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Deformations, Tiltings, and Decomposition Matrices

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Dedicated to Professor Vlastimil Dlab

Abstract. Let $\tilde{A}$ be a deformation of a k-algebra A_0 over an integral domain R, such that $\overline{A} = \tilde{A} \otimes_R K$ is a separable algebra for $K = \mathrm{QF}(R)$. When there is a lifting of a tilting complex T for A to a tilting complex $\tilde{T}$ for $\tilde{A}$, it defines a separable deformation $\tilde{B}$ of $B = \mathrm{End}_{D^b(\tilde{A})}(T)$, i.e., $\overline{B} = \tilde{B} \otimes_R K$ is also separable. Then the decomposition matrix $D_{\tilde{B}}$ of $\tilde{B}$ is obtained from the decomposition matrix $D_{\tilde{A}}$ of $\tilde{A}$ by operations on the columns determined by the summands of $\tilde{T}$ and multiplication of certain rows by (-1). We use this method on the Broué conjecture for the principal block of A_6, giving an explicit one-step tilting complex of length 3.

1 Introduction

Let A be a finite dimensional k algebra, where k is a field sufficiently large that $A/\operatorname{Rad} A$ is separable. For simplicity one might assume k algebraically closed and A basic, but we do not so in order to allow applications to the theory of representations of groups. The various A-modules will be finitely-generated.

Let R be an integral domain with a distinguished maximal ideal m such that $R/m \xrightarrow{\sim} k$, and a quotient field K.

Definition A *deformation* $\tilde{A}$ of A over (R, m) is a flat R-module $\tilde{A}$ together with an isomorphism

$$A \xrightarrow{\sim} \tilde{A} \otimes_R R/m.$$

The deformation is called *separable* if

$$\bar{A} = \tilde{A} \otimes_R K$$

is a separable algebra. We note that $\tilde{A}$ itself will generally not be separable.

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There are various examples in the literature of k-algebras with a separable deformation. Brauer tree algebras have a separable deformation over $k[t]$, as to various algebras for which the Donald-Flanigan problem has been solved. Group blocks have a separable deformation over a complete discrete valuation ring with residue field of characteristic p and quotient field of characteristic 0.

The examples given so far have been symmetric, but this is not necessary. The algebra A of dimension four whose quiver consists of two vertices and two arrows in opposite directions deforms to a matrix algebra. The algebra A is self injective but not symmetric. The join of two copies of A by identifying one vertex of each and adding a loop is not self injective but deforms to two matrix algebras.

2 Decomposition Matrices

In the sequel we consider identity idempotents e' of matrix algebras $M_d(k)$, and define $\deg e' = d$. This will be called "the degree of the idempotent". When we refer to a position in a complex of modules, we will write "the degree in the complex".

Assume for this section that R is either a ring with the idempotent lifting property, e.g.,a complete discrete valuation ring, or is the coordinate ring of an affine curve, e.g., $k[t]$. Let $\tilde{A}$ be an R-algebra which is a deformation of a finite-dimensional k-algebra A over (R,m), with $R/m \xrightarrow{\sim} k$. We assume that k is sufficiently large to ensure that

$$A/\operatorname{Rad}(A) \xrightarrow{\sim} \bigoplus_{i=1}^{\ell} M_{n_i}(k).$$

We know that $A/\operatorname{Rad}(A)$ can be embedded noncanonically in A. We choose such an embedding with image S, and let $e_1,\dots,e_\ell$ be the images of the identity elements of the matrix blocks. The $\{e_i\}$ form a set of orthogonal idempotents, which is complete, i.e., which sum to the identity of A. They are not all primitive unless all $n_i = 1$. Then, taking an etale cover of R if necessary when R is the coordinate ring of a curve, we may assume that the idempotents lift to $\tilde{e}_1,\dots,\tilde{e}_n$ in $\tilde{A}$, an orthogonal set of idempotents (generally not primitive even if all $n_i = 1$). This set is still complete because the lifting of the identity element of A to $\tilde{A}$ is unique.

A vector space basis of an algebra is called "well behaved" with respect to a set of idempotents if it is a union of bases for the different summands in a two-sided Peirce decomposition. We note that by various "straightening out theorems", a k-vector space basis well behaved w.r.t. $\{e_i\}$ lifts to a basis of $\tilde{A}$ well behaved w.r.t. $\{\tilde{e}_i\}$. This leads in the affine case to the main theorem of [10] that a degeneration of algebras corresponds to a degeneration of basis graphs. In particular, we can choose an algebra $S_R \xrightarrow{\sim} \bigoplus_{i=1}^{\ell} M_{n_i}(R)$ of $\tilde{A}$, such that $\tilde{e}_i$ is the identity element of $M_{n_i}(R)$. We write $\deg \tilde{e}_i = \deg e_i = n_i$.

Definition If A is an algebra with separable deformation $\tilde{A}$ over (R,m), $\tilde{e}_1,\dots,\tilde{e}_\ell$ are liftings of the identities of subalgebras S_i, and $f_1,\dots,f_m$ are the block idempotents of the blocks $\tilde{A}\otimes_R K$, then we define the decomposition matrix

$$D_{\tilde{A}} = [d_{ij}]^{m\times\ell}$$

where

$$d_{ij} = \frac{\deg f_i\tilde{e}_j}{\deg \tilde{e}_j}.$$

Note The f_i are canonical. Since we can refine $\{e_i\}$ to a complete orthogonal set of primitive idempotents, and such a set is determined up to conjugation by a unit in the algebra, this definition is independent of the choice of the e_i and of the choice of the lifting $\tilde{e}_i$. It is not difficult to see that for group blocks this coincides with standard definitions of the decomposition matrix, i.e., d_{ij} is the multiplicity of the Brauer character corresponding to e_i as a constituent in the ordinary character corresponding to f_i.

3 Tilting and Decomposition Matrices

We now assume our algebra A can be tilted to an algebra B. We recall the relationship between tilting and deformation for any deformation, not only for a separable deformation. First assume that T is a classical tilting module, and $\tilde{A}$ a deformation of k-algebras. It was proven in [2] that over an etale cover of an affine

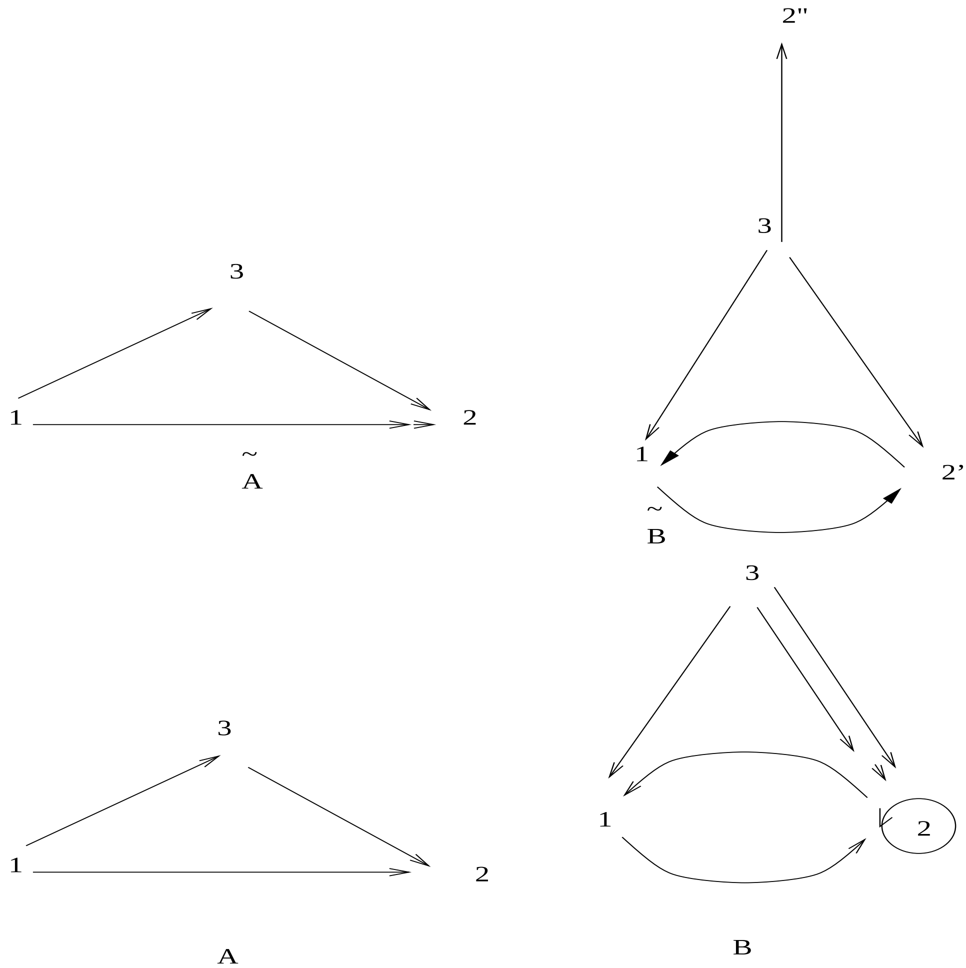

Figure 1

curve there is a deformed tilting module $\tilde{T}$ of $\tilde{A}$, lifting T, such that if we define $\tilde{B} = \mathrm{End}_{\tilde{A}}(\tilde{T})$, then $\tilde{B}$ is a flat deformation of $B = \mathrm{End}_A(T)$.

Example [2] We give an example of an algebra A with a tilting to B and a deformation to $\tilde{A}$. The diagrams in Figure 1 give the algebras $A, \tilde{A}, B, \tilde{B}$ in terms of the "basis graph" which is a directed graph, with a vertex for every idempotent in a complete orthogonal set of primitive idempotents and arrows for every basis element in $\mathrm{Rad}(A)$, according to the Peirce decomposition. The number of arrowheads indicate the depth in the radical series. In all four of the examples above, the basis graph determines the isomorphism class of the algebra. The basis graph behaves well under flat deformation (which preserves dimension) whereas the quiver behaves badly. The algebra $\tilde{A}$ is a copy of the Dynkin quiver algebra A_3 with linear orientation. In $\tilde{B}$ the arrows with solid arrowheads represent matrix units, and thus $\tilde{B}$ is Morita equivalent to the Dynkin quiver algebra A_3 with the middle vertex a source.

Jeremy Rickard [6] generalized the theorem in [2] in two directions: he allowed a tilting complex instead of just a tilting module, and he also allowed a wider range of parameter rings. We assume henceforward that T is a tilting complex of projective modules as defined in [5] and that $\tilde{A}$ is a separable deformation. Although Rickard's Theorem 4.1 [6] treats the case where R is a ring of an affine curve, we will give only the local case treated in Theorem 3.3 for simplicity of presentation.

Theorem *Let R be a complete commutative Noetherian local ring, and m the maximal ideal such that $R/m \xrightarrow{\sim} k$. Let A be a k-algebra with separable deformation $\tilde{A}$ and projective indecomposable modules $P_1, \ldots, P_\ell$. Let $T = \oplus T_j^*$ be a tilting complex for A with pairwise non-isomorphic summands. For each degree d in the complex T for which T has a non-zero term, set*

$$T_j^{*d} = \bigoplus_{t=1}^{t_d} Q_{t,j}^d,$$

where each $Q_{t,j}^d$ is isomorphic to an indecomposable projective module $P_{\sigma(j,t,d)}$. Let $\tilde{T}$ be a tilting complex for $\tilde{A}$ which specializes to T. Let $\tilde{B} = \mathrm{End}_{D^b(\tilde{A})}(\tilde{T})$. Then $\tilde{B}$ is also a separable deformation and for a suitable renumbering of the simples of B and of $\overline{B} = \tilde{B} \otimes_R K$,

$$D_{\tilde{B}} = S \cdot D_{\tilde{A}} M,$$

where S is a diagonal matrix with entries ± 1, and M is an invertible square matrix of integers with entries

$$m_{ij} = \sum_d (-1)^d \sum_t^{t_d} \delta_{i\sigma(j,t,d)}.$$

Proof By Rickard's Theorem 3.3 in [6], there is a unique tilting complex $\tilde{T}$ for $\tilde{A}$ such that $\tilde{T} \otimes_R k \xrightarrow{\sim} T$. Set $\overline{T} = \tilde{T} \otimes_R K$. Scalar extension preserves projectives, so $\overline{T}$ is still a bounded complex of finitely-generated projectives. Any chain map in $\mathrm{Hom}(\overline{T}, \overline{T}[v])$ determines a map in $\mathrm{Hom}(\tilde{T}, \tilde{T}[v])$ up to division by an element of R, so $\overline{T}$ satisfies the first condition for a tilting complex, and is thus a partial tilting complex. A partial tilting complex for a separable algebra must be homotopic to a complex $\hat{T}$ in which each simple module appears in a unique degree, since otherwise the condition

$$\mathrm{Hom}(\overline{T}, \overline{T}[v]) = 0, \quad v \neq 0$$

would not hold. Since the summands of $\tilde{T}$ generate the triangulated category $D^b(\tilde{A})$, and thus each projective module as a stalk complex, every simple of $\overline{A}$ can be generated, so $\overline{T}$ is a tilting complex for $\overline{A}$. Let d_i be the degree in the complex T in which N_i occurs in the complex $\hat{T}$ homotopic to $\overline{T}$. The degree of the corresponding simple module in $\tilde{B} \otimes K$ will be the total number of copies of N_i in degree d_i, after we have cancelled out the copies in other degrees via homotopy. We have taken only one copy of each summand in forming T, so this total multiplicity is the sum of the multiplicities for each summand T_j^*, i.e.,

$$d_i' = \sum d_{ij}'.$$

Since maps in a complex of modules for a separable algebra are the sum of zero maps and isomorphisms and since the isomorphisms cancel out under homotopy, the maps are irrelevant.

Let us fix N_i and T_j^*, and try to calculate d_{ij}'. Let d be a degree in the complex, and let $Q_{t,j}^d$ be one of the summands in T_j^{*d}, in degree d, isomorphic to $P_{\sigma(j,t,d)}$ for some index $s = \sigma(j,t,d)$ in $\{1, \ldots, \ell\}$. The number of copies of N_i in P_s is d_{is}, where $D_{\tilde{A}} = [d_{is}]$. Thus we get

$$d_{ij}'^{\,d} = \sum_{t=1}^{t_d} \sum_{s=1}^{\ell} (-1)^{d-d_i} d_{is} \delta_{s\sigma(j,t,d)}, \tag{1}$$

$$d_{ij}' = (-1)^{d_i} \sum_{d} \sum_{t=1}^{t_d} \sum_{s=1}^{\ell} (-1)^d d_{is} \delta_{s\sigma(j,t,d)}. \tag{2}$$

We now claim that d_{ij}' is in fact the (ij)-th entry in $D_{\tilde{B}}$. The idempotents corresponding to the simple modules are $\Pi_{T_j^*}$, the projections onto the j-th summand. Because we have taken a single copy of each summand, the degree of the idempotent $\Pi_{T_j^*}$ is one. We choose a complex homotopic to $\overline{T}$ in which all maps are zero. The endomorphism of $\overline{T}$ corresponding to N_i' is the projection on the sum of copies of N_i, which we denote $\overline{\Pi}_i^*$. The composition of the two idempotents $\overline{\Pi}_i \circ \Pi_{T_j^*}$ is the projection on the sum of the copies of N_i in $T_j^{*\prime}$. The degree of the idempotent $\overline{\Pi}_i \circ \Pi_{T_j^*}$ is the number of copies of N_i in this complex, i.e., d_{ij}'. This is then the (ij)-th entry in $D_{\tilde{B}}$. Let $s_i = (-1)^{d_i'}$ and let S be the diagonal matrix with entries s_i. Formula (2) for the entry d_{ij}' of $D_{\tilde{B}}$ shows that

$$D_{\tilde{B}} = S \cdot D_{\tilde{A}} \cdot M$$

where the (u,j) entry of M is $\sum_d \sum_t (-1)^d \delta_{u\sigma(j,t,d)}$. □

4 Factorizations into Elementary Tiltings for Symmetric Algebras

The matrix M does not actually determine the tilting complex. There are different possible "foldings" of the tilting complex, and different foldings correspond to different images of the simple modules under the induced stable equivalence. We will, in particular, be interested in factoring M into steps corresponding to tilting complexes of the following type, based on [7] and [4], where the "folding" is indicated by the partition $I' \cup I''$ of $I - I_0$, [8], [11], [12].

Definition Let I be the set of simples and consider a partition $\lambda = (I_0, I', I'')$, $I = I_0 \cup I' \cup I''$. If P_i, $i \in I$, are the indecomposable projectives, we define

$$\begin{array}{lll} P_i^* : & 0 \to P_i \to 0 & i \in I_0 \\ P_i^* : & 0 \to P_i \xrightarrow{g_i} Q_i \to 0 & i \in I' \\ P_i^* : & 0 \to R_i \xrightarrow{f_j} P_i \to 0 & i \in I'' \end{array}$$

where Q_i is the injective hull of the quotient of P_i by the largest submodule not containing elements from I_0, and R_i is the projective cover of the submodule of P_i whose cokernel is the largest quotient which does not contain simples from I_0.

Definition A map is *tight* if it does not have a nontrivial factorization through any sum of P_t with $t \in I_0$. Note that the maps in the P_i^* are tight.

If both I' and I' are nonempty, we require the following three conditions:
Condition 1: If $i \in I'$, $j \in I''$, and the composition $P_i \to Q_i \to R_j \to P_j$ is zero, then one of the two compositions of length two is zero.
Condition 2: If $t, s \in I_0$, $i \in I'$, $j \in I''$, and if $f_j : R_j \to P_j$ and $g_i : P_i \to Q_i$ are tight maps with a commutative square

$$\begin{array}{ccc} R_j & \xrightarrow{f_j} & P_j \\ \downarrow & & \downarrow \\ P_i & \xrightarrow{g_i} & Q_i \end{array}$$

then it is homotopic to zero, i.e., the diagonal map factors through a map from P_j to P_i.
Condition 3: If $i \in I'$, there are no S_j with $j \in I''$ in $\ker(m_i)$, and if $j \in I''$, there are no simples S_i with $i \in I'$ in $\operatorname{coker}(m_j)$.

Note In the example below we will see that in the cyclic defect case, Condition 3 implies Conditions 1 and 2.

Remark It is clear from the proof below that Condition 3 could be replaced by a weaker Condition 3′ which for each i, j would require either no S_j in $\ker(m_i)$ or no S_i in $\operatorname{coker}(m_j)$. However, since the algebra is symmetric, Conditions 3 and 3′ are equivalent.

Proposition 1 *If the partition λ satisfies Conditions 1–3, then $T(\lambda) = \oplus P_i^*$ is a tilting complex.*

Proof Since Q_i and R_i are sums of copies of projectives P_i with $i \in I_0$, we easily construct mapping cones for the other P_j, i.e., $j \in I' \cup I''$, which will generate these P_j. Thus the summands of P^* generate $D^b(A)$ as a triangulated category.

Thus we need only to show that $\operatorname{Hom}(T(\lambda), T(\lambda)[m]) = 0$ for $m \neq 0$.

We begin with summands

$$\begin{array}{ccc} & & P_j \\ & & \downarrow \\ R_i & \xrightarrow{f_i} & P_i \end{array}$$

Since P_j is projective and there is no map from P_j into coker f_i then any map must factor through R_i and thus is homotopic to zero. Dually, if there is a map

$$\begin{array}{l} P_j \\ \downarrow \\ P_i \xrightarrow{g_i} Q_i \end{array}$$

it can only be well defined if the composition is zero but then P_i maps into the kernel of g_i, and the map must be zero. Since each projective is also injective, we have no maps in the opposite direction either, by a dual argument.

The same arguments used above show that

$$\begin{array}{ccc} & R_i & \to P_i \\ & \downarrow & \\ R_j \xrightarrow{f_i} P_j & & \end{array}$$

is homotopic to zero, because the fact that coker(f_j) contains no simples from I_0 means that we can factor on the left. In the case of the opposite side

$$\begin{array}{ccc} R_i \to P_i & & \\ & \downarrow^{\ell} & \\ & R_j & \to P_j \end{array}$$

if it is well defined, both compositions with ℓ are zero. However, we have already shown above that if $\ell \circ f_i = 0$, then $\ell = 0$. The case on this opposite side is dual.

We are left with the “mixed” conditions. We begin with a well-defined chain map

$$\begin{array}{ccc} P_i & \xrightarrow{g_i} & Q_i \\ {\scriptstyle \ell_1}\downarrow & & \downarrow{\scriptstyle \ell_2} \\ R_j & \xrightarrow{f_j} & P_j \end{array}$$

We need to prove that there is a homotopy $h : Q_i \to R_j$ which gives $\ell_1 = h \circ g_i$ and $\ell_2 = f_j \circ h$. We first note that since coker f_j contains no copies of simples from I_0, there is an h_2 such that $\ell_2 = f_j \circ h_2$. If $\ell_1 = h_2 \circ g_i$, then we are finished. If not, we can replace ℓ_2 by 0 and ℓ_1 by $\ell_1' = \ell_1 - h_2 \circ g_i$. Since ker g_i contains no simples from I_0, there exists a factorization $\ell_1' = h_1 \circ g_i$. Since $f_j \circ h_1 \circ g_i = 0$ and $h_1 \circ g_i \neq 0$, we conclude by Condition 1 that $f_j \circ h_1 = 0$, and thus h_1 is a homotopy for the new chain map ℓ_1'. Thus $h = h_1 + h_2$ is a homotopy for the original chain map.

We must now consider the opposite direction

$$\begin{array}{ccc} R_j & \to & P_j \\ \downarrow & & \downarrow \\ P_i & \to & Q_i \end{array}$$

By Condition 2, the composition factors through a homomorphism $P_j \to P_i$ and thus the map is homotopic to zero, as desired.

Finally, we have the double shifts. First consider

$$\begin{array}{ccc} & & P_i \xrightarrow{g_i} Q_i \\ & & \ell \downarrow \\ R_j \xrightarrow{f_j} & P_j & \end{array}$$

By Condition 3, the simple S_i is not in the cokernel of f_j, and thus ℓ factors through R_j. Dually, the simple S_j is not in $\ker(g_i)$ and thus ℓ factors through Q_i. In either case the chain map is holomorphic to zero.

In the opposite direction

$$\begin{array}{ccc} R_j \xrightarrow{f_j} & P_j & \\ & \ell \downarrow & \\ & P_i \xrightarrow{g_i} Q_i & \end{array}$$

If S_j is not in $\ker(g_i)$, then $g_i \circ \ell$ is non-zero, and if S_i is not in $\mathrm{coker}(f_j)$, then $\ell \circ f_j$ is non-zero. In either case the chain map is not well-defined. □

Example If λ is a partition for the indices of a Brauer star algebra with indices $\{1, \ldots, e\}$, then a partition satisfying the three conditions will be one in which intervals of I', I_0, I'' alternate cyclically. Suppose $e = 12$, $I_0 = \{2, 4, 5, 11\}$, $I' = \{1, 9, 10\}$, $I'' = \{3, 6, 7, 8, 12\}$. The resulting Brauer tree algebra has the form

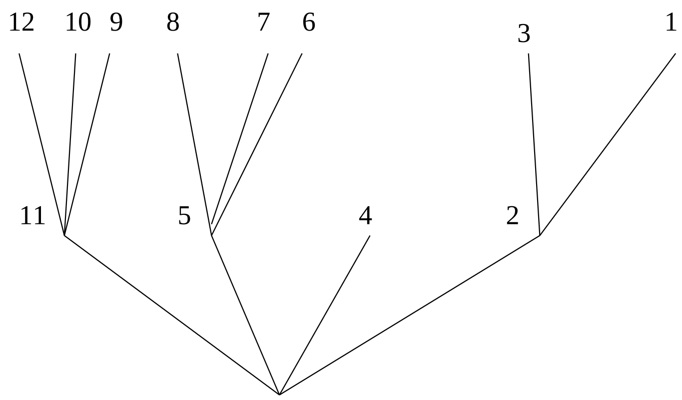

Figure 2

where the edges u at the exceptional vertex correspond to elements of I_0, the edges connected to u and less than u correspond to elements of I', and those numbered greater than u belong to I''. In [9], we show that the Green correspondents of the simple modules are determined by the I' and I''.

5 The Broué Conjecture

Let us consider the application of our main theorem to an example for the Broué conjecture. In the case of blocks of group algebras, our main theorem is a consequence of Theorem 3.1 in [1]. The principal blocks for a number of simple groups have been handled by Okuyama, so we consider one of them, namely that of A_6, for $p = 3$. The conjecture has already been settled for the block by Okuyama

[4] using different methods, but here we will give the tilting complex explicitly. The normalizer of the defect group is C_4 acting on $C_3 \times C_3$. The quiver of the principal block is

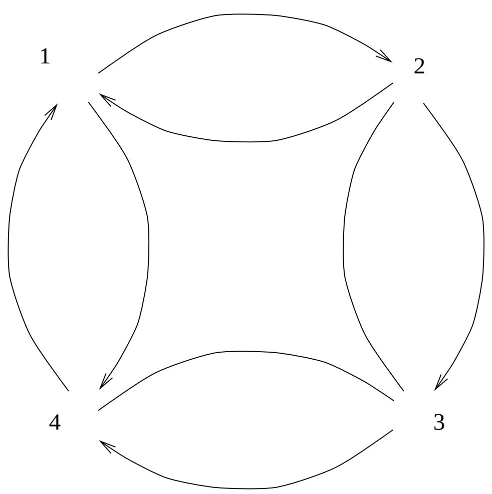

Figure 3

Let α_i be the clockwise arrow starting at i, and β_i the counterclockwise arrow starting at i. We have relations $\alpha^3 = 0$, $\beta^3 = 0$, $\alpha\beta = \beta\alpha$, for

$$\alpha = \sum \alpha_i$$

$$\beta = \sum \beta_i.$$

The decomposition matrix is

$$D_{\tilde{b}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The decomposition matrix of the block of A_6 is given, after a permutation of rows to bring the exceptional character to the bottom, by

$$D_{\tilde{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

There is a stable equivalence (induced by Green correspondence), given by

$$X_1 = 1 \qquad X_2 = \begin{matrix} & 2 \\ 3 & \\ 4 & \end{matrix} \qquad X_3 = \begin{matrix} & 3 & \\ 4 & & 2 \\ & 3 & \end{matrix} \qquad X_4 = \begin{matrix} 4 & \\ 3 & \\ & 2 \end{matrix}$$

Using Okayama's strategy of choosing I_0 with large Green correspondent, we take $I_0 = \{3\}$, $I' = \emptyset$, $I'' = \{1,2\}$ and get the tilting complex

$$\begin{array}{llcl} P_1^* : & 0 \to P_3 \oplus P_3 & \to P_1 \\ P_2^* : & 0 \to \quad P_3 & \to P_2 \\ P_3^* : & 0 \to \quad P_3 & \to 0 \\ P_4^* : & 0 \to \quad P_3 & \to P_4 \end{array}$$

The effect on the decomposition matrix is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

We now choose $I_0 = \{2,4\}$, $I' = \emptyset$, $I'' = \{1,3\}$. The map from P_2^* and P_4^* to P_1^* are isomorphisms on P_3, and similarly with the maps from P_2^* and P_4^* to P_3^*. So when we take mapping cones we get

$$\begin{array}{ll} P_1'^* : & \quad P_2 \oplus P_4 \to P_1 \\ P_2'^* : & P_3 \to P_2 \\ P_3'^* : & P_3 \to P_2 \oplus P_4 \\ P_4'^* : & P_3 \to P_4 \end{array}$$

The new matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

It differs from $D_{\tilde{B}}$ only by a permutation of the columns.

Okuyama [4] proves, using his Lemma 2.1, that this sequence of elementary tiltings produces the desired tilting. We comment that, given a candidate tilting complex such as P'^*, one can check directly whether or not it sends each of the X_i to the corresponding simple S_i. Let X_i^* be a projective resolution of X_i. For each X_i, there is a unique degree n_i such that $\mathrm{Hom}(P'^*, X_i^*[n_i]) \neq 0$, modulo homotopy. In this example, the degrees are $n_1 = 0$, $n_2 = 1$, $n_3 = 2$, $n_4 = 1$. By Okuyama's Lemma 1.3 (2), it then suffices to check that $\Omega^{n_i}(\mathrm{Hom}(P'^*, X_i^*[n_i]))$ is the simple module corresponding to $P_i'^*$, plus projectives.

We hope to turn this into a general method.

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