T$ are Cubist submanifolds with boundary, then the space

$$V^T_{T_1,T_2} = \bigoplus_{x_1 \in T_1 \cap \mathcal{X}, x_2 \in T_2 \mathcal{X}} 1_{G_{x_1}} V_T 1_{G_{x_2}}$$

is naturally a V_{T_1}-V_{T_2}-bimodule. We thus have naturally defined functors between the module categories of V_{T_1} and V_{T_2}.

Remark 87 Working with tilings of Euclidean space which admit a $\mathbb{Z}^w$-action, one can compactify. Indeed, passing to the completed algebra which contains elements $\sum_{x,y\in\mathcal{X}} v_{xy}$ of bounded degree, and taking fixed points of the $\mathbb{Z}^w$-action, we obtain an algebra $V_{\mathcal{C}/\mathbb{Z}^w}$ corresponding to a parallelohedral tiling of a w-dimensional torus $\mathbb{R}^w/\mathbb{Z}^w$. Such an algebra has one obvious difference from $V_{\mathcal{C}}$: it has a larger centre. Whilst the natural action of $S(H_k/ker(p_k))$ on $V_{\mathcal{C}}$ generates the centre of this algebra, the algebra $V_{\mathcal{C}/\mathbb{Z}^w}$ often has extra central elements, which arise as paths in the quiver of $V_{\mathcal{C}/\mathbb{Z}^w}$ representing non-trivial elements of the fundamental group of the torus. Is $V_{\mathcal{C}/\mathbb{Z}^w}$ finite over its centre, in general ?

Another homological difference caused by working on a torus is that the collections of modules of the algebras $V_{\mathcal{C}/\mathbb{Z}^w}$ no longer form highest weight categories.

It would be interesting to obtain information about representations of Cubist algebras corresponding to tilings of curved space.

Remark 88 Let M be a Riemannian manifold. We can think of the collection of categories

$$Cub_M = \{D^b(V_T), \quad T \text{ a Cubist tiling of } M\}$$

as a 2-category, whose arrows are exact functors, and whose 2-arrows are natural transformations between exact functors. The 2-category Cub_M is an invariant of M.

13. Blocks of symmetric groups

Let k be a field of characteristic $l > 0$. Describing the representations of a finite group over such a field is notoriously difficult, even for elementary groups, such as symmetric groups [6]. We conjecture that the Cubist algebras are of relevance here, as we have already proven in case $w = 2$ [8]:

Conjecture 89. *Let b_l be a block of a symmetric group over k, of weight w. Then*

$$b_l\text{-mod} \sim \frac{1}{2^w w!} U_{\mathcal{C}}\text{-mod},$$

as $l \to \infty$, for some Cubist subset $\mathcal{C}$ of $\mathbb{R}^{\frac{w(w+1)}{2}}$ of dimension w.

This conjecture is a slogan, rather than a precise mathematical statement. Therefore, let us be more precise.

If A is a locally finite dimensional algebra and $R \subset Irr(A)$ a set of irreducible modules, let A^R denote the basic algebra $\sum_{r,s\in R} e_r A e_s$, where $e_r \in A$ is an idempotent, such that Ae_r is a projective cover of r.

Let b be a block of a symmetric group of weight w. We conjecture that $b = b_{l-1}$ is one of a sequence b_1, b_2, b_3,..., of blocks of Hecke algebras of symmetric groups defined over k, that there exists a Cubist complex $\mathcal{C}$ whose vertices $\mathcal{X}$ lie on the lattice $\mathbb{Z}^w$, that there exists a commutative diagram of embeddings

$$\begin{tikzcd}
Irr(b_1) \arrow[r, hook] & Irr(b_2) \arrow[r, hook] & Irr(b_3) \arrow[r, hook] & \cdots \\
R_1 \arrow[r, hook] \arrow[u, hook] \arrow[d, hook] & R_2 \arrow[r, hook] \arrow[u, hook] \arrow[d, hook] & R_3 \arrow[r, hook] \arrow[u, hook] \arrow[d, hook] & \cdots \\
\mathcal{X}_1 \arrow[r, hook] & \mathcal{X}_2 \arrow[r, hook] & \mathcal{X}_3 \arrow[r, hook] & \cdots
\end{tikzcd}$$

where $\mathcal{X}_n \subset \mathcal{X}$ is the intersection of $\mathcal{C}$ with the subset $[-n, n]^w$ of $\mathbb{Z}^w$, where

$$b_n^{R_n}\text{-mod} \cong U_{\mathcal{C}}^{R_n}\text{-mod},$$

for all n, and where

$$\frac{|Irr(b_n)|}{|R_n|} \to 1, \quad \frac{|\mathcal{X}_n / C_2 \wr S_w|}{|R_n|}$$

as n tends to ∞.

Here, the group $C_2 \wr S_w$ acts on $\mathbb{Z}^w$ in the natural way, as a linear reflection group. This action restricts to an action on $[-n, n]^w$.

The conjecture is true in case $w \leq 2$ [8].

Remark 90 There is more to be said about the combinatorics of this conjecture. We shall limit ourselves to defining the projection, relative to which we expect the Cubist set to be defined. Further details can be found in a paper of J. Chuang and K.M. Tan [7].

Let L denote the natural permutation representation $\mathbb{R}\Omega$ of Σ_w of dimension w. Let $\bigwedge^2(L)$ be the second exterior power of L, a vector space of dimension $\frac{w(w+1)}{2}$. Let $E = S^2(L)^*$. Let O denote the orbit sum $\sum_{\omega\in\Omega} \omega$ under the Σ_w-action, an element of L. Then L embeds linearly in $L \oplus \bigwedge^2(L)$ via multiplication by $1 \oplus O$. Let $E = (L \oplus \bigwedge^2(L))^*$, and $H = L^*$, and let p denote the corresponding projection of E onto H.

It is this projection p, with respect to which we expect the Cubist set $\mathcal{C}$ of conjecture 89 to be defined.

Let us record here the existence of deformations of $U_{\mathcal{C}}$. Indeed, by definition, $V_{\mathcal{C}}$ has a natural w-dimensional central quadratic subspace, namely H_k. We therefore have (by [24]):

Proposition 91. *The Cubist algebra $U_{\mathcal{C}}$ has a natural w-parameter deformation $\widehat{U_{\mathcal{C}}}$.*

Facetious modules can be defined over any specialisation of $\widehat{U_{\mathcal{C}}}$, via their construction as standard objects. At a generic value of deformation parameter, they should become simple, so that the algebra $U_{\mathcal{C}}$ becomes semisimple.

At particular specialisations, we expect to obtain algebras which model blocks of symmetric groups over the l-adic integers. Indeed, working over $\mathbb{Z}_l$, and specialising a suitable deformation $\mathbb{Z}_l[z] \otimes U_{\mathcal{X}}$ at $z = l$, we obtain algebras which we expect to model symmetric group blocks over $\mathbb{Z}_l$. This is analogous to the conjectural structure theorem for l-adic blocks of symmetric groups, up to derived equivalence given in our paper "Tilting equivalences etc." [26].

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