BRAID GROUP ACTION ON THE REFOLDED TILTING COMPLEXES OF THE BRAUER STAR ALGEBRA

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Abstract. It was shown in earlier work that every Brauer tree algebra of type (e, m) can be reached from the Brauer star algebra of type (e, m) by a two-restricted tilting complex, i.e., a complex which is a direct sum of indecomposable complexes involving no more than two projectives. It was further shown that the different two-restricted tilting complexes correspond to an additional structure on the Brauer star, called a "pointing" which controls the "folding" of the complex. If we go out by one pointing and return by a different pointing, we get a self-equivalence of the Brauer star which we will call a "refolded" complex. In this paper, we show that the subgroup of the derived Picard group generated by the refolded complexes is in fact generated by certain elementary refolded complexes which satisfy the braid group relations for the braid group on the affine diagram $\tilde{A}_{e-1}$. The question of whether this action is faithful and whether the refolded complexes generate the entire derived Picard group remains open.

§1. Introduction.

If A is an algebra over a commutative noetherian ring k, the Picard group $\mathrm{Pic}(A)$ is the group of isomorphism classes of invertible $A \otimes A^{\circ}$-modules. Lately, as more attention has been paid to the bounded derived category $D^b(A)$, this has aroused interest in a corresponding concept in the derived category.

Let X be a bounded complex of left $A \otimes A^c irc$-modules which are projective both as A and as A° modules. We use the symbol "$\otimes$" to denote "$\otimes_k$". The bounded complex X is called *invertible* if there is a bounded complex Y of $A \otimes A^{\circ}$-modules (projective as A and A° modules) satisfying

$$X \otimes_A Y \xrightarrow{\sim} A \text{ in } D^b(A \otimes A^{\circ})$$

$$Y \otimes_A X \xrightarrow{\sim} A \text{ in } D^b(A \otimes A^{\circ}).$$

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Here A is identified with the stalk complex concentrated in degree zero.

The isomorphism classes of the invertible complexes X form a group under the operator $\otimes_A$. This group has been called the derived Picard group and denoted by $\operatorname{Tr}\operatorname{Pic}(A)$. A general discussion can be found in [RoZ] or in [Ye]. We embed $\operatorname{Pic}(A)$ into $\operatorname{Tr}\operatorname{Pic}(A)$ by sending an invertible $A \otimes A^\circ$-module M to a stalk complex concentrated in degree 0.

If M is an invertible $A \otimes A^\circ$ module, i.e., a representative of an element of $\operatorname{Pic}(A)$, then taking the tensor product over A with M gives a self-equivalence of the category $\operatorname{Mod}(A)$ of left A-modules with itself:

$$M \otimes_A - : \operatorname{Mod}(A) \to \operatorname{Mod}(A)$$

$$N \mapsto M \otimes_A N.$$

Similarly, if X is a representative of an element of $\operatorname{Tr}\operatorname{Pic}(A)$, then the tensor product over A with X gives a functor defining a self-equivalence of $D^b(A)$ with itself

$$X \otimes_A - : D^b(A) \to D^b(A).$$

Such a self-equivalence is called *a standard* self-equivalence. Thus in studying $\operatorname{Tr}\operatorname{Pic}(A)$, we are also studying certain self-equivalences of the derived category.

In this paper, we will be studying $\operatorname{Tr}\operatorname{Pic}(A)$ by considering a much larger class of complexes:

Definition. Let A and B be k-algebras. A *two-sided tilting complex* X from A to B is a bounded complex of $A \otimes_k B^\circ$-modules, projective both as A and as B°-modules, for which there exists a bounded complex Y of $B \otimes A^\circ$-modules, projective as B and as A°-modules, such that

$$X \otimes_B Y \xrightarrow{\sim} A \text{ in } D^b(A \otimes A^\circ),$$

and

$$Y \otimes_A X \xrightarrow{\sim} B \text{ in } D^b(B \otimes B^\circ).$$

The functors $Y \otimes_A -$ and $X \otimes_B -$ give equivalences of categories, (quasi) inverse to each other, between $D^b(A)$ and $D^b(B)$. Such equivalences are called *standard*, and it is not known if every equivalence between $D^b(A)$ and $D^b(B)$ is of this form.

We see from the definitions that the invertible complexes are simply the two-sided tilting complexes for which $A = B$. If X and X' are two-sided tilting complexes from A to B, and Y is the inverse of X, then $X' \otimes_B Y$ is the representative of an element of $\operatorname{Tr}\operatorname{Pic}(A)$. For any $[Z] \in \operatorname{Tr}\operatorname{Pic}(A)$, we can generate $[Z]$ as $[X' \otimes Y]$ by setting $X' = Z \otimes_A X$.

In this paper, we intend to study $\operatorname{Tr}\operatorname{Pic}(A)$ for a specific but very important algebra called the Brauer star algebra. The approach we will take is to study the subgroup $\mathcal{R}$ of $\operatorname{Tr}\operatorname{Pic}(A)$ generated by shifts, by $\operatorname{Pic}(A)$, and by isomorphism classes $[X' \otimes Y]$, where X' and Y have a very specific combinatorial relationship to each other. We manage to construct a large semigroup $\mathcal{R}^{\perp}$ inside $\mathcal{R}$, thus demonstrating that both $\mathcal{R}$ and $\operatorname{Tr}\operatorname{Pic}(A)$ are quite large. We also give an action of an affine braid group on $\mathcal{R}$.

The definition of $\mathcal{R}$ is based on previous work [Z], [SZ1], [SZ2], [SZ3] on classification of tilting complexes of the Brauer star algebra, and was inspired by a conversation with Zimmermann. The introduction of the braid group action is an extension of earlier work of Rouquier and Zimmermann [RoZ].

Before quoting the results of Rouquier and Zimmermann, we would like to review some notation about the bounded derived category $D^b(A)$. For any complex T of left A-modules, the shift functor $[n]$ shifts each component n places to the left, and multiplies the differential by $(-1)^n$. If X and Y are two complexes in $D^b(A)$, with differentials d_X and d_Y and $f : X \to Y$ is a chain map of complex, the *mapping cone*

$$\operatorname{Cone}(X \xrightarrow{f} Y)$$

is a complex whose component in degree i is $X_{i+1} \oplus Y_i$, and whose differential is given by

$$d^i : X_{i+1} \oplus Y_i \xrightarrow{\begin{bmatrix} -d_X^{i+1} & f^{i+1} \\ 0 & d_Y^i \end{bmatrix}} X_{i+2} \oplus Y_{i+1}.$$

The category $D^b(A)$ is a triangulated category in which the distinguished triangles are

$$X \to Y \to \mathrm{Cone}(X \to Y) \to X[1];$$

for the full definition, see [Ha]. What concerns us here is that any triangulated subcategory must be closed under taking shifts and mapping cones.

For actual calculations, we usually use not the two-sided tilting complex X, but rather a bounded complex of projective left A-modules T, which is isomorphic in $D^b(A)$ to X regarded only as a left A-module. The complex T is called either a *one-sided tilting complex*, to distinguish it from X, or simply a tilting complex. We give the definition in its usual form, going back to Rickard [R1].

Definition. Let A be a k-algebra. Let T be a bounded complex of finitely generated projective, left A-modules. T is called a *tilting complex* for A if

(1) $\mathrm{Hom}_{D^b(A)}(T, T[n]) = 0$ for $n \neq 0$

(2) The homotopy category $K^b(A)$ of bounded complexes of projectives is generated as a triangulated category by the direct summands of direct sums $\bigoplus_j T$, for all finite j.

Given a tilting complex T, let $B = \mathrm{End}_A(T)$. Let a two-sided complex of $A \otimes B^\circ$-modules, viewed as a complex of left A-modules, be represented by X. There exists a two-sided tilting complex X from A to B such that ${}_AX$ is isomorphic to T in $D^b(A)$. If X' is another two-sided tilting complex restricting to the same T, then there is an automorphism $\alpha \in \mathrm{Aut}(A)$ such that $X' =_\alpha A_1 \otimes_A X$, where ${}_\alpha A_1$ is the $A \otimes A^\circ$-module which is isomorphic to A except that the action of elements a from A is via multiplication by $\alpha(a)$. The module ${}_\alpha A_1$ is invertible with inverse ${}_1A_\alpha$, since ${}_\alpha A_1 \otimes_A {}_1A_\alpha \xrightarrow{\sim} A$. Thus if Y is the inverse of X, $[X' \otimes_B Y] \in \mathrm{Pic}(A)$.

For group algebras, the most basic case is that of blocks of cyclic defect group. Each such block corresponds to a tree, called the Brauer tree, with e edges, an exceptional vertex u_0 of multiplicity m, and a cyclic ordering on the edges at each vertex. The particular case where the tree is a star and the exceptional vertex is in the center is called the *Brauer tree.* The derived equivalence class depends on the numbers e and m. Rouquier and Zimmerman [RoZ] take as representative of a derived equivalence class the algebra with structure similar to that of $SL(2,p)$, with a linear tree and the exceptional vertex at the end. This provides a satisfactory theory when the multiplicity $m=1$ [KS]. For $e=2$, $\mathrm{Tr}\,\mathrm{Pic}(A)$ is generated, modulo shifts and $\mathrm{Pic}(A)$, by certain elementary self-equivalences which satisfy the relations of the braid group.

In this paper, in order to deal with the case $m \geq 1$, we present a different approach, taking as representative of the derived equivalence class the Brauer star algebra, to be defined in the next section. As in the case of $m=1$, certain group elements satisfying braid relations arise in a natural way. Since we will be using results from earlier papers in which the ground ring is a field, we will restrict to that case, though the result is presumably true more generally.

§2. The Brauer star algebra and its tilting complexes.

Let $e>1$ and $m \geq 1$ be integers. Let K be a field containing a primitive e-th root of unity. Let

$$S = K[x]/(x^{em+1})$$

be the truncated polynomial ring. The Brauer star algebra of type (e,m) is the skew group ring

$$A = S[C_e]$$

where $C_e = \langle d \rangle$ is a cyclic group of order e, and the automorphism of R is given by

$$d^{-1}xd = \xi x.$$

A has e uniserial indecomposable projectives $Q_1, \ldots, Q_e$, each generated by one of the primitive idempotents of KC_e. We number the indecomposables so that Q_i is the projective cover of the radical of Q_{i+1}, and designate the corresponding map by

$$h_{ii+1} : Q_i \to Q_{i+1}.$$

More generally, we denote by h_{ij} the A-homomorphism of maximal rank

$$h_{ij} : Q_i \to Q_j.$$

We denote by ε_i the non-identity map of maximal rank

$$\varepsilon_i : Q_i \to Q_i.$$

The map ε_i^m from Q_i to its socle will be denoted by s_i. In what follows we will drop the indices on h, s and ε, since it will always be clear from the context which projectives are involved.

Definition. [SZ1] A two-restricted tilting complex for A is a tilting complex whose irreducible direct summands are all shifts of the following complexes, with initial nonzero term in degree zero

$$S_i : 0 \to Q_i \to 0$$
$$T_{k\ell} : 0 \to Q_k \xrightarrow{h} Q_\ell \to 0.$$

These complexes and their shifts will be called *elementary.*

Let G be any Brauer tree of type (e, m), i.e., a tree with e edges and $e + 1$ vertices, such that one vertex u_0 has been designated as exceptional vertex and assigned multiplicity m, and there is given a designated cyclic ordering of the edges at each vertex, which we will indicate by embedding the tree G in the plane so that a counterclockwise circuit of the vertex determines the cyclic ordering on the edges. A *pointing* of the tree is the designation of a pair of adjacent vertices at each non-exceptional vertex, which we will indicate by marking the corresponding sector with a point.

Let us now choose one particular branch at the exceptional vertex, which we will designate by placing a point before that edge at the exceptional vertex. This point will be called the *enhancement point*. Let B' be the set of Brauer trees with enhancement points. A pointing together with an enhancement point will be called an *enhanced pointing*. Let B'' be the set of Brauer trees with enhanced pointings.

We then get a *standard numbering* of the vertices determined by the enhanced pointing. Starting at the point at the exceptional vertex we take a Green's walk around the tree, keeping the tree on our left [G]. We assign the numbers $1, \ldots, e$ to the non-exceptional vertices as we reach their points, and then assign the same number to the unique edge leading to that point.

The tilting complex $T = \oplus R_i$ corresponding to this enhanced pointing is built up recursively, starting at the exceptional vertex. For each edge from 0 to i, we set $R_i = S_i$. For any edge between i and j, with $0 < i < j$, assume we have defined R_i for every edge on a minimal path to the exceptional vertex. Then either R_i or R_j has been defined. If R_i has been defined, let R_j be that shift of T_{ij} for which Q_i is in the same degree in both R_i and R_j. If R_j has been defined, let R_j be that shift of T_{ij} for which Q_j is in the same degree in both R_i and R_j (see Example 1). All maps are the corresponding h_{ij}.

Example 1.

Fig. 1

$$
\begin{array}{llllll}
R_6: & & & & Q_6 & & \\
R_1: & & Q_1 & \to & Q_6 & & \\
R_5: & & Q_1 & \to & Q_5 & & \\
R_3: & & Q_3 & \to & Q_5 & & \\
R_2: & Q_2 \to & Q_3 & & & & \\
R_4: & & Q_4 & \to & Q_5 & & \\
R_7: & & & & Q_6 & \to & Q_7 \\
R_8: & & & & Q_8 & & \\
R_9: & & & & Q_8 & \to & Q_9
\end{array} \tag{1}
$$

The endomorphism ring of this tilting complex T is the Brauer tree algebra of the Brauer tree G^*. Considerable research has been done lately on the Brauer tree algebras and their self-equivalences [L], [M], [Ro1], [Ro2] [RoZ], [Z]. Definitions of this algebra appear in various forms in the literatures [A], [L], [M], but since we are concerned only with self-equivalences of the Brauer star we will not repeat them here. For R_i and R_j, $i \neq j$, adjacent at a non-exceptional vertex there is a unique map in each direction, with the composition mapping the top of R_i to its socle. At the exceptional vertex the dimension of the vector space $\mathrm{Hom}(R_i, R_i)$ is m for $i \neq j$ and $m+1$ for $i = j$.

We now give a construction of the B-module tilting complex $\hat{T}$ which gives a functor inverse to the functor defined by T.

Definition. [R1], [R2], [RS] The folded tree-to-star tilting complex $\hat{T}$ determined by a given enhanced pointing associates to each $i = 1, \dots, e$ a complex Q'_i determined by the path from the vertex i to the exceptional vertex, folded so that all indices are in ascending order. The complex can be determined recursively as follows: If $j_1, \dots, j_n$ is a path from the exceptional vertex to j_n, $n > 1$ then $Q'_{j_1} = R_{j_1}$, and Q'_{j_n} is obtained from $Q'_{j_{n-1}}$ by adding a copy of R_{j_n} at a displacement of one degree from $R_{j_{n-1}}$. If $j_n > j_{n-1}$, the copy of R_{j_n} is summed to the component of $Q_{j'_{n-1}}$ one degree to the left, and a map $\phi : R_{j_n} \to R_{j_{n-1}}$ is added to the differential, where ϕ is the unique morphism between R_{j_n} and $R_{j_{n-1}}$ in the direction of descending indices. If $j_{n-1} > j_n$, then R_{j_n} is summed to the component of $Q'_{j_{n-1}}$ one degree to the right, and the differential is adjusted by $\phi : R_{j_{n-1}} \to R_{j_n}$. The endomorphism ring of $\oplus Q'_i$ is then isomorphic to the Brauer star algebra, as was proven in [RS].

Example 2. We use the same Brauer tree as appeared in Example 1.

$$
\begin{aligned}
Q'_1 :&\quad R_6 \to R_1 \\
Q'_2 :&\quad R_6 \to R_1 \\
&\quad R_5 \stackrel{\nearrow}{\to} R_3 \to R_2 \\
Q'_3 :&\quad R_6 \to R_1 \\
&\quad R_5 \stackrel{\nearrow}{\to} R_3
\end{aligned}
\tag{2}
$$

$$Q'_4: \quad \begin{tikzcd} R_6 \arrow[r] & R_1 \\ R_5 \arrow[ur] \arrow[r] & R_4 \end{tikzcd}$$

$$Q'_5: \quad \begin{tikzcd} R_6 \arrow[r] & R_1 \\ R_5 \arrow[ur] & \end{tikzcd}$$

$$Q'_6: \quad R_6$$

$$Q'_7: \quad R_7 \to R_6$$

$$Q'_8: \quad R_8$$

$$Q'_9: \quad R_9 \to R_8$$

Any enhanced pointing of the tree determines an element σ of the symmetric group $\mathcal{S}_e$ given by $\sigma(i) = a_i$, where $a_1, \ldots, a_e$ are the edges encountered in a Green's walk around the tree [G]. In the example above, the permutation will be

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 1 & 5 & 3 & 2 & 4 & 7 & 8 & 9 \end{pmatrix} \tag{3}$$

By a theorem of Rickard, [R3], given a tilting complex T and an isomorphism $f : B \xrightarrow{\sim} \mathrm{End}_A(T)$, there is a unique two-sided tilting complex X from A to B such that $T \xrightarrow{\sim}_A X$ in $D^b(A)$, and such that, in the induced isomorphism $\theta : \mathrm{End}_A(T) \to \mathrm{End}_A({}_AX)$, the image of $f(b)$ for any $b \in B$ is the endomorphism of ${}_AX$ induced by multiplication by b. Keller has proven that this X is unique up to isomorphism in $D^b(A \otimes_k B^\circ)$ [Ke]. For any one-sided tilting complex T, we use this uniquely determined X to define a functor

$$F_T : D^b(A) \to D^b(B).$$

Now, given a Brauer tree with an enhanced pointing, determining a permutation σ, our algorithm determines a tilting complex T. Let X be the corresponding uniquely defined two-sided complex, and let

$$F_\sigma : D^b(A) \to D^b(B)$$

be the corresponding functor. We call F_σ a *tilting functor.* Since A is a symmetric algebra, $Y := \mathrm{Hom}_k(X,k)$ defines the inverse $\hat{F}_\sigma$ to F_σ.

Remark. Even if we had not chosen the particular unique X, the composition of the functors F_σ and $\hat{F}_\sigma$ would still have been an element of $\mathrm{Pic}(A)$, by Proposition 2.3 of [RoZ].

Let F^G_σ be the corresponding functor from $D^b(A)$ to $D^b(B)$ and let $\hat{F}^G_\sigma$ be the inverse functor given by the algorithm above.

Definition. A *basic refolded tilting functor* is a tilting functor giving a self-equivalence of the Brauer star algebra, corresponding to a functor $\hat{F}^G_\tau F^G_\sigma$, for two permutations σ, τ which correspond to different pointings of some Brauer tree G with a given enhancement point. A *refolded* tilting complex is a two-sided tilting complex corresponding to a composition of basic refolded tilting complexes. Note that $\hat{F}^G_\sigma F^G_\tau$ is inverse to $\hat{F}^G_\tau F^G_\sigma$. Let $\mathcal{R}$ denote the subgroup $TrPic(A)$ of generated by the shifts, $\mathrm{Pic}(A)$, and the refolded tilting complexes.

It is immediately clear that the set

$$\{\hat{F}^G_\tau F^G_{id} \mid G \in B',\ \tau \text{ is determined by a pointing of } G\}$$

of functors determine a generating set since $\hat{F}^G_\tau F^G_\sigma = (\hat{F}^G_\tau F^G_{id})(\hat{F}^G_\sigma F^G_{id})^{-1}$.

As our first result, we will construct a tilting complex corresponding to a functor $\hat{F}_\tau F_{id}$.

Proposition 1. *For any $\sigma \in S_e$ determined by a pointing of a tree $G \in B'$, the self-equivalence of the Brauer tree algebra A corresponding to the functor $\hat{F}^G_\sigma F^G_{id}$ is given by $\oplus Q'_i$, where*

$$\hat{F}^G F^G_{id}(Q_i) = Q'_i = (Q_{a_i} \xrightarrow{h} Q_{a_{j_{\ell-1}}} \xrightarrow{s} Q_{a_{j_{\ell-1}}} \xrightarrow{h} \dots Q_{a_{j_1}} \xrightarrow{s} \overset{\deg 0}{Q}_{a_{j_1}}),$$

where $i = j_\ell > j_{\ell-1} > \cdots > j_1$ *is a maximal descending sequence for which*

$$a_i < a_{j_{\ell-1}} < \ldots a_{j_1}.$$

Proof. By induction on the distance of i from the exceptional vertex. Note that the j_k are precisely the vertices at which the point is on the left as one goes out to i along a minimal path.

<u>$d=1$</u>: In this case the edge i is adjacent to the exceptional vertex. We get

$$Q_i \overset{F_{id}^G}{\mapsto} R_i = R'_{a_i} \overset{\hat{F}_\sigma^G}{\mapsto} \overset{\deg 0}{Q}_{a_i}.$$

<u>$d>1$</u>: Suppose that the proposition has already been proven for distance $d-1$, and that we have already shown that the sequence $i = j_\ell > \cdots > j_1$, is the maximal subsequence of the minimal path $i = i_n > i_{n-1} > \cdots > i_1$ for which the a_i are ascending. By the definition of the Green's walk for the pointing inducing the identity permutation, there is some s such that the minimal path for $i+1$ is

$$i+1 > i_s > i_{s-1} \cdots > i_1.$$

Case 1. $a_{i+1} < a_i$. This can occur only in the case $s = n$, since if $s < n$, i is on a later branch, and, by the numbering algorithm for the pointing inducing σ, must have a larger number than that of i. In the case $s = n$, by the assumption on $i = j_\ell > \cdots > j_1$, and the adjacency of $i+1$, we find that $i+1 > j_\ell > \cdots > j_1$ is the maximal descending sequence such that $a_{i+1} < a_i < \cdots < a_j$.

$$Q_{i+1} \overset{F_{id}^G}{\mapsto} R_{i+1} \to R_{i_n} \to \cdots \to \overset{\deg 0}{R}_{i_1}$$

By induction $(R_{i_n} \to \cdots \to R_{i_1}) \overset{\hat{F}_\tau^G}{\mapsto} Q'_i$.
Thus

$$\begin{aligned} Q_{i+1} \mapsto \operatorname{Cone}(R_{i+1} \to (R_{i_n} \to \cdots \to R_{i_1})) &\mapsto \operatorname{Cone}((Q_{a_{i+1}} \xrightarrow{h} Q_{a_i}) \xrightarrow{s} Q'_i) \\ &\mapsto (Q_{a_{i+1}} \xrightarrow{h} Q_{a_i} \xrightarrow{s} Q'_i) = Q'_{i+1}. \end{aligned}$$

Case 2. $a_{i+1} > a_i$. Find the smallest t such that $a_t < a_{i+1}$. If $t > 1$, then $a_t < a_{i+1} < a_{t-1}$. Thus the maximal sequence for $i+1$ is

$$i + 1 > j_{t-1} > \cdots > j_1.$$

Since $i+1$ follows immediately after i, the edge $i+1$ must be attached to one of the vertices i_p in the minimal path

$$i_n > i_{n-1} > \cdots > i_1.$$

It must be further from u_0 than j_{t-1} by the previous argument. Wherever it is attached, it establishes a new chain

$$i + 1 > j_{t-1} > \cdots > j_1,$$

since all the other a_{i_q}, $i_q > j_{t-1}$ are smaller, all being less than or equal to $a_t < a_{i+1}$

$$Q_{i+1} \overset{F^G_{id}}{\mapsto} (R_{i+1} \xrightarrow{\phi} R_{i_p} \to \cdots \to R_{i_1}).$$

By the induction hypothesis, $(R_{i_p} \to \cdots \to R_{i_1}) \overset{\hat{F}^G_\sigma}{\mapsto} Q'_{i_p}$. There are now two subcases.

Case 2a. $i_p = j_{t-1}$, so $a_{i+1} < a_{i_p}$. Then $R_{i+1} \overset{\hat{F}^G_\sigma}{\mapsto} (Q_{a_{i+1}} \to Q_{a_{i_p}})$. $\hat{F}^G_\sigma F^G_{id}(Q_{i+1})$ is the mapping cone of

$$\begin{array}{ccc} Q_{a_{i+1}} & \xrightarrow{h} & Q_{a_{j_{t-1}}} \\ & & \downarrow{\scriptstyle s} \\ & & Q'_{j_{t-1}} \end{array}$$

so

$$Q'_{i+1} = Q_{a_{i+1}} \xrightarrow{h} Q_{a_{j_{t-1}}} \xrightarrow{s} Q_{a_{j_{t-1}}} \xrightarrow{h} \ldots \overset{\deg 0}{Q}{}_{a_{j_1}},$$

as desired.

Case 2b. If $i_p > j_{t-1}$, then we have $\hat{F}^G_\sigma(R_{i+1}) = Q_{a_{i_p}} \to Q_{a_{i+1}}$. The relevant morphism $\hat{F}^G_\sigma(\phi) : \hat{F}^G_\sigma(R_{i+1}) \to \hat{F}^G_\sigma(R_{i_p} \to \cdots \to R_{i_1}) = Q'_{ip}$ is given by

$$\begin{array}{ccccccccc}
Q_{a_{i_p}} & \longrightarrow & Q_{a_{i+1}} & & & & & & \\
{\scriptstyle id}\downarrow & & \downarrow{\scriptstyle h} & & & & & & \\
Q_{a_{i_p}} & \xrightarrow{h} & Q_{a_{j_{t-1}}} & \xrightarrow{s} & Q_{a_{j_{t-1}}} & \longrightarrow & \dots & \longrightarrow & Q_{a_{j_1}}
\end{array}$$

Taking the mapping cone and removing the factor $Q_{a_{i_p}} \xrightarrow{id} Q_{a_{i_p}}$, we get

$$Q'_{i+1} = Q_{a_{i+1}} \xrightarrow{h} Q_{a_{j_{t-1}}} \xrightarrow{s} Q_{a_{j_{t-1}}} \to \cdots \to Q_{a_{j_1}},$$

as desired.

Corollary. *$H_\sigma = \hat{F}^G_\sigma F^G_{id}$ is independent of the choice of G.*

Remark. If σ is the permutation corresponding to a Brauer tree $G \in B'$ with a given enhanced pointing, and if σ' corresponds to $G' \in B'$ with the same tree and the same pointing but a different enhancement point, then $\hat{F}^G_\sigma F^G_{id}$ and $\hat{F}^{G'}_{\sigma'} F^{G'}_{id}$ determine the same element of $\operatorname{Tr}\operatorname{Pic}(A)$, subject to a cyclic renumbering of the indices of the Q_i.

Example 4. We carry out this procedure for the permutation σ given above (3). In (3) and (4) we omit the maps, which are alternately h and s.

$$\begin{array}{r}
Q_6 \\
Q_1 \to Q_6 \to Q_6 \\
Q_5 \to Q_6 \to Q_6 \\
Q_3 \to Q_5 \to Q_5 \to Q_6 \to Q_6 \\
Q_2 \to Q_3 \to Q_3 \to Q_5 \to Q_5 \to Q_6 \to Q_6 \\
Q_4 \to Q_5 \to Q_5 \to Q_6 \to Q_6 \\
Q_7 \\
Q_8 \\
Q_9
\end{array} \tag{4}$$

Remark. Not every permutation σ is induced by a pointing. For example, if $e = 3$, (2 3 1) is not induced by any enhanced pointing.

Lemma 1. *For any permutation σ representing an enhanced pointing, there is a tree corresponding to σ with all points on the left as one goes out along any minimal path.*

Proof. Let G be a tree such that $H_\sigma = \hat{F}^G_\sigma F^G_{id}$ and such that the total distances of all vertices from the exceptional vertex is minimal. Now suppose that at some vertex the point is not on the left. All those branches to the left of the point can be moved down to the next vertex closer to the exceptional vertex, yet the numbering will not be affected because the points will be encountered in the same order. Contradiction to the minimality.

Definition. The length $\ell(\sigma)$ of a permutation σ is the minimal number of adjacent transpositions $(j\ j+1)$ required to generate σ.

Lemma 2. *If σ is a non-trivial corresponding to an enhanced pointing of a Brauer tree, there exists a transposition $\tau = (i\ i{+}1)$, such that $a_i > a_{i+1}$ in σ and $\sigma = \sigma' \circ \tau$, where σ' is a permutation corresponding to a Brauer tree, such that $\ell(\sigma') = \ell(\sigma) - 1$.*

Proof. We associate to σ the normalized tree G given in Lemma 1. If G is a star, σ is trivial. Therefore, G contains at least one non-exceptional, non-terminal vertex. Let i be a non-exceptional, non-terminal vertex at maximal distance from the exceptional vertex. Then $i + 1$ is a terminal vertex, and, by the normalization of G, we have $a_i > a_{i+1}$. We can move all the other branches at i to $i + 1$, without changing the permutation, as in the change from (a) to (b) in Fig. 2. Now the point at i is on the left and the point at $i + 1$ is on the right. If we now reverse the points as in (c) of Fig. 2, so that the point at i is on the right and the point at

$i+1$ is on the left, we get a new permutation σ', $\sigma'(j) = b_j$, where

$$\begin{aligned} &b_j = a_j \qquad j \neq i, i+1 \\ &b_i = a_{i+1} \\ &b_{i+1} = a_i \end{aligned}$$

so that $b_i < b_{i+1}$.

We have defined σ' by giving a pointing of G', so it is defined by a pointing of a Brauer tree. We now verify that $\sigma = \sigma' \circ \tau$:

$$\begin{aligned} &\sigma' \circ \tau(j) = \sigma'(j) = a_j \qquad j \neq i, i+1 \\ &\sigma' \circ \tau(i) = \sigma'(i+1) = b_{i+1} = a_i \\ &\sigma' \circ \tau(i+1) = \sigma'(i) = b_i = a_{i+1} \end{aligned}$$

By standard results in the theory of symmetric groups, exchanging adjacent numbers a_i and a_{i+1} in σ which are out of order produces a permutation of length $\ell(\sigma) - 1$.

(a) (b) (c)

Fig. 2

Proposition 2. *The subgroup $\mathcal{R}$ of refolded tilting complexes is generated by shifts, by* $\mathrm{Pic}(A)$ *and by $H_\tau = \hat{F}_\tau^{G_i} F_{id}^{G_i}$ for τ adjacent transpositions $(i\ i+1)$. We may take G_i to be the Brauer tree with valency $e-1$ at the exceptional vertex, and one edge attached to the i-th edge after the enhancement point.*

Proof. Let σ be a permutation given by a pointing. Since the H_σ are independent of the choice of tree, we take the tree given in Lemma 2 which has a pointing giving σ. It is standard theory of permutations [H] that any permutation σ can be written as a product of transpositions $\sigma = \tau_1 \dots \tau_n$ in such a way that τ_i is always switching the indices of two adjacent elements which are in numerical order in $\sigma_{i-1} = \tau_1 \circ \cdots \circ \tau_{i-1}$. Here $n = \ell(\sigma)$. Note the reversal in the order of the indices. We want to show that

$$H_\sigma = H_{\tau_n} \circ \cdots \circ H_{\tau_1}.$$

We can work by induction on n. It obviously suffices to prove that if $\sigma = \sigma' \circ \tau$ as in Lemma 2, then

$$H_\sigma = H_\tau \circ H_{\sigma'}.$$

Since H_τ leaves everything fixed except Q'_i and Q'_{i+1}, we need consider only those. We have

$$\sigma'(i) = b_i$$
$$\sigma'(i+1) = b_{i+1}$$

with $b < b_{i+1}$.

The effect of H_{τ_i} is

$$Q'_i \overset{H_{\tau_i}}{\mapsto} Q''_i = Q'_{i+1}$$
$$Q'_{i+1} \overset{H_{\tau_i}}{\mapsto} Q''_{i+1} = \mathrm{Cone}(\mathrm{Cone}(Q'_i \to Q'_{i+1}) \to Q'_{i+1})$$

We must show that these are the complexes determined by σ in Proposition 1. Here $\sigma(i) = a_i = b_{i+1}$ and $\sigma(i+1) = a_{i+1} = b_i$.

$\underline{Q''_i}$: Let $i+1 > j_{\ell-1} > \cdots > j_1$ be the sequence of indices in Q'_{i+1}. Since $b_{i+1} > b_i$ in $\tau^{\sigma'}$, we have $j_{\ell-1} < i$. Now consider σ. Corresponding to

$$i > j_{\ell-1} \cdots > j_1,$$

we have

$$b_{i+1} < b_{j_{\ell-1}} \cdots < b_{j_1}$$

which is again a maximal order-reversing subsequence. Thus $Q''_i = Q'_{i+1}$.

$\underline{Q''_{i+1}}$: We have $a_{i+1} < a_i$, and edge $i+1$ is attached to vertex i, so we are in Case 1 of Proposition 1.

For $i+1$ the new sequence is

$$i+1 > i > j_{t-1} > \cdots > j_1$$

$$b_i < b_{i+1} < b_{j_{t-1}} < \cdots < b_{j_1}.$$

We get the new Q''_{i+1} as the cone of

$$\begin{tikzcd} Q_{b_i} \arrow[r, "h"] & Q_{b_{i+1}} \arrow[d, "s"] & & & \\ & Q_{b_{i+1}} \arrow[r, "h"] & Q_{j_{t-1}} \arrow[r] & \dots \arrow[r] & Q_{j-1} \end{tikzcd}$$

This completes the proof that R is generated by the H_{τ_i}. □

We now demonstrate that these generators satisfy the braid relations for the braid group on the Weyl diagram consisting of e points in a circle. For a general reference to the braid group determined by a Weyl diagram, see [H].

Remark. Let $\mathcal{R}^{\perp}$ be the semigroup of $\mathcal{R}$ generated by the H_τ. It is clear from Proposition 1 that these elements have representative tilting complexes whose components are all in non-positive degrees, and that multiplying by an H_τ only increases total dimension. Thus no monomial is zero.

Proposition 3. *The subgroup $\mathcal{R}$ of refolded complexes has an action by $\tilde{B}_e$, the braid group on the Euclidean diagram $\tilde{A}_{e-1}$ obtained by completing A_{e-1} to a cycle.*

Proof. We define a homomorphism of $\tilde{B}_e$ into $\mathcal{R}$ by sending a half-twist switching i and $i+1$ to H_τ, where $\tau = (i\ i+1)$. In order to show that this is a well-defined homomorphism, it will suffice to show that the H_τ satisfy the braid relations

Case 1. τ, τ' are disjoint transpositions $\tau = (i\; i{+}1)$. $\tau' = (j\; j{+}1)$ (where $e{+}1 = 1$), then both $H_{\tau'} H_{\tau}$ and $H_{\tau} H_{\tau'}$ give

$$\begin{array}{ccccc}
 & & & & Q_{i+1} \\
Q_i & \xrightarrow{h} & Q_{i+1} & \xrightarrow{s} & Q_{i+1} \\
 & & & & Q_{i+2} \\
 & & & & \vdots \\
 & & & & Q_{j+1} \\
Q_j & \xrightarrow{h} & Q_{j+1} & \xrightarrow{s} & Q_{j+1} \\
 & & & & Q_{j+2}
\end{array}$$

Case 2. Let τ, τ' be adjacent transpositions, i.e., $\tau = (i-1\; i)$ $\tau' = (i\; i+1)$ then we calculate

$$H_{\tau'} \circ H_{\tau} =$$

$$\begin{array}{ccccccccc}
 & & & & & & & & Q_i \\
 & & & & & & & & Q_{i+1} \\
Q_{i-1} & \to & Q_i & \to & Q_i & \to & Q_{i+1} & \to & Q_{i+1}
\end{array}$$

and

$$H_{\tau} \circ H_{\tau'} \circ H_{\tau} =$$

$$\begin{array}{ccccccccc}
 & & & & & & & & Q_{i+1} \\
 & & & & Q_i & \to & Q_{i+1} & \to & Q_{i+1} \\
Q_{i-1} & \to & Q_i & \to & Q_i & \to & Q_{i+1} & \to & Q_{i+1}
\end{array}$$

On the other hand,

$H_\tau \circ H_{\tau'} =$

$$\begin{array}{ccccc} & & & & Q_{i+1} \\ Q_{i-1} & \to & Q_{i+1} & \to & Q_{i+1} \\ Q_i & \to & Q_{i+1} & \to & Q_{i+1} \end{array}$$

and finally, as desired

$$H_{\tau'} \circ H_\tau \circ H_{\tau'} = H_\tau \circ H_{\tau'} \circ H_\tau.$$

This homomorphism then determines a braid group action by left multiplication. We are left with the question of whether when $m \neq 1$ these braid generators, together with shifts and $\mathrm{Pic}(A)$, generate the entire derived Picard group, and whether the homomorphism is one-to-one. If we omit one of the generators of $\mathcal{R}$, the corresponding subgroup $\mathcal{R}_\tau$ has an action by an ordinary braid group. Since the word problem has been solved for the ordinary braid group, it may be more accessible to try to show that this action is faithful; Zimmerman suggested in private correspondence that it may be possible to achieve this with an extension of the methods in [KS] A question also arises as to the relationship between the refolded tilting complexes for the Brauer star algebra and those of the linear algebra used in [RoZ].

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