CATEGORIES OF CATEGORIES IN REPRESENTATION THEORY.

WILL TURNER

Abstract. We discuss the roles 2-categories play in representation theory, with reference to examples rooted in classical mathematics.

1. Intro

Suppose that G is a group, and k a commutative ring. The collection kG-mod of representations of G over k possesses a number of pleasing homological features. Firstly, kG-mod is an abelian category. Secondly, so long as k is a field, kG-mod is a Calabi-Yau category of dimension zero, meaning that $Hom(P,M) \cong Hom(M,P)^*$ naturally in representaions M, and projective representations P. Thirdly, kG-mod is a symmetric monoidal category. Here, the monoidal structure is provided by the tensor product operator, taking a pair of representations V, W to their tensor product $V \otimes_k W$.

Symmetric monoidal categories also naturally appear in geometry and topology. For example, for a natural number n, the cobordism category n-Cob, whose objects are $n-1$-dimensional closed topological manifolds and whose morphisms are n-dimensional cobordisms between $n-1$-dimensional closed manifolds, is a symmetric monoidal category. The monoidal structure is provided by disjoint union operator, taking a pair of closed manifolds M, N to their disjoint union $M \amalg N$.

An n-dimensional topological quantum field theory was defined by M. Atiyah to be a symmetric monoidal functor from n-Cob to the category of vector spaces [2]. It was observed that 2-dimensional topological quantum field theories could be identified with commutative symmetric algebras [23]. Subsequently, Turaev identified 3-dimensional topological quantum field theories with certain braided monoidal categories, known as modular categories [29]. Examples of modular categories are provided by certain representation categories of quantum groups.

So monoidal categories are of some importance in algebra and topology. In this paper, we increase the categorical dimension by one, and look at monoidal 2-categories.

Why should we study these ? A first answer is: because we can. According to category theoretic dogma, whenever we see a mathematical equality, we should look to replace it with an isomorphism. Such a philosophy suggests that we work with categories of higher valency if possible, since they carry more information. Rather than working with sets and functions we should look to consider categories and functors, since between functors we have natural transformations which carry extra information. Rather than working with categories, functors, and natural transformations, we should look to consider 2-categories, 2-functors, and 2-natural transformations, since between 2-natural transformations we have further transformations carrying yet more information. And so on. The world we enter is known as higher category theory, or homotopical algebra [24], [10]. Upon entering the world of homotopical algebra, previously simple notions become complicated or apparently absurd: 2 times 2 does not necessarily make 4. Our head begins to spin as we climb the categorical ladder. But surely we can at least step up a couple of rungs, without becoming too dizzy.

A discourse has taken place between M. Kapranov & V. Voevodsky, B. Day & R. Street, J. Baez & Neuchl, and S. Crans , over the definition of a braided monoidal 2-category [19], [14], [6], [13]. Crans' paper also contains definitions of sylleptic monoidal 2-categories, and symmetric monoidal 2-categories.

Baez and Langford have proved that certain 2-tangle 2-categories admit braided monoidal structures in the sense of Crans, and possess duals in a sense which they define [5]. This is promising, because some of the work on 2-categories has a topological motivation: to construct four dimensional topological invariants, analogous to the three dimensional invariants of Turaev & Reshetikhin, and Turaev & Viro [29]. The 2-tangle 2-categories have a four dimensional flavour, just as tangle monoidal categories have a three dimensional flavour.

There are intriguing connections between this project of overturning four dimensional invariants and classical representation theory, suggested by the *categorification* program of I. Frenkel [12], [17]. To categorify a mathematical structure is to interpret of it in the world of homotopical algebra. In the Frenkel program, certain classical categories (such as categories of infinite dimensional representations for semisimple Lie algebras, or categories of representations of affine Hecke algebras, or categories of sheaves on algebraic varieties) are acted on by certain classical functors. Upon taking their complexified Grothendieck groups, the categories become vector spaces, and the functors become linear maps. The resulting vector space thus defines a representation of some algebra, which happens to be the universal enveloping algebra of a semisimple Lie algebra, or a quantization of such an enveloping algebra.

A second approach to categorification has been advocated by J. Baez [3]. Here, the relation to classical mathematics is quite different. To obtain strict categorifications, one considers categories internal to a classical category. Here, an internal category is an object X_0 (resembling the collection of objects in a category), and an object X_1 (resembling the morphisms in a category), along with various structures on the pair (X_0, X_1), and conditions on those structures (resembling the axioms for a category). In this setting, a strict 2-vector space is defined to be a category internal to the category of vector spaces; a strict 2-group is defined to be a category internal to the category of groups; a strict 2-Lie algebra is defined to be a category internal to the category of Lie algebras. And so on.

The internalisation approach has the appeal of an elegant principle, which can be applied generally as a guide to incorporating homotopical ideas into classical mathematics. On the other hand, categorifications arising around the Frenkel program, such as Khovanov's categorification of the Jones polynomial [20], or Chuang and Rouquier's categorification of the representation theory of $\mathfrak{sl}_2$ [11], can appear somewhat miraculous, and have a streak of mystery running through them.

An important difference between the various approaches to categorification is their homological complexity. Categories appearing in the Frenkel program are usually far from semisimple, and thus carry homological intricacies as well as homotopical ones. In other approaches, subtle homological features are often suppressed. For example, a 2-vector space in the sense of Kapranov and Voevodsky is a semisimple category, and therefore homologically bland. A 2-vector space in the sense of Baez and A. Crans is merely a 2-term chain complex; here, homological information is being encoded in a homotopical manner.

In this paper we explore the 2-categorical landscape, making excursions from classical mathematics. Let us give a brief summary of the material which is included, with numerical references to the sections in which that material can be found: we introduce a variety of monoidal 2-categories whose origins lie in algebra and topology; we give examples of monoidal 2-categories associated to collections of algebras (4), to collections of groups (4), to collections of categories (5), to topological quantum field theories (6), and to rings (8); in this article we only describe data for these 2-categories; in a separate article, we will address coherence laws and thus provide proofs of a number of theorems which are only stated here; we discuss the representation theory of 2-categories (7); we describe a homotopical setting for algebraic structures on the set of natural numbers (8,9); we discuss motivation for studying 2-categories lying in topological quantum field theory (6), and modular representation theory (10).

Although its concerns are superficially quite different, this paper arose from my attempts to understand the paper of Joe Chuang and Raphaël Rouquier on categorification of $\mathfrak{sl}_2$ [11]. I am grateful to both Joe and Raphaël, for discussion and insight.

2. Data for Monoidal 2-categories.

Here we record the basic data required for the definition of various types of 2-categories. The definitions are due to Kapranov and Voevodsky, Day and Street, Baez and Neuchl, and Crans. We do not write down the all relevant coherence laws here. Instead, we give references to where the coherence laws can be found in the literature.

Here is the data required for the definition of a strict 2-category $\mathcal{T}$:

- A collection of objects $O \in \mathcal{T}$;
- For every pair of objects $O, O' \in \mathcal{T}$, a category $\mathcal{T}(O, O')$.

• For every triple of objects $O, O', O'' \in \mathcal{T}$, a functor

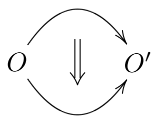
$$\mathcal{T}(O,O') \times \mathcal{T}(O',O'') \to \mathcal{T}(O,O'').$$

• For every object O an arrow i_O, called the *identity arrow*.

The objects in $\mathcal{T}(O,O')$ are the arrows in $\mathcal{T}$:

$$O \longrightarrow O'$$

The morphisms in $\mathcal{T}(O,O')$ are the 2-arrows in $\mathcal{T}$:



A semistrict monoidal 2-category is a strict 2-category with extra structure. Here is the additional data for the definition of a semistrict monoidal 2-category $\mathcal{T}$:

• An object $I \in \mathcal{T}$.

• For any two objects $O, O' \in \mathcal{T}$ an object $O \diamond O' \in \mathcal{T}$.

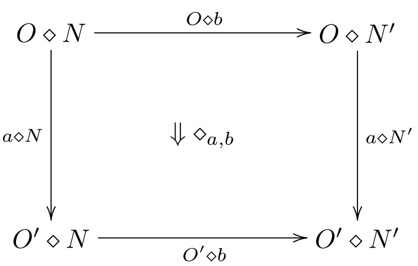
• For any arrow $a : O \to O'$ and any object $N \in \mathcal{T}$ an arrow $a \diamond N : O \diamond N \to O' \diamond N$.

• For any object $O \in \mathcal{T}$ and any arrow $b : N \to N'$ an arrow $O \diamond b : O \diamond N \to O \diamond N'$.

• For any 2-arrow $\alpha : a \Rightarrow a'$ and any object $N \in \mathcal{T}$ a 2-arrow $\alpha \diamond N : a \diamond N \Rightarrow a' \diamond N$.

• For any object $O \in \mathcal{T}$ and any 2-arrow $\beta : b \to b'$ a 2-arrow $O \diamond \beta : O \diamond b \Rightarrow O \diamond b'$.

• For any two arrows $a : O \to O'$ and $b : N \to N'$ a 2-isomorphism $\diamond_{a,b}$, called the *tensorator*, where



We define $a \diamond b$ to be the composite arrow $(a \diamond N)(O' \diamond b)$, from $O \diamond N$ to $O' \diamond N'$.

A braided monoidal 2-category is a monoidal 2-category with extra structure. Here is the additional data for the definition of a braided monoidal 2-category:

• For any two objects $O, N \in \mathcal{T}$ an arrow $\sigma_{O,N} : O \diamond N \to N \diamond O$, called the *braiding*.

• For any two arrows $a : O \to O'$ and $b : N \to N'$, a 2-isomorphism $\sigma_{a,b}$, where

$$\begin{array}{ccc} O \diamond N & \xrightarrow{a\diamond b} & O' \diamond N' \\ {\scriptstyle \sigma_{O,N}}\downarrow & \Downarrow \sigma_{a,b} & \downarrow{\scriptstyle \sigma_{O',N'}} \\ N \diamond O & \xrightarrow[b\diamond a]{} & N' \diamond O' \end{array}$$

• For any three objects $O, N, M \in \mathcal{T}$, a 2-isomorphism $\sigma_{O|M,N}$, where

$$\begin{array}{ccc} O \diamond N \diamond M & \xrightarrow{\sigma_{O,N\diamond M}} & N \diamond M \diamond O \\ {\scriptstyle \sigma_{O,N}\diamond M}\searrow & \Downarrow \sigma_{O|M,N} & \nearrow{\scriptstyle N\diamond\sigma_{O,M}} \\ & N \diamond O \diamond M & \end{array}$$

• For any three objects $O, N, M \in \mathcal{T}$, a 2-isomorphism $\sigma_{O,N|M}$, where

$$\begin{array}{ccc} O \diamond N \diamond M & \xrightarrow{\sigma_{O\diamond N,M}} & M \diamond O \diamond N \\ {\scriptstyle O\diamond\sigma_{N,M}}\searrow & \Downarrow \sigma_{O,N|M} & \nearrow{\scriptstyle \sigma_{O,M}\diamond N} \\ & N \diamond O \diamond M & \end{array}$$

A sylleptic monoidal 2-category is a braided monoidal 2-category with extra structure. Here is the additional data for the definition of a semistrict sylleptic monoidal 2-category $\mathcal{T}$:

• For any objects $O, N \in \mathcal{T}$, a 2-isomorphism $v_{O,N}$, called the *syllepsis*, where

$$\begin{array}{ccc} O \diamond N & = & O \diamond N \\ {\scriptstyle \sigma_{O,N}}\searrow & \Downarrow v_{O,N} & \nearrow{\scriptstyle \sigma_{N,O}} \\ & N \diamond O & \end{array}$$

A semistrict symmetric monoidal 2-category is a sylleptic monoidal category satisfying additional coherence laws. No extra data is required.

A monoidal 2-category with duals is a monoidal 2-category with extra structure. Here is the additional data for a monoidal 2-category with duals:

• For every 2-arrow $\alpha : a \Rightarrow b$ there is a 2-arrow $\alpha^* : b \Rightarrow a$ called the *dual* of α.

• For every arrow $a : O \to N$ there is an arrow $a^* : N \Rightarrow O$ called the *dual* of a, a 2-arrow $i_a : 1_O \Rightarrow aa^*$ called the *unit* of a, and a 2-arrow $e_a : a^*a \Rightarrow 1_N$ called the *counit* of a.

• For every object $O \in \mathcal{T}$ there is an object O^* called the *dual* of O, an arrow $i_O : I \to O \diamond O^*$ called the *unit* of O, an arrow $e_O : O^*O \to I$ called the *counit* of O, and a 2-arrow $T_O : (i_O \diamond O)(O \diamond e_O) \Rightarrow 1_O$ called the *triangulator* of O.

3. Coherence laws

The coherence laws for our higher categorical structures will be considered in detail in the sequel to this paper. For now, we satisfy ourselves with a few mutterings and references. The coherence laws for a strict 2-category are classical. They are as follows:

∘ If $a : O \to O'$ is an arrow in $\mathcal{T}$, then $a = i_O.a = a.i_{O'}$.

∘ If a, a', a'' are arrows in $\mathcal{T}$, then $(a.a').a'' = a.(a'.a'')$

The coherence laws for a monoidal 2-category are due to Kapranov and Voevodsky [19]. Coherence laws for braided, sylleptic, and symmetric monoidal 2-categories can be found in Crans' paper [13].

Braided monoidal 2-categories with duals are braided monoidal 2-categories, which are also monoidal categories with duals, satisfying additional coherence laws, due to Baez and Langford [5]. Sylleptic monoidal 2-categories with duals are sylleptic monoidal 2-categories which are also braided monoidal categories with duals, satisfying an additional coherence law. Symmetric monoidal 2-categories with duals are symetric monoidal 2-categories which are also sylleptic monoidal categories with duals.

4. Algebras and groups.

Let k be a commutative ring. There is a well-known example of a symmetric monoidal 2-category, whose objects are associative algebras over k. We call this $\mathcal{T}_{alg}$. Here we go through the data which defines $\mathcal{T}_{alg}$ in detail.

Here is data for the 2-category $\mathcal{T}_{alg}$:

• Objects of $\mathcal{T}_{alg}$ are k-algebras.

• Given two algebras A, A', the category $\mathcal{T}_{alg}(A, A')$ is the category of A-A'-bimodules. Up to equivalence, this category is independent of the isomorphism type of A and A'.

• Given three algebras A, A', A'', the functor

$$\mathcal{T}_{alg}(A, A') \times \mathcal{T}_{alg}(A', A'') \to \mathcal{T}_{alg}(A, A'')$$

takes a pair of bimodules (M, M') to the bimodule $M \otimes_{A'} M'$, and pair of bimodule homomorphisms (f, f') to the bimodule homomorphism $f \otimes f'$. Up to natural isomorphism, this functor is independent of the isomorphism type of A, A' and A''.

Remark 1 Since we work with usual tensor product of bimodules, the 2-category defined above is a weak 2-category. We define a strict 2-category $\mathcal{T}_{alg}$ by taking the strictification of the weak 2-category described above. This is an annoying technicality, but we are bound by the fact that the definitions of braided monoidal 2-categories in the literature are semistrict, rather than weak.

Here is the extra data for the monoidal structure on $\mathcal{T}_{alg}$:

• The object I is given by the ring k, thought of as a k-algebra.

• For any two algebras A, A', the object $A \diamond A'$ is defined to be the tensor product $A \otimes_k A'$.

• For an A-A'-bimodule M and an algebra B, we define $M \diamond B = M \otimes_k B$, an $A \otimes_k B$-$A' \otimes_k B$-bimodule.

• For an algebra A and a B-B'-bimodule N, we define $A \diamond N = A \otimes_k N$, an $A \otimes_k B$-$A \otimes_k B'$-bimodule.

• For a bimodule homomorphism f, and an algebra B, we define $f \diamond B$ to be the bimodule homomorphism $f \otimes_k 1_B$.

• For an algebra A and a bimodule homomorphism g, we define $A \diamond g$ to be the bimodule homomorphism $1_A \otimes_k g$.

• For bimodules ${}_AM_{A'}$ and ${}_BN_{B'}$, we define the tensorator to be the bimodule isomorphism which is the composition of the natural isomorphisms

$$(A \otimes_k N) \otimes_{A \otimes_k B'} (M \otimes_k B') \to (A \otimes_A M) \otimes_k (N \otimes_{B'} B') \to (M \otimes_k N) \to$$

$$(M \otimes_{A'} A') \otimes_k (B \otimes_B N) \to (M \otimes_k B) \otimes_{A' \otimes_k B} (A' \otimes_k N).$$

Remark 2 Again, to obtain a semistrict monoidal structure on $\mathcal{T}_{alg}$, we strictify the product $\diamond$.

To avoid unnecessary complication in the rest of the paper, we will omit qualifications about strictification. Throughout, for all the 2-categories considered, we take it to be understood we have strictified the product on our arrows, and our product $\diamond$.

Here is the extra data for a braiding on $\mathcal{T}_{alg}$:

• Given two algebras A and B, the braiding is defined to be the bimodule $A \otimes_k B^{\sigma^{A,B}}$, on which $A \otimes B$ acts freely on the left, and on which $B \otimes_k A$ acts freely on the right via the isomorphism $\sigma^{A,B} : A \otimes_k B \cong B \otimes_k A$.

• Given two bimodules M and N, the braiding is defined to be the isomorphism $M \otimes_k N \cong N \otimes_k M$ which takes $m \otimes n$ to $n \otimes m$.

• Given three algebras A, B, and C, we have an equality between the isomorphism $\sigma^{A,B \otimes_k C}$ and $(\sigma^{A,B} \otimes_k 1_C)(1_B \otimes_k \sigma^{A,C})$. This lifts to an isomorphism between the bimodule $(A \otimes_k B \otimes_k C)^{\sigma^{A,B \otimes_k C}}$ and the bimodule

$$(A \otimes_k B \otimes_k C)^{\sigma^{A,B} \otimes 1_C} \otimes_{(B \otimes_k A \otimes_k C)} (B \otimes_k A \otimes_k C)^{1_B \otimes_k \sigma^{A,C}}.$$

• Given three algebras A, B, and C, we have an equality between the isomorphism $\sigma^{A\otimes_k B,C}$ and $(1_A \otimes_k \sigma^{B,C})(\sigma^{A,C} \otimes_k 1_B)$. This lifts to an isomorphism between the bimodule $(A \otimes_k B \otimes_k C)^{\sigma^{A\otimes_k B,C}}$ and the bimodule

$$(A \otimes_k B \otimes_k C)^{1_A \otimes_k \sigma^{B,C}} \otimes_{(A\otimes_k C \otimes_k B)} (A \otimes_k C \otimes_k B)^{\sigma^{A,C} \otimes_k 1_B}.$$

Here is the extra data for a sylleptic structure on $\mathcal{T}_{alg}$:

• Given two algebras A and B, the syllepsis $v_{A,B}$ is defined to be the bimodule isomorphism, given by the composition of natural isomorphisms

$$A \otimes B \to (A \otimes B) \otimes_{A\otimes B} (A \otimes B) \to (A \otimes B)^{\sigma^{A,B}} \otimes_{B\otimes A} (B \otimes A)^{\sigma^{B,A}}.$$

Theorem 3. *$\mathcal{T}_{alg}$ is a symmetric monoidal 2-category.*

Let us now assume k to be a field. The monoidal category $\mathcal{T}_{alg}$ does not have duals. The problem is that adjoints to $M \otimes_B -$ are not necessarily given by tensoring with a bimodule. What is worse, although a right adjoint to $M \otimes_B -$ does exist, namely $Hom_B(M, -)$, the right adjoint of this functor $Hom_B(M, -)$ may not exist, and even if it does, it may not be naturally isomorphic to $M \otimes_B -$.

There are situations in which it is possible to resolve these difficulties, in which we assume a Calabi-Yau property for the algebras involved. For example, suppose $\mathcal{G}$ denotes a collection of finite groups, which is closed under taking direct products. Then for suitable choice of bimodules M over group algebras, adjoints to $M \otimes_{kG} -$ *are* given by tensoring with a bimodule. Pursuing this, we can make an analogous construction, which does define a neat example of a symmetric monoidal 2-category with duals. We call this symmetric monoidal 2-category $\mathcal{T}_\mathcal{G}$.

Here is the data for the 2-category $\mathcal{T}_\mathcal{G}$:

• The objects of $\mathcal{T}_\mathcal{G}$ are groups in $\mathcal{G}$.

• Given two groups $G, G' \in \mathcal{G}$, the category $\mathcal{T}_\mathcal{G}(G, G')$ is defined to be the category of exact finite dimensional kG-kG'-bimodules with a nondegenerate $G \times G'^{op}$-invariant bilinear form. Objects in this category are kG-kG'-bimodules M, projective as a left kG-module and as a right kG'-module, with a nondegenerate bilinear form $\langle , \rangle$, such that

$$\langle gmg', n \rangle = \langle m, g^{-1} n g'^{-1} \rangle,$$

for $g \in G$, $g' \in G'$, and $m, n \in M$. Morphisms in this category are bimodule homomorphisms.

• Given three groups $G, G', G'' \in \mathcal{G}$, the functor

$$\mathcal{T}_\mathcal{G}(G, G') \times \mathcal{T}_\mathcal{G}(G', G'') \to \mathcal{T}_\mathcal{G}(G, G'')$$

takes a pair of bimodules (M, M') to the bimodule $M \otimes_{kG'} M'$, and a pair of bimodule homomorphisms (f, f') to the bimodule homomorphism $f \otimes f'$.

• The bimodule ${}_{kG}kG_{kG}$ defines the identity arrow from G to G. The bilinear form takes an element $g \otimes g'$ of $kG \otimes_k kG$ to 1 if gg' is the identity, and to zero otherwise, for $g, g' \in G$.

Remark 4 Having a nondegenerate $G \times G'^{op}$-invariant bilinear form on a finite dimensional kG-kG'-bimodule M is equivalent to having a bimodule isomorphism between M and $Hom(M,k)$. Indeed, if $\langle -,- \rangle$ is such a bilinear form, then the corresponding bimodule isomorphism is given by $m \mapsto \langle m,- \rangle$. Conversely, if $\phi : M \to Hom(M,k)$ is a bimodule isomorphism, then $\langle x,y \rangle = \phi(x)(y)$ defines an invariant nondegenerate bilinear form.

Let us check that the composition of two arrows in $\mathcal{T}_{\mathcal{G}}$ is an arrow in $\mathcal{T}_{\mathcal{G}}$. Indeed, suppose ${}_{kG}M_{kG'}$ and ${}_{kG'}M'_{kG''}$ are two bimodules in $\mathcal{T}_{\mathcal{G}}$. We have bimodule isomorphisms

$$M \cong Hom_k(M,k)^{op}, \quad M' \cong Hom_k(M',k)^{op}.$$

We thus have canonical isomorphisms

$$Hom(M \otimes_{kG'} M', k) \cong Hom_{kG'}(M, Hom(M',k)) \cong Hom_{kG'}(M, M'^{op}).$$

Since M is exact, and group algebras are symmetric algebras, $Hom_{kG'}(M,-)$ is isomorphic to the functor $Hom_k(M,k) \otimes_{kG'} -$ (see [26], 2.2.4), and we have canonical isomorphisms

$$Hom_{kG'}(M, M'^{op}) \cong Hom(M,k) \otimes_{kG'} M'^{op} \cong M^{op} \otimes_{kG'} M'^{op}.$$

Thus, we have a bimodule isomorphism between $M \otimes_{kG'} M'$ and the opposite of its dual, so $M \otimes_{kG'} M'$ can be thought of as an arrow in $\mathcal{T}_{\mathcal{G}}$.

The extra data on $\mathcal{T}_{\mathcal{G}}$ which gives it the structure of a symmetric monoidal 2-category is defined just like the analogous structure on $\mathcal{T}_{alg}$. The product $G \diamond H$ of two groups is defined to be the direct product $G \times H$ in $\mathcal{G}$. Since the group algebra of $G \times H$ is canonically isomorphic to $kG \otimes_k kH$, all previous definitions go through as for $\mathcal{T}_{alg}$.

Here is the data to give $\mathcal{T}_{\mathcal{G}}$ the structure of a symmetric monoidal 2-category with duals:

• If we have a 2-arrow which is given by a bimodule homomorphism $f : M \to N$, we define its dual f^* to be the composition bimodule homomorphism

$$N \longrightarrow Hom(N,k) \xrightarrow{Hom(f,k)} Hom(M,k) \longrightarrow M$$

• Given an arrow which corresponds to a bimodule ${}_{kG}M_{kG'}$, we define its dual to be $M^* = M^{op}$. We have a natural map from kG to $Hom_{kG'}(M^*,M^*)$. We have a canonical isomorphism from $Hom_{kG'}(M^*,M^*)$ to $M^{**} \otimes_{kG'} M^*$. Since $M^{**} \cong M$, composing these maps gives us a map from kG to $M \otimes_{kG'} M^*$ which we define to be the unit of M. We define the counit dually.

• Let G be a group. We define the dual of G to be G^{op}, which is isomorphic to G via the map which takes an element to its inverse. We have a group homomorphism $\Delta^l : G \to G \times G^{op}$, which takes a group element g to the pair (g, g^{-1}). The module $k\Delta^l(G)\backslash G \times G^{op}$ is a k-$kG \times G^{op}$ bimodule, which is the unit of G. Since the unit is a permutation module, it is naturally self-dual, and therefore admits an invariant nondegenerate bilinear form.

We have a group homomorphism $\Delta^r : G \to G^{op} \times G$, which takes a group element g to the pair (g^{-1}, g). The permutation module $kG^{op} \times G/\Delta^r(G)$ is a $kG^{op} \times kG$-k-bimodule, which is the counit of G. Since the counit is a permutation module, it is naturally self-dual, and therefore admits an invariant nondegenerate bilinear form.

Consider $\Delta^l(G)\backslash G \times G^{op} \times G/\Delta^r(G)$. A set of representatives for this double coset space is the collection of elements $\{(g,1,1), g \in G\}$. Furthermore, the elements $(g,1,1)$, $(1,g,1)$ and $(1,g,g)$ all represent the same double coset space. The map

$$kG \to k\Delta^l(G)\backslash G \times G^{op} \times G/\Delta^r(G)$$

which takes g to $(g,1,1)$ is thus a bimodule isomorphism. The composition of this map with the canonical isomorphism between $k\Delta^l(G)\backslash G \times G^{op} \times G/\Delta^r(G)$ and

$$((k\Delta^l(G)\backslash G \times G^{op}) \otimes_k kG) \otimes_{kG \times G^{op} \times G} (kG \otimes_k (kG^{op} \times G/\Delta^r(G)))$$

is a bimodule isomorphism, which we define to be the triangulator of G.

Theorem 5. *$\mathcal{T}_G$ is a symmetric monoidal 2-category with duals.*

5. Dual numbers.

The dual numbers $\Lambda = \mathbb{C}[d]/d^2$ form a super Hopf algebra, whose coproduct sends d to $d \otimes 1 + 1 \otimes d$. Their category of representations Λ-mod is a symmetric monoidal category. In this section, we will use topological arguments to define a natural lift of Λ to a symmetric monoidal 2-category $\mathcal{T}_\Lambda$. This will be a model case for a more subtle construction given in the following section of the paper.

The dual numbers can be thought of as the universal enveloping algebra of the one dimensional fermionic Lie algebra $\mathbb{C}^{0|1}$. We wish to describe a categorification of the dual numbers, à la Frenkel.

One reason for trying this is that, as such a small example, it is quite elementary. Furthermore, there are many modules over Λ with topological realisations, as the simplicial or singular chain complex associated to an oriented simplicial complex, or topological space. Thinking of it this way, we should try to lift the symmetric monoidal category Λ-mod to a symmetric monoidal 2-category $\underline{\Lambda}$-cat, and define a 2-functor $\underline{C}$ from the 2-category Cat of all categories to $\underline{\Lambda}$-cat, such that the following diagram commutes

$$\begin{array}{ccc} Cat & \xrightarrow{\underline{C}} & \underline{\Lambda}\text{-cat} \\ \downarrow{\scriptstyle \mathcal{N}} & & \downarrow{\scriptstyle K} \\ Simp^+ & \xrightarrow{C} & \Lambda\text{-mod} \end{array}$$

Here, $\mathcal{N}$ is the nerve construction, which associates an oriented simplicial complex to a category, whilst C is the augmented chain complex associated to a simplicial complex, defined over $\mathbb{C}$.

We thus have many geometric realisations of representations of the dual numbers, which we wish to lift to functorial representations in a uniform way. For example, the $n-1$-simplex Δ_{n-1} can be realised as the nerve of the free category on a linear quiver with n vertices. Its augmented chain complex is the n-fold super-tensor product of the regular Λ-representation.

In this section we demonstrate a natural way to achieve this lift. A drawback is that the categories appearing in our construction are homologically trivial. This is a shame: in previous approaches to categorifying Lie algebra representations, it has been found that nontrivial categories appearing in Lie theory provide a further reaching theory. In the following section we show how our construction can be de-trivialised, by incorporating the data of a topological quantum field theory.

We could work with derived categories, and exact functors between these, as conventional wisdom has it that the natural lift of a representation is usually a triangulated category. However in this instance, no triangulated categories throw themselves forward. Instead, we work with the category $\mathbb{Z}_{\leq}$. The objects of $\mathbb{Z}_{\leq}$ are integers, whilst there is a single morphism from z to z' whenever $z \leq z'$. The category $\mathbb{Z}_{\leq}$ is an object very well suited for study of simplicial topology, and we therefore consider this to be the atomic object in our theory. We have a natural shift functor S on the $\mathbb{Z}_{\leq}$, given by addition of 1. There is a unique natural transformation $q : 1 \Rightarrow S$.

Before giving details, let us provide a sketch of our construction. Roughly speaking, we define

$$\underline{C}(\mathcal{C}) = Fun(\mathbb{Z}_{\leq}, \mathcal{C})$$

to be the collection of paths through the category $\mathcal{C}$, and identify the action of a monoidal category $\underline{\Lambda}$ on $\underline{C}(\mathcal{C})$, for every $\mathcal{C}$. We define

$$\underline{C}(\mathcal{C}) \diamond \underline{C}(\mathcal{D}) = \underline{C}(\mathcal{C} \star \mathcal{D}),$$

where $\mathcal{C} \star \mathcal{D}$ is the category whose collection of objects is the disjoint union of the collections of objects in $\mathcal{C}$ and $\mathcal{D}$, and whose collection of arrows is the disjoint union of the arrows in $\mathcal{C}$ and $\mathcal{D}$, along with the collection of pairs (c, d), where c is an object in $\mathcal{C}$, and d is an object in $\mathcal{D}$. We think of (c, d) as defining a morphism from c to d.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, we can thus define both $\underline{C}(\mathcal{C}) \diamond \underline{C}(\mathcal{D})$ and $\underline{C}(\mathcal{D}) \diamond \underline{C}(\mathcal{C})$. We have a natural correspondence between paths across $\mathcal{C} \star \mathcal{D}$ and paths across $\mathcal{D} \star \mathcal{C}$, which sends an arrow of the form (c, d) to (d, c), and preserves paths across $\mathcal{C}$ and $\mathcal{D}$. We thus recover an equivalence of $\underline{\Lambda}$-categories,

$$\underline{C}(\mathcal{C}) \diamond \underline{C}(\mathcal{D}) \cong \underline{C}(\mathcal{D}) \diamond \underline{C}(\mathcal{C}).$$

It is this product, and these equivalences, which define a braiding on the collection of $\underline{\Lambda}$-categories.

Let us be more precise. Let Q_n denote the linearly ordered quiver with n vertices. Any full subcategory of $\mathbb{Z}_{\leq}$ with n objects is equivalent to the free category $\langle Q_n \rangle$ on Q_n. Consider the collection of functorial embeddings of $\langle Q_i \rangle$ in $\langle Q_n \rangle$, for $i \leq n$. We depict some of these embeddings in Figure 1.

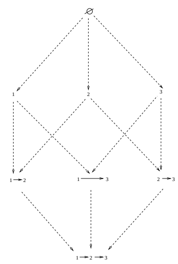


Figure 1. Subcategories of $\langle Q_3 \rangle$.

There is an obvious similarity between this diagram and the augmented chain complex of an oriented $n-1$-simplex Δ_{n-1}, as in Figure 2.

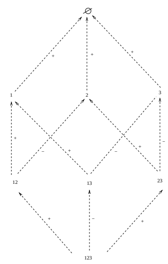


Figure 2. The augmented chain complex $C(\Delta_2)$ associated to a triangle.

It is this similarity which underlies the construction of the nerve $\mathcal{N}$ of a category. The associated chain complex $C(\Delta_{n-1})$ has a canonical basis given by faces of the simplex; the differential d takes a face to a signed sum of the faces on its boundary. We have $d^2 = 0$.

Let us interpret this chain complex in a functorial way. Let k be a field. We think of each functorial embedding of $\langle Q_i \rangle$ as corresponding to a copy of the abelian category $\mathbb{Z}_{\leq}$-mod of functors from $\mathbb{Z}_{\leq}$ to the category of k-vector spaces. We take the direct sum of these module categories, over all functorial embeddings of $\langle Q_i \rangle$ in $\langle Q_n \rangle$. There is a natural endofunctor D of this category, lifting the differential d. How do we categorically interpret the signs appearing in this chain complex ? We interpret each negative sign as corresponding to a shift S, which we now interpret as an exact endofunctor of $\mathbb{Z}_{\leq}$-mod, or equivalently as a $\mathbb{Z}_{\leq}$-$\mathbb{Z}_{\leq}$-bimodule. If we think of S, and the identity functor 1 as bimodules, we can think of q as a bimodule homomorphism $q : 1 \Rightarrow S$.

The equation $D^2 = 0$ is not satisfied here. Rather, we have a natural transformation $T : D^2 \Rightarrow D^2.S$, such that $T^2.1_{s^2} = 1_{D^2}.q^2$, as natural transformations from D^2 to D^2S^2. The functor D^2 can be thought of as a sum of components move from one corner of a square to its opposite in an $n-1$-dimensional cube; each component can then be written $S^i \oplus S^{i+1}$. The transformation T takes a component $S^i \oplus S^{i+1}$

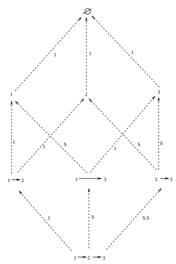


FIGURE 3. Shifts defining D.

to $S^{i+1} \oplus S^{i+2}$; it is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ \\ q^2 & 0 \end{pmatrix}$$

The square of such a matrix is obviously q^2. The natural transformation T plays the role of the equation $d^2 = 0$.

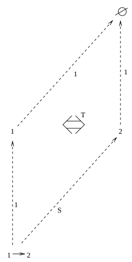


FIGURE 4. The transformation T.

What relations hold between the functors T ? We have the braid relations:

$$T_{12}T_{23}T_{12} = T_{23}T_{12}T_{23} : D^3 \Rightarrow D^3S^3.$$

These can be pictured as follows:

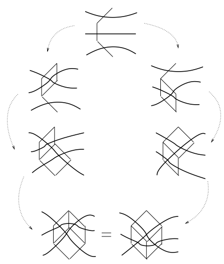


FIGURE 5. Braid relations.

We define $\underline{\Lambda}$ to be the free monoidal category on an object D, a central invertible object S, an arrow

$$T : D^2 \Rightarrow D^2S$$

and a central arrow $q : 1 \Rightarrow S$, modulo the braid relations, and the relation

$$T^2 = q^2.$$

We define an $\underline{\Lambda}$-category to be a 2-functor from $\underline{\Lambda}$ in the category of categories.

Let us associate a $\underline{\Lambda}$-category $\underline{C}(\mathcal{C})$ to every category $\mathcal{C}$.

Consider the collection of compactly supported functors $Fun_c(\mathbb{Z}_{\geq}, \mathcal{C})$, namely those functors from the free category on $\mathbb{Z}_{\leq}$ to $\mathcal{C}$, which take every integer much less than zero to the same object $i \in \mathcal{C}$, which every arrow between integers much less than zero to 1_o, which take every integer much greater than zero to $t \in \mathcal{C}$, and every arrow between integers much greater than zero to 1_t. We identify functors in $Fun_c(\mathbb{Z}_{\leq}, \mathcal{C})$, if they are identical, up to a shift S^i in $\mathbb{Z}_{\leq}$ for some $i \in \mathbb{Z}$. We define

$$\underline{C}(\mathcal{C}) = (\mathbb{Z}_{\leq}\text{-mod})^{\oplus Fun_c(\mathbb{Z}_{\leq}, \mathcal{C})}$$

to be a direct sum of copies of $\mathbb{Z}_{\leq}$-mod, one for every compactly supported functor from that category to $\mathcal{C}$. So we have one copy of $\mathbb{Z}_{\leq}$-mod corresponding to the empty set, one copy of $\mathbb{Z}_{\leq}$ for every object in $\mathcal{C}$, one copy for every arrow in $\mathcal{C}$, and so on.

We can identify a compactly supported functor from $\mathbb{Z}_{\leq}$ to $\mathcal{C}$ as a copy of Q_n^f in $\mathcal{C}$, for some n. Therefore, by what we have already said, $\underline{C}(\mathcal{C})$ is naturally a $\underline{\Lambda}$-category.

How do we form the product of two such $\underline{\Lambda}$-categories, $\underline{C}(\mathcal{C})$ and $\underline{C}(\mathcal{D})$? We define $\mathcal{C} \star \mathcal{D}$ to be the category whose collection of objects is the disjoint union of the collections of objects in $\mathcal{C}$ and $\mathcal{D}$, and whose collection of arrows is the disjoint union of the arrows in $\mathcal{C}$ and $\mathcal{D}$, along with the collection of pairs (c, d), where c is an object in $\mathcal{C}$, and d is an object in $\mathcal{D}$. We think of (c, d) as defining a morphism from c to d. Composition of arrows in $\mathcal{C}$ and arrows in $\mathcal{D}$ within $\mathcal{C} \star \mathcal{D}$ descends from composition within $\mathcal{C}$ and $\mathcal{D}$ themselves. The composition of (c, d) with any arrow from d to d' is equal to (c, d'); the composition of any arrow from c' to c with (c, d) is equal to (c', d), for $c, c' \in \mathcal{C}$, $d, d' \in \mathcal{D}$.

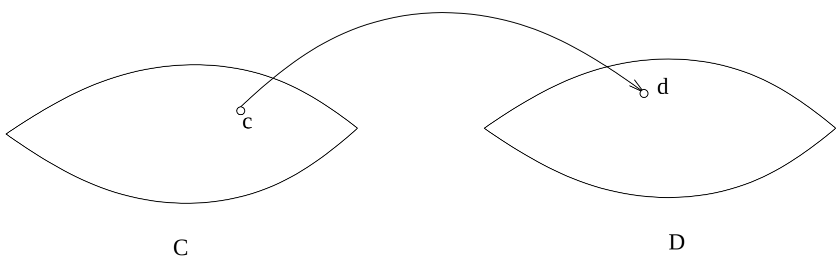


FIGURE 6. The $\star$ product of two categories.

We define

$$\underline{C}(\mathcal{C}) \diamond \underline{C}(\mathcal{D}) = \underline{C}(\mathcal{C} \star \mathcal{D}).$$

Let us define $K(\mathbb{Z}_{\leq}\text{-mod})$ to be $\mathbb{C}$, and $K(S^i)$ to be multiplication by $(-1)^i$. We obtain a chain complex $K(\underline{C}(\mathcal{C}))$ by extending K additively across $\underline{C}(\mathcal{C})$.

Lemma 6. *The vector space $K(\underline{C}(\mathcal{C}))$ is a chain complex with differential $K(D)$. We have $K(\underline{C}(\mathcal{C})) = C(\mathcal{N}\mathcal{C})$ as Λ-modules. We have*

$$K(\underline{C}(\mathcal{C}) \diamond \underline{C}(\mathcal{D})) = K(\underline{C}(\mathcal{C})) \otimes K(\underline{C}(\mathcal{D}))$$

as Λ-modules.

We define $\mathcal{T}_\Lambda$ to be the collection of $\underline{\Lambda}$-categories of the form $\underline{C}(\mathcal{D})$.

Theorem 7. *$(\mathcal{T}_\Lambda, \diamond)$ is a symmetric monoidal 2-category.*

The $\star$ product is reminiscent of Feynman calculus. It has other interesting features. For example, if we define $\mathbb{Q}_{\leq}$ to be the category whose objects are rational numbers, and with a single morphism from q to q' whenever $q \leq q'$, then we have the following lemma:

Lemma 8. *The set of real numbers $\mathbb{R}$ can be identified with the set of decompositions*

$$\mathbb{Q}_{\leq} = \mathbb{Q}_l \star \mathbb{Q}_r$$

of $\mathbb{Q}_{\leq}$ as a product of two nonempty subcategories $\mathbb{Q}_l, \mathbb{Q}_r \subset \mathbb{Q}_{\leq}$ where $\mathbb{Q}_{\leq}$ contains no initial object.

Proof. Decompositions of $\mathbb{Q}_{\leq}$ of this form can be identified with Dedekind cuts of $\mathbb{Q}$. □

The $\star$ product is associative:

Lemma 9. *Given three categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, we have an equivalence between $(\mathcal{C} \star \mathcal{D}) \star \mathcal{E}$ and $\mathcal{C} \star (\mathcal{D} \star \mathcal{E})$.*

Forming the product $\mathcal{C} \star \mathcal{D}$ of two categories is analogous to gluing manifolds along a boundary. Anon we present a lemma to justify this claim.

Suppose $X \subset \mathbb{R}^N \times [0,1]$ is a manifold. We define $X^{\sim}$ to be the category whose objects are elements of X, with a single morphism from x to x' if there is a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = x'$, and $\pi(\gamma(y)) \leq \pi(\gamma(y'))$ whenever $y \leq y' \in [0,1]$. Here π denotes projection onto the $N+1^{th}$ component of $\mathbb{R}^N \times [0,1]$.

Let $X_l = X \cap (\mathbb{R}^N \times [0, \frac{1}{2}])$ and $X_r = X \cap (\mathbb{R}^N \times [\frac{1}{2}, 1])$. We thus obtain X by gluing X_l and X_r along $X_\partial = X \cap (\mathbb{R}^N \times \frac{1}{2})$.

Lemma 10. *Suppose X_∂ is path connected. Then $X^{\sim}$ and $X_l^{\sim} \star X_r^{\sim}$ are equivalent.*

This analogy between the product $\star$ and gluing of manifolds suggests the $\star$ product may be compatible with topological quantum field theory. This happens to be the case, as we shall observe in the next section.

6. Topological quantum field theories.

A n-dimensional topological quantum field theory (TQFT) is defined to be a symmetric monoidal functor from the cobordism category (n-Cob, $\amalg$) to the category of finite dimensional vector spaces ($Vect, \otimes$).

According to this definition, a two dimensional topological quantum field theory Z can be identified with a commutative symmetric algebra, which is the vector space Z_{S^1} corresponding to the circle; the product on Z_{S^1} is given by the linear map from $Z_{S^1 \amalg S^1}$ to Z_{S^1} corresponding to the cobordism defined by a pair of troosers. Three and four-dimensional TQFTs are more serpentine beasts.

Consider the following diagram:

$$\text{Groups} \longrightarrow \text{Braided monoidal categories with duals} \longrightarrow \text{3d TQFTs}$$

$$\text{Groups} \dashrightarrow \text{Braided monoidal 2-categories with duals} \dashrightarrow \text{4d TQFTs}$$

The first row describes the passage from a group to a three dimensional topological quantum field theory via braided monoidal categories. To obtain a braided monoidal category from a finite group G, we take the module category of the Drinfeld double $kG \ltimes kG^*$ of the group algebra of G. A more functorial way to describe this category is as the Drinfeld centre of the category of finite dimensional representations of G. A similar construction works when G is a semisimple Lie group; one obtains the braided monoidal category of representations of the quantum group associated to G. To obtain a $3d$ TQFT from this category, we apply a surgical procedure devised by Turaev and Reshetikhin.

The second row describes a conjectural passage from a group G to a four dimensional topological quantum field theory via braided monoidal 2-categories. The second row was conjectured by Crane and Frenkel to be obtained from the first row, via a mythical process called categorification.

In general, Baez and Dolan hypothesise that n dimensional TQFTs can be defined in an n-categorical framework, via n-categories with duals [4].

There is special interest in the four dimensional case due to its relations with classical representation theory. Papers of Khovanov, of Bernstein, Frenkel and Khovanov, and Stroppel, have established links between categories of infinite dimensional representations of semisimple Lie algebras, and knot invariants such as the Jones polynomial [8], [28]. In their work, the enveloping algebra U of the Lie algebra $\mathfrak{g}$ of the group G lifts to a 2-category $\underline{U}$. They construct representations

$$\underline{U} \circlearrowright \underline{V}$$

of $\underline{U}$ on abelian categories $\underline{V}$ of classical representations, such as categories $\mathcal{O}$ for semisimple Lie algebras, or representation categories of Hecke algebras. Taking the complexified Grothendieck group of such an abelian category, we obtain a complex vector space $V = \mathbb{C}K_0(\underline{V})$, with the action of various linear maps. Miraculously, these maps define an action of U on V.

$$U \circlearrowleft V.$$

The interpretation here in terms of 2-categories is due to Chuang and Rouquier [11], and is more explicitly drawn out in Rouquier's recent article [25]. In Chuang and Rouquier's paper on categorification of $\mathfrak{sl}_2$, deep results concerning symmetric group representations are proved in this idiom. An $\mathfrak{sl}_2$-categorification is a representation of a certain 2-category $\underline{U}(\mathfrak{sl}_2)$. Symmetric group algebras show up as 2-arrows in this 2-category. And what is the relation with four dimensional topology ? The collection $\underline{U}(\mathfrak{g})$-cat of all representations of $\underline{U}(\mathfrak{g})$, or some close relation, is hoped to form a braided monoidal 2-category.

In this section we adopt a different approach to categorification, which does not refer directly to representation theory. By topological techniques, we associate a braided monoidal 2-category with duals $\mathcal{T}_Z$ to any classical topological quantum field theory Z. In particular, by applying our construction to the three dimensional TQFTs devised by Turaev et al, we obtain a braided monoidal 2-category from any finite group, or any complex semisimple Lie algebra.

How does our construction work ? As is suggested by the construction of the previous section, in which we categorified chain complexes, we work with the category of categories, with monoidal structure given by $\star$. We exploit the data of Z to obtain a rich 2-categorical structure. In case Z is the 2-dimensional TQFT corresponding to the commutative symmetric algebra Λ, we obtain an object which contains Khovanov's algebra H, and is therefore closely related to the theory of $\mathfrak{sl}_2$-categorifications [8], [9].

Let $Z : (n\text{-Cob}, \amalg) \to (Vect_k, \otimes_k)$ be an n-dimensional topological quantum field theory. To every closed $n-1$-manifold X, we associate a vector space Z_X, and to every n-dimensional cobordism Y from X to X', we associate a linear map $Z_Y : Z_X \to Z_{X'}$.

Let $\mathcal{C}$ be a category. We define a new category $\mathcal{C}_Z$, whose objects O are open simplicial manifolds of dimension $n-1$ in the nerve of $\mathcal{C}$. In other words, an object O of $\mathcal{C}_Z$ is a simplicial subcomplex of the nerve of $\mathcal{C}$, whose geometric realisation is a manifold of dimension $n-1$, with boundary. The $d-1$-cells of O correspond to functors from $\langle Q_d \rangle$ to $\mathcal{C}$, so that inclusion of cells $\Delta_{d_1-1} \subset \Delta_{d_2-1}$ corresponds to inclusion of categories $\langle Q_{d_1} \rangle \subset \langle Q_{d_2} \rangle$.

We insist that every connected component of O has at least one $n-1$-simplex. We identify two objects O in $\mathcal{C}_Z$ if they correspond to the same combinatorial data in $\mathcal{C}$.

To define morphisms in $\mathcal{C}_Z$, note that when two objects O and O' have the same boundary ∂, we have a simplicial complex of dimension $n-1$, given by $X_{O,O'} = O \cup_\partial O'$. Note that $X_{O,O'}$ is a closed simplicial $n-1$-manifold, that is to say a simplicial

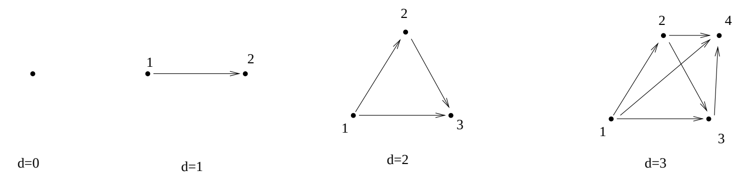


FIGURE 7. Simplexes in the nerve of $\mathcal{C}$.

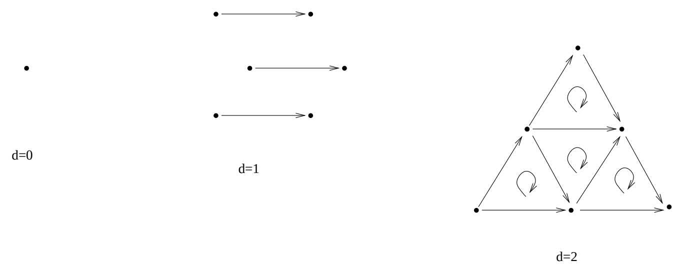


FIGURE 8. Open simplicial manifolds in the nerve of $\mathcal{C}$.

complex whose geometric realisation is an $n-1$-manifold without boundary. We define $Hom_{\mathcal{C}_Z}(O, O')$ to be $Z_{X_{O,O'}}$ when O and O' have the same smooth boundary, and to be $\{0\}$ otherwise.

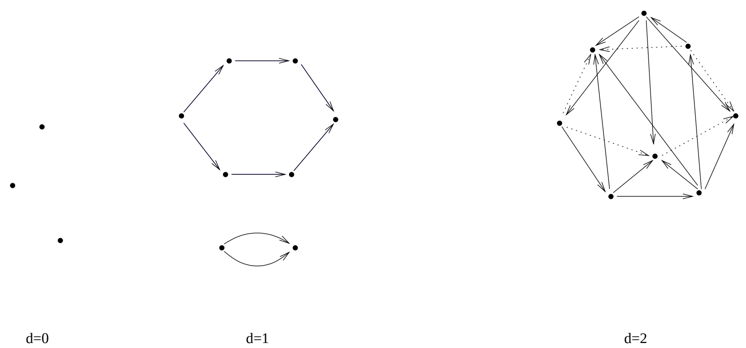


FIGURE 9. Closed simplicial manifolds in the nerve of $\mathcal{C}$.

When three objects O, O' and O'' have the same boundary ∂, we have a cobordism $Y_{O,O',O''}$ between $X_{O,O'} \amalg X_{O',O''}$ and $X_{O,O''}$, which can be thought of as the process of gluing the two manifolds along O' at a given time. This cobordism defines a linear map

$$Z_{Y_{O,O',O''}} : Z_{X_{O,O'}} \otimes Z_{X_{O',O''}} \to Z_{X_{O,O''}},$$

which we take to define composition

$$Hom(O, O') \otimes Hom(O', O'') \to Hom(O, O'')$$

in $\mathcal{C}_Z$. Otherwise, we define the composition be zero.

The axioms of a topological quantum field theory ensure the associativity of composition in $\mathcal{C}_Z$.

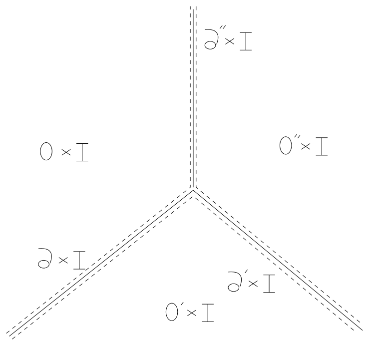


FIGURE 10. Cobordism $Z_{Y_{O,O',O''}}$.

Lemma 11. *The blocks of $\mathcal{C}_Z$ correspond to closed $n-2$-dimensional simplicial manifolds in the nerve of $\mathcal{C}$.*

For any finite set I of objects with smooth boundary in $\mathcal{C}_Z$, the ring

$$R_{\mathcal{C},Z} = \oplus_{i,j \in I} Hom_{\mathcal{C}_Z}(O_i, O_j)$$

is a finite dimensional symmetric algebra. The endomorphism ring of a single object in $\mathcal{C}_Z$ is a commutative symmetric algebra.

Example 12 Let Z be the 2-dimensional TQFT associated to the commutative symmetric algebra $k[x]/x^2$. Let $\mathcal{C}$ be the free category $\langle Q_{2d} \rangle$.

Then $R_{\mathcal{C},Z}$ contains a copy of Khovanov's algebra H [21]. Thus objects of $\mathcal{C}_Z$ correspond to idempotents in H.

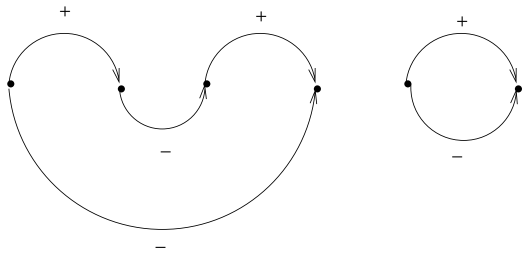


FIGURE 11. An idempotent in Khovanov's algebra H.

The collection of categories $\mathcal{C}_Z$ looks like a promising candidate for a monoidal 2-category with duals. In fact, associated to Z we construct a braided monoidal 2-category with duals named $\mathcal{T}_Z$, whose objects are categories, and whose 2-arrows resemble objects of $\mathcal{C}_Z$. We exploit the 2-category of 2-tangles of Baez and Langford.

Before proceeding to define this gadget, let us make some remarks on the $\star$ product.

Let $\mathbb{S}$ denote a collection of categories with finitely many objects, no two of which are equivalent, such that $\mathbb{S}$ is closed under taking opposites, closed under taking full subcategories and closed under the $\star$ product. For example, we could take $\mathbb{S}$

to be the collection of free categories $\langle Q\rangle$ on finite quivers Q, with an initial and terminal vertex. The product $\langle Q\rangle \star \langle Q'\rangle$ is then $\langle Q \star Q'\rangle$, where $Q \star Q'$ is defined to be the quiver $Q \amalg Q'$, with a single additional arrow from the terminal vertex of Q to the initial vertex of Q'

Since no two categories in $\mathbb{S}$ are equivalent, the $\star$ product $\mathcal{C}_1 \star \mathcal{C}_2 \star ... \star \mathcal{C}_n$ of n categories $\mathcal{C}_1, ..., \mathcal{C}_n$ in $\mathbb{S}$ is uniquely defined.

Definition 13. *We call a category $\mathcal{C}$ in $\mathbb{S}$* indecomposable *if it is nonempty, and in any decomposition $\mathcal{C} = \mathcal{C}^l \star \mathcal{C}^r$, either $\mathcal{C}^l$ or $\mathcal{C}^r$ is the empty category.*

Lemma 14. *Any category $\mathcal{C}$ in $\mathbb{S}$ has a unique decomposition*

$$\mathcal{C} = \mathcal{C}_1 \star \mathcal{C}_2 \star ... \star \mathcal{C}_n,$$

with $\mathcal{C}_i$ indecomposable.

Proof. Given a second decomposition

$$\mathcal{C} = \mathcal{C}'_1 \star \mathcal{C}'_2 \star ... \star \mathcal{C}'_o$$

of this kind, we have a decomposition

$$\mathcal{C}_n = (\mathcal{C}_n \backslash \mathcal{C}'_o) \star (\mathcal{C}_n \cap \mathcal{C}'_o),$$

which implies $\mathcal{C}_n = \mathcal{C}_n \cap \mathcal{C}'_o$ by the indecomposability of $\mathcal{C}_n$. Similarly, we find $\mathcal{C}'_o = \mathcal{C}_n \cap \mathcal{C}'_o$, and thus $\mathcal{C}_n = \mathcal{C}'_o$. Inductively, we find that our two decompositions of $\mathcal{C}$ as a $\star$ product are identical. $\square$

Lemma 15. *Given any two categories $\mathcal{C}$ and $\mathcal{D}$, we can identify the objects of $(\mathcal{C} \star \mathcal{D})_Z$ with the objects of $(\mathcal{D} \star \mathcal{C})_Z$.*

Proof. We identify the objects of $(\mathcal{C} \star \mathcal{D})_Z$ with the objects of $(\mathcal{D} \star \mathcal{C})_Z$ via a correspondence σ defined by the following recipe:

Take a simplicial $n-1$-manifold M in $(\mathcal{C}\star\mathcal{D})_Z$; write it as a union $M = (M \cap \mathcal{C}) \cup_\theta M_i \cup_{\theta'} (M \cap \mathcal{D})$ where M_i is the union of $n-1$-simplices in M which lie in neither $\mathcal{C}$ nor $\mathcal{D}$ and where $\theta = M_i \cap \mathcal{C}, \phi = M_i \cap \mathcal{D}$; form a new complex $\sigma(M_i)$ in $\mathcal{D}\star\mathcal{C}$ which is obtained by reversing all arrows from $\mathcal{C}$ to $\mathcal{D}$ in M_i whilst preserving the directions of all arrows within θ and ϕ; form the union $\sigma(M) = (M \cap \mathcal{D}) \cup_\phi \sigma(M_i) \cap (M \cap \mathcal{C})$ which is an object of $(\mathcal{D} \star \mathcal{C})_Z$. $\square$

To define the 2-category $\mathcal{T}_Z$, we exploit the monoidal category of tangles, and the monoidal 2-category of 2-tangles. The idea behind higher dimensional tangles is simple: the collection of numbers forms a monoid under addition; a number can be represented as a collection of points in $\mathbb{R}^2$; a tangle represents the evolution of such a collection of points through time; the collection of tangles forms a monoidal category, whose arrows point with the arrow of time; a tangle can be represented as a collection of strings in $\mathbb{R}^3$; a 2-tangle represents the evolution of such a collection of strings through time; the collection of 2-tangles forms a monoidal 2-category whose 2-arrows point with the arrow of time; and so on. For details, consult the paper of Baez and Langford [5].

Given a natural number n, we let $\mathbf{n}$ denote the set $\{1, ..., n\}$. Here is data for the 2-categorical structure on $\mathcal{T}_Z$:

• Objects of $\mathcal{T}_Z$ are categories in $\mathbb{S}$ with finitely many objects and morphisms.

• Let $\mathcal{C}, \mathcal{D}$ be objects of $\mathcal{T}_Z$, with decompositions

$$\mathcal{C} = \mathcal{C}_1 \star \mathcal{C}_2 \star ... \star \mathcal{C}_m,$$

$$\mathcal{D} = \mathcal{D}_1 \star \mathcal{D}_2 \star ... \star \mathcal{D}_n$$

into indecomposable pieces, such that $m + n = 2l$. An arrow $v = (t, O)$ from $\mathcal{C}$ to $\mathcal{D}$ is a tangle $t : \mathbf{m} \to \mathbf{n}$ such that $\mathcal{C}_i = \mathcal{D}_j$ when $i \in \mathbf{m}$ is connected to $j \in \mathbf{n}$ by t, $\mathcal{C}_i = \mathcal{C}_j^{op}$ when $i \in \mathbf{m}$ is connected to $j \in \mathbf{m}$ by t, and $\mathcal{D}_i = \mathcal{D}_j^{op}$ when $i \in \mathbf{m}$ is connected to $j \in \mathbf{n}$ by t; and a sequence $O = (O_1, ..., O_d)$ of objects of

$$(\mathcal{C}_{\omega_1} \star \mathcal{C}_{\omega_2} \star ... \star \mathcal{C}_{\omega_l})_Z \subset \mathcal{C}_Z,$$

where $\omega_1 < \omega_2 < ... < \omega_l$ is an increasing sequence of elements of the $\mathbf{m}$ connected to $\mathbf{n}$ by t.

By Lemma 15, an object in the sequence O can be identified with an object of

$$(\mathcal{D}_{\omega_1'} \star \mathcal{D}_{\omega_2'} \star ... \star \mathcal{D}_{\omega_l'})_Z \subset \mathcal{D}_Z,$$

where $\omega_1' < \omega_2' < ... < \omega_l'$ is an increasing sequence of elements of $\mathbf{n}$ connected to $\mathbf{m}$ by t.

• Let $v = (t, O)$ and $v' = (t', O')$ be arrows from $\mathcal{C}$ to $\mathcal{D}$ whose sequences O and O' have lengths d and d' respectively. A 2-arrow $\theta = (\tau, s, \alpha)$ from v to v' is a 2-tangle τ from t to t'; also a tangle $s : \mathbf{d} \to \mathbf{d}'$ such that O_i,O_j' intersect the same components $\mathcal{C}_\omega$ of $\mathcal{C}$ and have the same boundary ∂_{ij} when $i \in \mathbf{d}$ is connected to $j \in \mathbf{d}'$ by s, such that O_i, O_j intersect the same components $\mathcal{C}_\omega$ of $\mathcal{C}$ and have the same boundary ∂_{ij} when $i \in \mathbf{d}$ is connected to $j \in \mathbf{d}$ by s, and such that O_i', O_j' intersect the same components $\mathcal{D}_\omega$ of $\mathcal{D}$ and have the same boundary ∂_{ij}' when $i \in \mathbf{d}$ is connected to $j \in \mathbf{d}'$ by s; also elements $\alpha_{ij} \in Z_{O_i \cup_{\partial_{ij}} O_j'}$ for $i \in \mathbf{d}$ connected to $j \in \mathbf{d}'$ by s, elements $\alpha_{ij} \in Z_{O_i \cup_{\partial_{ij}} O_j}$ for $i \in \mathbf{d}$ connected to $j \in \mathbf{d}$ by s, and elements $\alpha_{ij} \in Z_{O_i' \cup_{\partial_{ij}} O_j'}$ for $i \in \mathbf{d}'$ connected to $j \in \mathbf{d}'$ by s.

We think of the element α_{ij} as lying on the braid from i to j.

We next describe how to multiply arrows in $\mathcal{T}_Z$. Suppose $(t, O) : \mathcal{C} \to \mathcal{D}$ and $(u, P) : \mathcal{D} \to \mathcal{E}$ are arrows, where $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ have m, n and o indecomposable factors. We define their product $(t, O)(u, P)$ to be the pair $(tu, (OP))$, where tu denotes the composition of tangles t and u; and where (OP) consists of the sequence obtained by placing the sequence P after the sequence O before removing all those objects which do not lie in

$$(\mathcal{C}_{\omega_1} \star \mathcal{C}_{\omega_2} \star ... \star \mathcal{C}_{\omega_l})_Z \subset \mathcal{C}_Z,$$

where $\omega_1 < \omega_2 < ... < \omega_l$ is an increasing sequence of elements of the $\mathbf{m}$ connected to $\mathbf{o}$ by tu, as well as those objects which do not lie in

$$(\mathcal{E}_{\omega_1'} \star \mathcal{E}_{\omega_2'} \star ... \star \mathcal{E}_{\omega_l'})_Z \subset \mathcal{E}_Z,$$

where $\omega'_1 < \omega'_2 < ... < \omega'_l$ is an increasing sequence of elements of the $\mathbf{m}$ connected to $\mathbf{o}$ by tu.

We next describe how to define horizontal composition of 2-arrows in $\mathcal{T}_Z$. Suppose we have a 2-arrow (τ, s, α) from (t, O) to (u, P) and a 2-arrow (τ', s', α') from (t', O') to (u', P'). The horizontal composition of (τ, s, α) and (τ', s', α') is defined to be the 2-arrow $((\tau\tau'), (ss'), (\alpha, \alpha'))$, where $(\tau\tau')$ is the 2-tangle obtained composing the 2-tangles τ and τ' horizontally; where (s, s') is the tangle obtained by placing the tangle s' next to the tangle s; and where (α, α') is the collection of elements $\alpha_{ij}, \alpha'_{ij}$ which correspond to pairs of objects lying in the sequences (OO') and (PP').

To define vertical composition of 2-arrows, we use cobordisms. Suppose we have a 2-arrow (τ, s, α) from (t, O) to (u, P) and a 2-arrow (τ', s', α') from (u, P) to (v, Q). The vertical composition of (τ, s, α) and (τ', s', α') is defined to be $(\tau\tau', ss', \alpha\alpha')$, where $\tau\tau'$ is the vertical product of the 2-tangles τ and τ'; where ss' is the composition of the tangles; and where $\alpha\alpha'$ is defined by composing 2-arrows along on the same braid of ss' in $\mathcal{C}_Z$.

The identity arrow of $\mathcal{C}$ in $\mathcal{T}_Z$ is given by the pair $(1_\mathcal{C}, \emptyset)$ consisting of the identity tangle $1_\mathcal{C}$ whose strings are in one-one correspondence with indecomposable components of $\mathcal{C}$, and the empty sequence $\emptyset$.

Here is additional data for the definition of a monoidal structure on $\mathcal{T}_Z$:

- The object I is given by the category $\emptyset$ with no objects.
- For two objects $\mathcal{C}, \mathcal{C}'$ of $\mathcal{T}_Z$, we define their product to be the category $\mathcal{C} \star \mathcal{C}'$.
- Given an arrow (t, O) and a category $\mathcal{D}$ in $\mathcal{T}_Z$, we define the arrow $(t, O) \star \mathcal{D}$ to be the pair $((t1_\mathcal{D}), O)$, where $(t1_\mathcal{D})$ is the tensor product in the 2-tangle 2-category of the identity tangle, whose strings are in one-one correspondence with indecomposable components of $\mathcal{D}$, and t.
- For a category $\mathcal{C}$ and an arrow (u, P) in $\mathcal{T}_Z$, we define $\mathcal{C} \star (u, P)$ to be $((1_\mathcal{C} u), P)$ in a similar way.
- Given a 2-arrow (τ, s, α) and a category $\mathcal{D}$ in $\mathcal{T}_Z$, we define the 2-arrow $(\tau, s, \alpha) \star \mathcal{D}$ to be the triple $((\tau 1, (s1), \alpha)$.
- Given a 2-arrow (τ, s, α) and a category $\mathcal{C}$ in $\mathcal{T}_Z$, we define the 2-arrow $\mathcal{C} \star (\tau, s, \alpha)$ to be the triple $((1\tau), (1s), \alpha)$.
- Given any two arrows $(t, O) : \mathcal{C} \to \mathcal{C}'$ and $(u, P) : \mathcal{D} \to \mathcal{D}'$, we should define the corresponding tensorator. Let us first identify the composition of $(t, O) \star \mathcal{D}$ and $\mathcal{C}' \star (u, P)$. It can be identified with $((ut), O \amalg P)$ via a suitable 2-arrow $(\tau, 1, \emptyset)$. The composition of $\mathcal{C} \star (u, P)$ and $(t, O) \star \mathcal{D}'$ can likewise be identified with $((ut), O \amalg P)$. We define the tensorator to be the composition of these identifications.

Here is the data for a braiding on $\mathcal{T}_Z$:

- Given any two categories $\mathcal{C}$ and $\mathcal{D}$, we define $\sigma_{\mathcal{C},\mathcal{D}}$ to be the arrow $(\sigma_{C,D}, \emptyset)$, where $\sigma_{C,D}$ is the braid which associates indecomposable components of $\mathcal{C}$ and $\mathcal{D}$

with themselves, so that in projection onto the plane, all braids from $\mathcal{C}$ to $\mathcal{C}$ cross over the top of all braids from $\mathcal{D}$ to $\mathcal{D}$, whilst braids from $\mathcal{C}$ to $\mathcal{C}$ and braids from $\mathcal{D}$ to $\mathcal{D}$ do not cross. Thus $\sigma_{C,D}$ defines the braiding on the 2-category of 2-tangles.

• Given two arrows $a = (t, O)$ and $b = (u, P)$, we define the $\sigma_{a,b}$ to be the triple $(\sigma_{t,u}, 1, 1)$, where $\sigma_{t,u}$ comes from the braiding on the 2-category of 2-tangles, and the 1s correspond to identity elements.

• Given three categories $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, we define $\sigma_{\mathcal{C}|\mathcal{D},\mathcal{E}}$ to be $(\sigma_{C|D,E}, 1, 1)$, where $\sigma_{C|D,E}$ comes from the braiding on the 2-category of 2-tangles, and the 1s correspond to identity elements.

• Given three categories $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, we define $\sigma_{\mathcal{C},\mathcal{D}|\mathcal{E}}$ to be $(\sigma_{C,D|E}, 1, 1)$, where $\sigma_{C,D|E}$ comes from the braiding on the 2-category of 2-tangles, and the 1s correspond to identity elements.

Here is the additional data for duals on $\mathcal{T}_Z$:

• The dual of a 2-arrow (τ, s, α) is defined to be (τ^*, s^*, α^*), where τ^* is the dual of τ in the tangle 2-category, where s^* is the dual of s in the tangle category, and where α^* is obtained from α by identifying $Z_{O_i \cup_{\partial_{ij}} O'_j}$ with $Z_{O'_j \cup_{\partial_{ij}} O_i}$ for various objects O_i, O'_j.

• The dual of an arrow (t, O) is defined to be (t^*, O), where t^* is the dual of t in the 2-tangle 2-category.

• For a category $\mathcal{C} \in \mathcal{T}_Z$, we define the dual $\mathcal{C}^*$ of $\mathcal{C}$ to be its opposite category $\mathcal{C}^{op}$.

The unit $i_{\mathcal{C}} : \emptyset \to \mathcal{C}^{op} \star \mathcal{C}$ is defined to be the pair $(i_C, \emptyset)$, where i_C is the counit in the 2-tangle 2-category which connects a point corresponding to the indecomposable factor $\mathcal{C}_i$ of $\mathcal{C}$ to a point corresponding to the indecomposable factor $\mathcal{C}_i^{op}$ of $\mathcal{C}^{op}$.

The counit $e_{\mathcal{C}} : \mathcal{C}^{op} \star \mathcal{C} \to \emptyset$ is defined to be the pair $(e_C, \emptyset)$, where e_C is the counit in the 2-tangle 2-category which connects a point corresponding to the indecomposable factor $\mathcal{C}_i$ of $\mathcal{C}$ to a point corresponding to the indecomposable factor $\mathcal{C}_i^{op}$ of $\mathcal{C}^{op}$.

The composition $(i_{\mathcal{C}} \star \mathcal{C})(\mathcal{C} \star e_{\mathcal{C}})$ can be identified with the pair $(i_C \otimes 1_C)(1_C \otimes e_C), \emptyset)$. We define the triangulator of $\mathcal{C}$ to be the triple $(\tau, 1, 1)$, where τ is the triangulator in the tangle 2-category.

Theorem 16. *$\mathcal{T}_Z$ is a braided monoidal 2-category with duals.*

7. Shades of representation theory

If a is a thing, then a representation of a is an image of a within some other thing b. Let us be more specific. Suppose $\mathcal{C}$ is a category. Given a category $\mathcal{A}$, a representation of $\mathcal{A}$ with values in $\mathcal{C}$ is nothing but a functor from $\mathcal{A}$ to $\mathcal{C}$. For example, a linear representation of a group G is a functor from the one object category whose morphisms are elements of G to the category of vector spaces. A sheaf on a topological space X is a functor from the category whose morphisms are containments of open sets in X to the category of vector spaces, satisfying certain properties. A topological quantum field theory is a functor from a cobordism category to the category of vector spaces, satisfying certain properties.

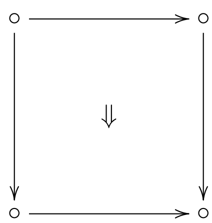
The category of vector spaces is a good choice for the target of a representation for a number of reasons: its simplicity; its modest geometric behaviour; its structure as an abelian category; its structure as a tensor category; its notion of duality; the notion of an inner product; its physical resonance. These features are all good, but the category of vector spaces is by no means the only possible choice of representing category $\mathcal{C}$. For example, the theory of group representations which take values in the category of sets is the theory of permutation groups. Integral representations of groups take values in the category of abelian groups.

When we consider representations of 2-categories, it is clear that a representation should be defined to be a 2-functor, taking values in some 2-category, perhaps satisfying some extra conditions. Where should such representations take their values? One choice is in the 2-category $\mathcal{T}_{alg}$, or in variants such as $\mathcal{T}_{\mathcal{G}}$, or the 2-category of abelian categories, or the 2-category of triangulated categories. These 2-categories have many properties which are desirable for good representation theory: they are fairly well understood; there are stratification properties given by recollement [7]; there are various notions of dimension, such as homological dimension, or Brauer's defect of a block, or Rouquier's dimension of a triangulated category [27]; we often have monoidal structures given by tensor product; we can define a Calabi-Yau property [18]; there is some physical resonance [15]. These 2-categories also have one obvious deficiency: whilst they carry a great deal of geometrical information, they are not intuitively geometric objects in themselves.

In any case, 2-categories like $\mathcal{T}_{alg}$ seem to provide an adequate setting for representation theory of 2-categories. When we define representations of double categories like $\underline{\mathbb{N}}$ in the following section, they will take values in 2-categories such as $\mathcal{T}_{alg}$.

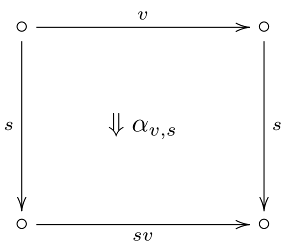
8. Rings.

Double categories are variants of strict 2-categories in which there are both vertical and horizontal arrows, whilst 2 arrows are squares



In a double category, we can compose horizontal arrows with horizontal arrows, vertical arrows with vertical arrows, and we can compose 2-arrows either vertically or horizontally [16].

Let R be a ring, and V an R-module. We have a natural double category $\underline{V}$ associated to V, with one object $\circ$, whose horizontal arrows correspond with elements $v \in V$, and whose vertical arrows correspond with elements $r \in R$. The horizontal product is given by addition in V, and the vertical product is given by multiplication in R. We define the 2-arrows in $\underline{V}$ to be squares of the form



The distributivity of multiplication is encoded in the horizontal product of two 2-arrows:

$$\alpha_{v,s}\alpha_{v',s} = \alpha_{v+v',s},$$

which only makes sense because $s(v + v') = sv + sv'$ for $s \in R, v \in V$. Vertical multiplication of 2-arrows is defined in the obvious way:

$$\genfrac{}{}{0pt}{}{\alpha_{v,s}}{\alpha_{sv,t}} = \alpha_{v,st}.$$

We denote by $\underline{R}$ the double category associated to the regular R-module. The nerve of $\underline{R}$ is a topological invariant of R.

Double categories live one step up the categorical ladder from categories. Therefore, rings naturally live one categorical step down the categorical ladder from groups. As the representations of a group form a symmetric monoidal category, so we would like the representations of a ring to form a symmetric monoidal 2-category. To create such an effect, we define a representation of $\underline{R}$ to be a 2-functor from $\underline{R}$ to a 2-category $\mathcal{T}$. Such a 2-functor corresponds to the following data:

- An object $t \in \mathcal{T}$.
- An arrow $h_r \in \mathcal{T}$ for every $r \in R$.
- An arrow v_s for every $s \in R$.
- A 2-arrow $\alpha_{r,s}$ from $h_r v_s$ to $v_s h_{rs}$.

This data is required to be compatible with the various relations in $\underline{R}$. Since double categories are strict in nature, their representations are also considered in a strict sense. So compatibility merely means that $h_r h_{r'}$ is equal to $h_{r+r'}$; that $v_s v_{s'}$ is equal to $v_{ss'}$; and that the relations between 2-arrows in $\underline{R}$ also hold between the corresponding homomorphisms $\alpha_{r,s} \in \mathcal{T}$.

If G is a group, then the category of representations of G forms a symmetric monoidal category. The following theorem gives a two-dimensional analogue of this statement.

Theorem 17. *The collection* $\underline{R}$-cat *of representations of* $\underline{R}$ *in a* 2*-category* $\mathcal{T}$ *forms a* 2*-category.*

If $\mathcal{T}$ *is a symmetric monoidal* 2*-category, then* $\underline{R}$-cat *is a symmetric monoidal* 2*-category.*

This theorem looks well enough, but has little content if we cannot give examples of representations of double categories $\underline{R}$. In the remainder of this section, we give a handful of examples. In fact, when R is a ring, defining horizontal arrows which correspond to the arrow -1 in $\underline{R}$ raises some technical difficulties. It is simpler to work with slightly weaker structures than rings, such as the natural numbers $\mathbb{N}$. We can define a double category $\underline{\mathbb{N}}$ according to the recipe above. This double category has many natural representations.

To begin with, let us describe some representations of $\underline{\mathbb{N}}$ which take values in the 2-category of categories. Let A be an algebra, and M an A-module. The following data defines a representation of $\underline{\mathbb{N}}$ in the 2-category of categories:

- The object is A-mod.
- The horizontal arrows are endofunctors $h_r = - \oplus_k M^{\oplus r}$ of A-mod, for $r \in \mathbb{N}$; the vertical arrows are $- \otimes_k k^{\oplus s}$, for $s \in \mathbb{N}$.
- The functors $h_r v_s$ and $v_s h_{rs}$ are both naturally isomorphic to $- \otimes_k k^{\oplus s} \oplus M^{\oplus rs}$. The 2-arrow $\alpha_{r,s}$ describes the resulting natural isomorphism between these functors.

Let G be a finite group, and M a kG-module. We have further representations of $\underline{\mathbb{N}}$ in the 2-category of categories and given as follows:

- The object is kG-mod.
- The horizontal arrows are endofunctors $h_r = - \otimes_k M^{\otimes r}$ of kG-mod, for $r \in \mathbb{N}$; the vertical arrows are the non-additive endofunctors $-^{\otimes s}$, for $s \in \mathbb{N}$.
- The functors $h_r v_s$ and $v_s h_{rs}$ are both naturally isomorphic to $-^{\otimes s} \otimes_k M^{\otimes rs}$. The 2-arrow $\alpha_{r,s}$ describes the resulting natural isomorphism between these functors.

What about more complicated objects, such as $\mathbb{N}[\sqrt{2}]$? These too can have many representations. For example, if A is an algebra, and M is an A-module, then we obtain representations of $\underline{\mathbb{N}[\sqrt{2}]}$ as follows:

- The object is $A_1 \oplus A_{\sqrt{2}}$-mod.
- The horizontal arrows are endofunctors of $A_1 \oplus A_{\sqrt{2}}$-mod given by

$$h_{r+r'\sqrt{2}} = - \oplus \begin{pmatrix} M^{\oplus r} \\ \\ M^{\oplus r'} \end{pmatrix};$$

The vertical arrows are endofunctors of $A_1 \oplus A_{\sqrt{2}}$-mod given by

$$v_{s+s'\sqrt{2}} = \begin{pmatrix} k^{\oplus s} & k^{\oplus 2s'} \\ \\ k^{\oplus s'} & k^{\oplus s} \end{pmatrix} \otimes_k -$$

• The functors $h_{r+r'\sqrt{2}}v_{s+s'\sqrt{2}}$ and $v_{s+s'\sqrt{2}}h_{(r+r'\sqrt{2})(s+s'\sqrt{2})}$ are both naturally isomorphic to

$$\left(\begin{pmatrix} k^{\oplus s} & k^{\oplus 2s'} \\ & \\ k^{\oplus s'} & k^{\oplus s} \end{pmatrix} \otimes_k - \right) \oplus \begin{pmatrix} M^{\oplus r_1 s_1 + 2 r_2 s_2} \\ \\ M^{\oplus r_2 s_1 + r_1 s_2} \end{pmatrix}$$

The 2-arrow $\alpha_{r+r'\sqrt{2},s+s'\sqrt{2}}$ describes the resulting natural isomorphism between these functors.

The representations above do not take values in a symmetric monoidal 2-category. They take values in the 2-category of categories which, unlike the 2-category of algebras and bimodules, does not have any obvious monoidal structure. Ideally, we would like to define representations which take values in a symmetric monoidal 2-category with duals, so that the representation 2-categories themselves are symmetric monoidal 2-categories with duals, given the topological rigidity which is implied by such extra structure.

Here, we go so far as to define representations which take values in a symmetric monoidal category. Indeed, we define representations of $\underline{\mathbb{N}}$ with values in $\mathcal{T}_{alg}$ as follows:

Whenever we have a k-vector space X with a map $k \to X$, we define the infinite tensor product $X^{\otimes \mathbb{N}}$ to be the injective limit of the sequence

$$k \to X \to X^{\otimes 2} \to ...$$

which is obtained by tensoring up the map $k \to X$.

Let G be a group, and let M be a kG-module, and $k \to M$ a map of kG-modules.

Let $G_\infty = G \oplus G \oplus G \oplus ...$ be a direct sum of $\mathbb{N}$ copies of G. Let $F = kG_\infty$ be the group algebra of kG_∞, thought of as a kG_∞-kG_∞ bimodule. We have a map $k \to kG$ which takes 1 to the identity, and an isomorphism $F \cong kG^{\otimes \mathbb{N}}$.

For $s \in \mathbb{N}$, we have a group embedding ϕ_s of G_∞ in itself, which takes a group element $(g_1, g_2, g_3, ...)$ to $(g_1, ..., g_1, g_2, ..., g_2, g_3, ..., g_3, ...)$, where there are s copies of g_1 appearing in this list, s copies of g_2, and so on. Note that $\phi_s \phi_{s'} = \phi_{ss'}$.

We have a representation of $\underline{\mathbb{N}}$ in the symmetric monoidal category $\mathcal{T}_{alg}$ defined by the following data:

• The object is kG_∞.

• The horizontal arrows are kG_∞-kG_∞-bimodules $h_r = (kG \otimes_k M^{\otimes r})^{\otimes \mathbb{N}}$, for $r \in \mathbb{N}$, where $\underline{g} = (g_j)_{j \in \mathbb{N}}$ acts where it acts on the left of the i^{th} factor as

$$\underline{g}(x \otimes m_1 \otimes ... \otimes m_r) = (g_i x \otimes g_i m_1 \otimes ... \otimes g_i m_r),$$

and on the right of the i^{th} factor as

$$(x \otimes m_1 \otimes ... \otimes m_r)\underline{g} = (x g_i \otimes m_1 \otimes ... \otimes m_r);$$

the vertical arrows are the bimodules $v_s = {}^{\phi_s}F$, for $s \in \mathbb{N}$.

• The bimodules $h_r v_s$ and $v_s h_{rs}$ are both isomorphic to $(kG^{\otimes s} \otimes_k M^{\otimes rs})^{\otimes \mathbb{N}}$, on which $\underline{g}$ acts on the left of the i^{th} factor as

$$\underline{g}(x_1 \otimes ... \otimes x_s \otimes m_1 \otimes ... \otimes m_{rs}) = (g_i x_1 \otimes ... \otimes g_i x_s \otimes g_i m_1 \otimes ... \otimes g_i m_{rs}),$$

and on which $\underline{g}$ acts on the right of the i^{th} factor as

$$(x_1 \otimes ... \otimes x_s \otimes m_1 \otimes ... \otimes m_{rs})\underline{g} = (x_1 g_{si+1} \otimes ... \otimes x_s g_{si+s} \otimes m_1 \otimes ... \otimes m_{rs}).$$

The 2-arrow $\alpha_{r,s}$ describes the resulting natural isomorphism between these bimodules.

Remark 18 Since the group G_∞ is infinite for G nontrivial, the above representations do not take values in a symmetric monoidal category with duals $\mathcal{T}_{\mathcal{G}}$. For the representations to take value in a monoidal 2-category with duals we would have to define a suitable extension of $\mathcal{T}_{\mathcal{G}}$ in which infinite groups were included.

9. More structure on $\mathbb{N}$

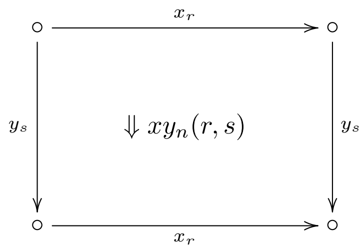
In the last section, we observed that rings live a rung up the categorical ladder from groups, and defined representation 2-categories associated to rings. The collection of natural numbers is somewhat special, and can be interpreted as living yet further up the categorical ladder, as we demonstrate in this section.

In $\mathbb{N}$, we consider multiplication to be a feature independent of addition, although it is of course related to addition. The relation between addition and multiplication is then described by distributivity, which finds a homotopical interpretation in $\underline{\mathbb{N}}$. However $\mathbb{N}$ has an additional feature: we can take powers. Whilst the taking of powers is related to multiplication, the taking of powers can be considered to an algebraic feature which is independent of multiplication.

Can we interpret the power operation, and its relations to multiplication and addition, in a homotopical manner? The answer is yes. We can define a triple category $\underline{\underline{\mathbb{N}}}$ which encodes the operations of addition, multiplication, and powers independently as follows:

In our triple category $\underline{\underline{\mathbb{N}}}$, we have a single object $\circ$. We have arrows in three independent directions, which we call x-arrows, y-arrows, and z-arrows. The x-arrows are given by elements x_r, for $r \in \mathbb{N}$, with product given by $x_r x_{r'} = x_{r+r'}$. The y-arrows are given by elements y_s, for $s \in \mathbb{N}$, with product given by $y_s y_{s'} = y_{ss'}$. The z-arrows are given by elements z_t, for $t \in \mathbb{N}$, with product given by $z_t z_{t'} = z_{tt'}$.

We have 2-arrows in three planes, labelled xy, yz, and xz. The xy 2-arrows



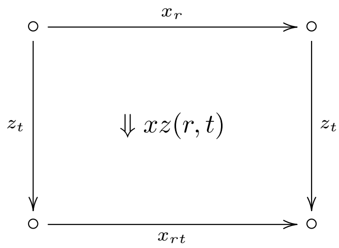
are squares $xy_n(r,s)$, for $r,s,n \in \mathbb{N}$, whose x-arrows are identically x_r, and whose y-arrows are identically y_s. The product in the x-direction is

$$xy_n(r,s)xy_{n'}(r',s) = xy_{nn'}(r+r',s).$$

The product in the y-direction is

$$xy_n(r,s)xy_{n'}(r,s') = xy_{nn'}(r,ss').$$

The xz 2-arrows



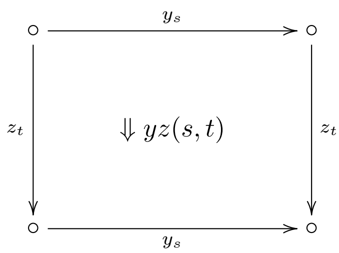
are squares $xz(r,t)$, for $r,t \in \mathbb{N}$, whose z-arrows are identically z_t, and whose x-arrows are x_r and x_{rt}. The product in the x-direction is

$$xz(r,t)xz(r',t) = yz(r+r',t).$$

The product in the z-direction is

$$xz(r,t)xz(r,t') = yz(r,tt').$$

The yz 2-arrows



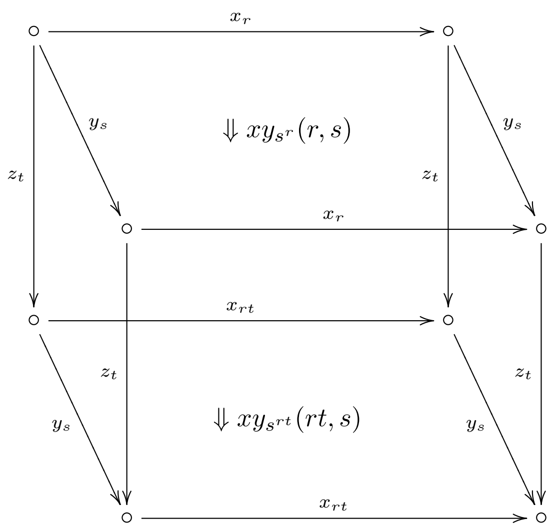
are squares $yz(s,t)$, for $s,t \in \mathbb{N}$, whose y-arrows are identically y_s, and whose z-arrows are identically z_t. The product in the y-direction is

$$yz(s,t)yz(s',t) = yz(ss',t).$$

The product in the z-direction is

$$yz(s,t)yz(s,t') = yz(s,tt').$$

Finally, we have 3-arrows $xyz(r,s,t)$, for $r,s,t \in \mathbb{N}$,



whose xz 2-arrows are identically $xz(r,t)$, whose yz 2-arrows are identically $yz(s,t)$, and whose xy 2-arrows are $xy_{s^r}(r,s)$ and $xy_{s^{rt}}(rt,s)$. The product in the x-direction is

$$xyz(r,s,t)xyz(r',s,t) = xyz(r+r',s,t).$$

The product in the y-direction is

$$xyz(r,s,t)xyz(r,s',t) = xyz(r,ss',t).$$

The product in the z-direction is

$$xyz(r,s,t)xyz(rt,s,t') = xyz(r,s,tt').$$

The nerve of $\underline{\underline{\mathbb{N}}}$ is a topological invariant of $\mathbb{N}$, which encodes some of its algebraic structure. If we wish to rear a more complicated beast, we can use higher products $\times_i$, which are defined on $\mathbb{N}$ recursively by $x \times_0 y = x+y$ and $x \times_i y = x \times_{i-1} x \times_{i-1} \ldots \times_{i-1} x$, where x appears y times in the formula. In particular, for $i \leq 2$ we have $x \times_0 y = x+y$, $x \times_1 y = xy$, $x \times_2 y = x^y$.

10. Alperin's weight conjecture.

In our introduction, we emphasised topological motivation for studying 2-categories. This is standard. However, there are algebraic reasons to take higher categories seriously lying in classical representation theory, which we wish to emphasise in our conclusion. Chuang and Rouquier's categorification of $\mathfrak{sl}_2$ gives us an example, since it provides solutions to deep problems in modular representation theory. To approach from a different angle, let us consider a great conjecture in modular representation theory. The conjecture is one of an ensemble of conjectures, due to a number of authors (Alperin & McKay, Alperin, Knörr & Robinson, Dade, Isaacs-Navarro, Uno). We state here Alperin's weight conjecture, which has character-theoretic reformulations due to Knörr and Robinson [22]:

Conjecture 19. *(Alperin,* [1]*) Let p be a prime number, and let k be an algebraically closed field of characteristic p. The number of isomorphism classes of irreducible representations over k of a finite group G, is equal to the sum over all conjugacy classes of p-subgroups P of G of the numbers of isomorphism classes of projective irreducible representations over k of $N_G(P)/P$.*

Note that this is a statement about collections of collections of irreducible modules, since there is a sum taken over all p-subgoups.

There is an important special case: when the Sylow p-subgroups of G are abelian. Suppose P is such a Sylow subgroup of G. Then Alperin's conjecture is equivalent to the statement that the number of isomorphism classes of irreducible modules of G is equal to the number of irreducible representations of $N_G(P)$. A conjecture of Broué predicts a much stronger statement holds: the bounded derived categories of kG and $kN_G(P)$ are equivalent. We could say that this conjecture of Broué is a categorification of Alperin's weight conjecture, since by passing to complexified Grothendieck groups, and taking dimensions, we find that the truth of Broué's conjecture would imply the truth of Alperin's, for groups whose Sylow p-subgroups are abelian. However, putting aside the desire to fling a piece of ugly jargon at everything to which it might apply, the word categorification seems a bit inappropriate in this context, since to state Broué's conjecture, we do not ascend the categorical ladder. A group can be thought of as a category with one object. The collection of representations of a group reflects this, being a category itself. General statements about group representations are therefore intrinsically categorical, and the conjecture of Broué gives a beautiful expression of this fact.

What about the general form of Alperin's weight conjecture, concerning groups whose Sylow p-subgroups are possibly nonabelian ? Is there a strong categorical statement which would imply this ? Here is where the canker gnaws: not yet. Since the weight conjecture concerns collections of collections of representations, we should expect that any homological strengthening of it will concern categories of categories, or more properly, 2-categories of categories. This is a suggestion from modular group representation theory that we should look to understand the 2-categorical landscape better. The Alperin weight conjecture is so simple to state, and so general, it suggests there is some vital homological principle lurking beneath its surface. There are approaches to prove it via the classification of finite simple groups, but these seem to defy the existence of such a principle.

The results described in this paper are concerned with classical mathematical objects, such as groups and rings. However, our efforts do little to reach towards the weight conjecture. Our reason for mentioning the conjecture here is that it is a bright star in the firmament, which belongs the galaxy of 2-categories.

References

1. J. L. Alperin, *Weights for finite groups*, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), Proc. Sympos. Pure Math., vol. 47, Amer. Math. Soc., Providence, RI, 1987, pp. 369–379. MR MR933373 (89h:20015)

2. Michael Atiyah, *Topological quantum field theories*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 68, 175–186 (1989). MR MR1001453 (90e:57059)
3. John C. Baez and Alissa S. Crans, *Higher-dimensional algebra. VI. Lie 2-algebras*, Theory Appl. Categ. **12** (2004), 492–538 (electronic). MR MR2068522 (2005m:17039)
4. John C. Baez and James Dolan, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. **36** (1995), no. 11, 6073–6105. MR MR1355899 (97f:18003)
5. John C. Baez and Laurel Langford, *Higher-dimensional algebra. IV. 2-tangles*, Adv. Math. **180** (2003), no. 2, 705–764. MR MR2020556 (2005f:57010)
6. John C. Baez and Martin Neuchl, *Higher-dimensional algebra. I. Braided monoidal* 2-*categories*, Adv. Math. **121** (1996), no. 2, 196–244. MR MR1402727 (97h:18009)
7. A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR MR751966 (86g:32015)
8. Joseph Bernstein, Igor Frenkel, and Mikhail Khovanov, *A categorification of the temperley-lieb algebra and schur quotients of $U(\mathfrak{sl}_2)$ via projective and zuckerman functors*, Selecta Math. (N.S.) **5** (1999), no. 2, 199–241. MR MR1714141 (2000i:17009)
9. J. Brundan and C. Stroppel, *Highest weight categories arising from khovanov's diagram algebra iii: category o*, arXiv:0812.1090.
10. Eugenia Cheng and Aaron Lauda, *Higher-dimensional categories: an illustrated guide book*, available at http://www.cheng.staff.shef.ac.uk/.
11. Joseph Chuang and Raphaël Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Ann. of Math. (2) **167** (2008), no. 1, 245–298. MR MR2373155 (2008m:20011)
12. Louis Crane and Igor B. Frenkel, *Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases*, J. Math. Phys. **35** (1994), no. 10, 5136–5154, Topology and physics. MR MR1295461 (96d:57019)
13. Sjoerd E. Crans, *Generalized centers of braided and sylleptic monoidal* 2-*categories*, Adv. Math. **136** (1998), no. 2, 183–223. MR MR1626844 (99j:18006)
14. Brian Day and Ross Street, *Monoidal bicategories and Hopf algebroids*, Adv. Math. **129** (1997), no. 1, 99–157. MR MR1458415 (99f:18013)
15. Michael R. Douglas, *D-branes, categories and $N = 1$ supersymmetry*, J. Math. Phys. **42** (2001), no. 7, 2818–2843, Strings, branes, and M-theory. MR MR1840318 (2003b:81158)
16. Charles Ehresmann, *Catégories structurées. III. Quintettes et applications covariantes*, Topol. et Géom. Diff. (Sém. C. Ehresmann), Vol. 5, Institut H. Poincaré, Paris, 1963, p. 21. MR MR0191939 (33 #166)
17. I. Frenkel, *Representation theory beyond affine lie algebras: a perspective*, http://www.msri.org/publications/ln/msri/2002/ssymmetry/ifrenkel/1/index.html.
18. V. Ginzburg, *Calabi-yau algebras*, arXiv:math/0612139v3.
19. M. M. Kapranov and V. A. Voevodsky, 2-*categories and Zamolodchikov tetrahedra equations*, Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991), Proc. Sympos. Pure Math., vol. 56, Amer. Math. Soc., Providence, RI, 1994, pp. 177–259. MR MR1278735 (95f:18011)
20. Mikhail Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426. MR MR1740682 (2002j:57025)
21. ______, *A functor-valued invariant of tangles*, Algebr. Geom. Topol. **2** (2002), 665–741 (electronic). MR MR1928174 (2004d:57016)
22. Reinhard Knörr and Geoffrey R. Robinson, *Some remarks on a conjecture of Alperin*, J. London Math. Soc. (2) **39** (1989), no. 1, 48–60. MR MR989918 (90k:20020)
23. Joachim Kock, *Frobenius algebras and 2D topological quantum field theories*, London Mathematical Society Student Texts, vol. 59, Cambridge University Press, Cambridge, 2004. MR MR2037238 (2005a:57028)
24. Tom Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004. MR MR2094071 (2005h:18030)
25. R. Rouquier, 2-*kac-moody algebras*, arXiv:0812.5023v1.
26. Raphaël Rouquier, *Block theory via stable and Rickard equivalences*, Modular representation theory of finite groups (Charlottesville, VA, 1998), de Gruyter, Berlin, 2001, pp. 101–146. MR MR1889341 (2003g:20018)
27. ______, *Dimensions of triangulated categories*, J. K-Theory **1** (2008), no. 2, 193–256. MR MR2434186
28. Catharina Stroppel, *Categorification of the Temperley-Lieb category, tangles, and cobordisms via projective functors*, Duke Math. J. **126** (2005), no. 3, 547–596. MR MR2120117 (2005i:17011)

29. V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, vol. 18, Walter de Gruyter & Co., Berlin, 1994. MR MR1292673 (95k:57014)