Exceptional Modules are Tree Modules

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ABSTRACT. We consider representations of a quiver over an arbitrary field. Recall that an indecomposable representation M without self-extensions is said to be exceptional. We are going to show that exceptional representations can be exhibited using matrices involving as coefficients just 0 and 1. Actually, if d is the dimension of M, there exists such a matrix presentation with precisely d–1 non-zero coefficients: the corresponding "coefficient quiver" is a tree.

Let k be a field, let Q be a (finite) quiver and kQ the path algebra of Q over k. Recall that a representation M of Q over k is of the form $M = (M_x, M_\alpha)_{x,\alpha}$; here, for every vertex x of Q, there is given a finite-dimensional k-space M_x, for every arrow $\alpha\colon x \to y$, there is given a linear transformation $M_\alpha\colon M_x \to M_y$. A representation M of Q over k (or better, the corresponding direct sum $\bigoplus_x M_x$) is just an arbitrary (finite-dimensional) kQ-module.

Let d_x be the dimension of M_x, and $d = \sum d_x$; we call d the *dimension* of M. A *basis* $\mathcal{B}$ of M is by definition a subset of the disjoint union of the various k-spaces M_x such that for any vertex x the set $\mathcal{B}_x = \mathcal{B} \cap M_x$ is a basis of M_x. Let us assume that such a basis $\mathcal{B}$ of M is given. For any arrow $\alpha\colon x \to y$, we may write M_α as a $(d_y \times d_x)$-matrix $M_{\alpha,\mathcal{B}}$ whose rows are indexed by $\mathcal{B}_y$ and whose columns are indexed by $\mathcal{B}_x$. We denote by $M_{\alpha,\mathcal{B}}(b, b')$ the corresponding matrix coefficients, where $b \in \mathcal{B}_x$, $b' \in \mathcal{B}_y$, these matrix coefficients $M_{\alpha,\mathcal{B}}(b, b')$ are defined by $M_\alpha(b) = \sum_{b' \in \mathcal{B}} M_{\alpha,\mathcal{B}}(b, b')\, b'$. By definition, the *coefficient quiver* $\Gamma(M, \mathcal{B})$ of M with respect to $\mathcal{B}$ has the set $\mathcal{B}$ as set of vertices, and there is an arrow (α, b, b') provided $M_{\alpha,\mathcal{B}}(b, b') \neq 0$. Note that there is a long-standing tradition in matrix theory to focus attention to such coefficient quivers, see e.g. [BR]. In his lectures [CB2] at the Banach center in Warsaw in 1988, Crawley-Boevey has drawn attention to the use of coefficient quivers for dealing with different types of matrix problems, as they arise in representation theory, in particular for dealing with representations of quivers.

We will call an indecomposable representation of Q over k a *tree module* provided there exists a basis $\mathcal{B}$ of M such that the coefficient quiver $\Gamma(M, \mathcal{B})$ is a tree. Note that for a tree module M of dimension d, there is a basis $\mathcal{B}$ of M such that precisely $d-1$ matrix coefficients are non-zero, and one may assume that all these coeffients are equal to 1 (see section 1). Thus, any tree module can be exhibited by 0-1-matrices.

We recall that an indecomposable representation M of Q over k is said to be *exceptional* (or to be a "partial tilting module", or a "stone") provided $\mathrm{Ext}^1_{kQ}(M, M) = 0$. The aim of this note is to present the following result:

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Theorem. *Let k be a field, let Q be a quiver. Any exceptional representation of Q over k is a tree module.*

The proof will use Schofield's inductive construction of all exceptional modules and a detailed consideration of the case of a directed quiver with 2 vertices (recall that a quiver is said to be *directed*, provided there is no directed cyclic path).

The first considerations below should be well-known, but there does not seem to exist a reference: We will show that we may restrict to the case of a directed quiver, since the support of any exceptional module is directed (for an exceptional representation M, we may define its *support* as the full subquiver given by all vertices x with $M_x \neq 0$).

On the other hand, starting with a directed quiver, all the preprojective and all the preinjective indecomposable representations are exceptional, thus Theorem above shows that they are tree modules. In particular, for the representation finite quivers, all the indecomposable representations are tree modules. In this special case, the result is due to Crawley-Boevey [CB2]. When Gabriel [G] exhibited matrix presentations of these modules, he used 0-1-matrices with few non-zero entries, but in four instances, the number of non-zero entries still can be reduced. Corresponding matrix presentations can be found in section 6.

1. Tree modules

Let M be a representation of Q over k. Here are some obvious properties of the coefficient quiver.

(1) *If M is indecomposable and $\mathcal{B}$ is a basis of M, then $\Gamma(M, \mathcal{B})$ is connected. If M is decomposable, then there exists a basis $\mathcal{B}$ of M such that $\Gamma(M, \mathcal{B})$ is not connected.*

Proof: Clearly, any decomposition of $\Gamma(M, \mathcal{B})$ yields a direct decomposition of M. Conversely, if we deal with a direct decomposition $M = M' \oplus M''$, and if $\mathcal{B}'$ is a basis of M' and $\mathcal{B}''$ a basis of M'', then let $\mathcal{B}$ be the disjoint union of the various sets $\mathcal{B}'_x$ and $\mathcal{B}''_x$. Clearly, $\mathcal{B}$ is a basis of M and $\Gamma(M, \mathcal{B})$ is the disjoint union of the quivers $\Gamma(M', \mathcal{B}')$ and $\Gamma(M'', \mathcal{B}'')$. (Note that the zero module is said to be decomposable, and the empty quiver is said to be not connected; thus the last assertion is valid also in the trivial case.)

Remark 1: Note that, in general, $\Gamma(M, \mathcal{B})$ *may be connected whereas M is decomposable.* For example, consider the representation $M = (k^2, k, [1, 1])$ of the quiver $\circ \to \circ$, and let $\mathcal{B}$ be given by the canonical bases of k^2 and k. Then $\Gamma(M, \mathcal{B})$ is connected, but M is decomposable.

Remark 2: In general, *the shape of the coefficient quiver $\Gamma(M, \mathcal{B})$ not only depends on the representation M, but also on the choice of $\mathcal{B}$, even if we choose*

$\mathcal{B}$ so that the number of non-zero matrix coefficients is minimal. As typical example, consider a quiver of type D_4 and take for M the indecomposable representation of dimension 5. On the left, we exhibit two different matrix presentations and on the right, the corresponding coefficient quivers: even the underlying graphs are different.

$$\begin{tikzcd}
 & & k \\
k \arrow[r, "{\begin{bmatrix}1\\1\end{bmatrix}}"] & k^2 \arrow[ur, "{[1\ 0]}"] \arrow[dr, "{[0\ 1]}"'] & \\
 & & k
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \circ \arrow[r] & \circ \\
\circ \arrow[ur] \arrow[dr] & & \\
 & \circ \arrow[r] & \circ
\end{tikzcd}$$

$$\begin{tikzcd}
 & & k \\
k \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"] & k^2 \arrow[ur, "{[1\ 0]}"] \arrow[dr, "{[1\ 1]}"'] & \\
 & & k
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\circ \arrow[r] & \circ \arrow[r] \arrow[dr] & \circ \\
 & \circ \arrow[r] & \circ
\end{tikzcd}$$

Two bases $\mathcal{B}, \mathcal{B}'$ of M are said to be *proportional* provided for every element $b \in \mathcal{B}$, there is a (non-zero) scalar $\lambda(b) \in k$ such that $\lambda(b)b$ belongs to $\mathcal{B}'$. Note that if $\mathcal{B}, \mathcal{B}'$ are proportional bases, then we may identify the coefficient quivers $\Gamma(M, \mathcal{B})$ and $\Gamma(M, \mathcal{B}')$.

(2) *Let $\mathcal{B}$ be a basis of M such that $\Gamma(M, \mathcal{B})$ is a tree. Then there is a basis $\mathcal{B}'$ of M which is proportional to $\mathcal{B}$ such that all the non-zero coefficients $M_{\alpha,\mathcal{B}'}(b, b')$ are equal to 1.*

Proof: Fix one of the elements b_0 of $\mathcal{B}$, let $\lambda(b_0) = 1$. Let us assume that b, b' are neighbours in the coefficient quiver $\Gamma(M, \mathcal{B})$ and that $\lambda(b)$ is already defined. In case there is an arrow (α, b, b') in $\Gamma(M, \mathcal{B})$, let $\lambda(b') = \lambda(b) \cdot M_{\alpha,\mathcal{B}}(b, b')$; otherwise, there will be an arrow (α, b', b) and we define $\lambda(b') = \lambda(b) \cdot M_{\alpha,\mathcal{B}}(b', b)^{-1}$. Since $\Gamma(M, \mathcal{B})$ is a tree, this procedure defines non-zero scalars $\lambda(b)$ for all $b \in \mathcal{B}$, in a unique way. We set $\mathcal{B}'$ as the set of elements $\lambda(b)b$ with $b \in \mathcal{B}$. It is easy to check that $M_{\alpha,\mathcal{B}'}(\lambda(b)b, \lambda(b')b') = 1$ in case $M_{\alpha,\mathcal{B}}(b, b') \neq 0$, and that $M_{\alpha,\mathcal{B}'}(\lambda(b)b, \lambda(b')b') = 0$ otherwise.

Remark 3. Given a special biserial algebra, the string modules (or "modules of the first kind") are always tree modules.

Remark 4. We may consider the coefficient quiver as a coloured quiver, with colours given by Q: any arrow of the form (α, b, b') will be endowed with colour α, and any vertex of the form $b \in B_x$ has colour x.

(By definition, a *coloured quiver* is just a morphism $c\colon \Gamma \to \Gamma'$ of quivers. To be precise, let Γ be a quiver with Γ_0 as set of vertices, Γ_1 as set of arrows, and similarly, let Γ' be a quiver with Γ'_0 as set of vertices, Γ'_1 as set of arrows. For any arrow α (in Γ_1 or Γ'_1), we denote by $g(\alpha)$ its tail and by $h(\alpha)$ its head. A quiver morphism $c = (c_0, c_1)$ is given by two maps $c_0 \colon \Gamma_0 \to \Gamma'_0$ and $c_1 \colon \Gamma_1 \to \Gamma'_1$ such that $g(c_1(\alpha)) = c_0(g(\alpha))$ and $h(c_1(\alpha)) = c_0(h(\alpha))$, for any arrow α in Γ_1. If $c \colon \Gamma \to \Gamma'$ is a coloured quiver, then a vertex x of Γ is said to be *of colour* $c_0(x)$, similarly an arrow α of Γ is said to be *of colour* $c_1(\alpha)$. Note that if Γ has at least one vertex and is connected, then clearly the colours of the arrows allow to recover the colours of the vertices.)

In case we deal with a representation M with a basis $\mathcal{B}$ such that all the coefficients $M_{\alpha,\mathcal{B}}(b, b')$ are 0 or 1, then the coefficient quiver considered as a coloured quiver, allows to recover M.

Remark 5. We should stress that our use of the notion of a tree module follows the considerations of Crawley-Boevey in [CB2], but deviates from the terminology used elsewhere. In particular, the "tree modules" in his paper [CB1] form a much narrower class of modules: these "tree modules" are also tree modules as defined above, but the converse is not true.

2. Extensions of representations of quivers

The set of vertices of the quiver Q will be denoted by Q_0, its set of arrows by Q_1. Given an arrow α, we denote by $g(\alpha)$ its tail, and by $h(\alpha)$ its head.

Let M, M' be representations of the quiver Q over the field k. For any vertex x, let d_x be the dimension of M_x and d'_x the dimension of M'_x.

Let us recall [R1] that the group $\operatorname{Ext}^1_{kQ}(M, M')$ can be constructed as follows: consider the k-spaces

$$C^0(M, M') = \bigoplus_{x \in Q_0} \operatorname{Hom}_k(M_x, M'_x)$$
$$C^1(M, M') = \bigoplus_{\alpha \in Q_1} \operatorname{Hom}_k(M_{g(\alpha)}, M'_{h(\alpha)}),$$

and let

$$\delta_{MM'} \colon C^0(M, M') \longrightarrow C^1(M, M')$$

be given by

$$\delta_{MM'}((f_x)_x) = (f_y M_\alpha - M'_\alpha f_x)_{\alpha \colon x \to y}.$$

The kernel of $\delta_{MM'}$ is just the set $\operatorname{Hom}_{kQ}(M, M')$ of homomorphisms, the cokernel is $\operatorname{Ext}^1_{kQ}(M, M')$.

Now assume that we have fixed a basis $\mathcal{B}$ of M, and a basis $\mathcal{B}'$ of M'. We obtain a basis of $C^1(M, M')$ consisting of matrix units: For $1 \leq s \leq d_{g(\alpha)}$ and $1 \leq t \leq d_{h(\alpha)}$, let us denote by $E_{MM'}(\alpha, s, t)$ the corresponding matrix unit in $\operatorname{Hom}_k(M_{g(\alpha)}, M'_{h(\alpha)})$ (the matrix with one coefficient equal to 1, namely the coefficient in the column with index s and the row with index t, the remaining coefficients being zero). Let $\mathcal{E}_{MM'}$ be the set of all such matrix units. Since the set $\mathcal{E}_{MM'}$ is a basis of $C^1(M, M')$, we may choose a subspace $\Phi = \Phi_{MM'}$ of $C^1(M, M')$ generated by a subset of $\mathcal{E}_{MM'}$ such that Φ is a direct complement for the image of $\delta_{MM'}$. We see that *there is a subset of $\mathcal{E}_{MM'}$ such that the residue classes modulo the image of $\delta_{M,M'}$ form a basis of* $\operatorname{Ext}^1_{kQ}(M, M')$.

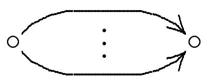
A pair X, Y of representations is said to be *orthogonal* provided we have

$$\operatorname{Hom}_{kQ}(X, Y) = 0 = \operatorname{Hom}_{kQ}(Y, X).$$

Let X, Y be orthogonal representations of Q over k, such that $\operatorname{End}_{kQ}(X) = k = \operatorname{End}_{kQ}(Y)$. Let $\dim_k \operatorname{Ext}^1_{kQ}(X, Y) = n$. We consider the full subcategory $\mathcal{F} = \mathcal{F}(X, Y)$ of the category of all representations of Q given by all representations M which fit into an exact sequence of the form

$$0 \to Y^v \to M \to X^u \to 0,$$

with arbitrary natural numbers u, v. It is well-known (see for example [R1]) that *$\mathcal{F}$ is equivalent to the category of representations of the quiver $\Theta(n)$ over k,*



with n arrows.

The existence of such an equivalence can be seen quite easily: Clearly, $\mathcal{F}$ is an abelian length category. Since $\operatorname{Ext}^1_{kQ}(X, Y)$ has dimension n, there exists an exact sequence

$$0 \to Y^n \to Z \to X \to 0$$

such that Z has no direct summand isomorphic to Y. Then Z is indecomposable and belongs to $\mathcal{F}$. It is easy to see that $Y \oplus Z$ is a progenerator for $\mathcal{F}$ and that the endomorphism ring of $Y \oplus Z$ is just the path algebra of $\Theta(n)$.

We are going to study this equivalence in more detail. Consider an exact sequence of the form

$$0 \to Y^v \to M \to X^u \to 0,$$

thus an element of $\operatorname{Ext}^1_{kQ}(X^u, Y^v)$. We may write $X^u = X \otimes k^u$, $Y^v = Y \otimes k^v$, with $\otimes = \otimes_k$. As we have noted, the vector space $\operatorname{Ext}^1_{kQ}(X \otimes k^u, Y \otimes k^v)$ is the cokernel of the map $\delta_{X\otimes k^u, Y\otimes k^v}$, but this map may be identified with $\delta_{XY} \otimes 1_{\operatorname{Hom}(k^u,k^v)}$:

$$\begin{array}{ccc}
C^0(X \otimes k^u, Y \otimes k^v) & \xrightarrow{\delta_{X\otimes k^u, Y\otimes k^v}} & C^1(X \otimes k^u, Y \otimes k^v) \\
\| & & \| \\
C^0(X, Y) \otimes \operatorname{Hom}(k^u, k^v) & \xrightarrow{\delta_{XY}\otimes 1_{\operatorname{Hom}(k^u,k^v)}} & C^1(X, Y) \otimes \operatorname{Hom}(k^u, k^v).
\end{array}$$

Thus, the cokernel of $\delta_{X\otimes k^u, Y\otimes k^v}$ may be identified with

$$\operatorname{Ext}^1_{kQ}(X,Y)\otimes \operatorname{Hom}(k^u,k^v).$$

This is of course not surprising, but it shows the following: Let Φ be a direct complement in $C^1(X,Y)$ of the image of δ_{XY}. We may use the elements of $\Phi\otimes\operatorname{Hom}(k^u,k^v)$ in order to obtain representatives of all the exact sequences

$$0\to Y^v\to M\to X^u\to 0.$$

In order to deal with the elements of $\Phi\otimes\operatorname{Hom}(k^u,k^v)$, we choose a k-basis $\phi_1,\dots,\phi_n$ of Φ. An arbitrary element of $\Phi\otimes\operatorname{Hom}(k^u,k^v)$ is of the form

$$\sum_{i=1}^n \phi_i\otimes\gamma_i \quad \text{with} \quad \gamma_i\in\operatorname{Hom}(k^u,k^v),$$

thus it is given by the n-tuple $(\gamma_1,\dots,\gamma_n)$ of $(u\times v)$-matrices. Note that we may consider arbitrary base changes in k^u and in k^v: different bases of k^u and k^v correspond to different direct decompositions of X^u and Y^v into indecomposables. This shows explicitly in which way the classification problem for $\mathcal{F}(X,Y)$ is related to the classification of representations of the quiver $\Theta(n)$.

In particular, we see the following: As we have mentioned above, we may start with a subspace Φ of $C^1(X,Y)$ having a basis $\phi_1,\dots,\phi_n$ which is a subset of $\mathcal{E}_{XY}$, thus consisting of element of the form $E_{XY}(\alpha,s,t)$. If we deal with a representation $(U,V;\alpha_i)$ of $\Theta(n)$ with dimension vector $[u,v]$, then the element $\sum_{i=1}^n\phi_i\otimes\alpha_i$ in $C^1(X^u,Y^v)$ gives a corresponding representation M in $\mathcal{F}(X,Y)$. If we assume that the linear transformations $\alpha_1,\dots,\alpha_n$ are given in matrix form, then these matrices occur once as blocks in $\sum_{i=1}^n\phi_i\otimes\alpha_i$, all other blocks being zero.

3. Representations without self-extensions.

The path algebra kQ has as k-basis the set of all paths in Q. We denote by kQ^+ the ideal generated by all paths of length at least 1. A representation M of Q over k is said to be *nilpotent*, provided there is a natural number n such that $(kQ^+)^nM=0$. Note that in case Q is not directed, there are simple representations which are not nilpotent. A module of finite length is nilpotent if and only if all its composition factors are nilpotent. The nilpotent simple representations are those of the form $S(x)$, where x is a vertex of Q; by definition, $S(x)_x=k$, $S(x)_y=0$, for $y\neq x$, and $S_\alpha=0$ for every arrow α.

A representation M of Q is said to be *sincere*, provided $M_x\neq 0$ for any vertex x of Q. Of course, M is said to be *faithful* provided there is no non-zero element $r\in kQ$ with $rM=0$.

Proposition 1. *Let S be a simple representation of Q. If S is not nilpotent, then* $\operatorname{Ext}^1_{kQ}(S,S) \neq 0$.

Proof: We may assume that S is sincere. Let d be the dimension vector of S. As we have mentioned in the previous section, $\operatorname{Ext}^1_{kQ}(S,S)$ may be calculated as the cokernel of the map

$$\delta_{SS} \colon C^0(S,S) \longrightarrow C^1(S,S).$$

The dimension of $C^0(S,S)$ is $\sum d_x^2$, that of $C^1(S,S)$ is $\sum_\alpha d_{g(\alpha)}d_{h(\alpha)}$. Since the kernel of δ_{SS} is the set of all endomorphisms of S and therefore non-zero, it is sufficient to see that we have

$$\sum_x d_x^2 \;\leq\; \sum_\alpha d_{g(\alpha)}d_{h(\alpha)}.$$

Given any vertex x of Q, let x^- be the set of arrows α with $h(\alpha) = x$. We claim that the map

$$\phi_x = (S_\alpha)_{\alpha\in x^-} \,:\, \bigoplus_{\alpha\in x^-} S_{g(\alpha)} \longrightarrow S_x$$

is surjective, for any x. Namely, if ϕ_x is not surjective, for some x, we define a subrepresentation $U(x)$ of S as follows: let $U(x)_x$ be the image of ϕ_x, and let $U(x)_y = S_y$ for $y \neq x$. Since S is simple, we see that $U(x) = 0$. But this means that x is the only vertex (since we have assumed that S is sincere) and that S_α is the zero map, for any arrow α, thus S is nilpotent.

Since ϕ_x is surjective, we see that

$$d_x \leq \sum_{\alpha\in x^-} d_{g(\alpha)},$$

and therefore

$$\sum_x d_x^2 \;\leq\; \sum_x \Big(\sum_{\alpha\in x^-} d_{g(\alpha)} \Big) \cdot d_x = \sum_\alpha d_{g(\alpha)} \cdot d_{h(\alpha)}.$$

This completes the proof.

Recall that the support of an object A of finite length in an abelian length category is the set of isomorphism classes of composition factors of A (see [R3]). The proposition shows that the support of a representation of Q without self-extensions may be identified with a subset of Q_0, or, as we prefer, with the full subquiver given by these vertices.

Proposition 2. *Let M be a sincere representation of the quiver Q with $\mathrm{Ext}^1_{kQ}(M,M) = 0$. Then Q has no oriented cycles and M is a faithful kQ-module.*

Proof: Let $\mathcal{A}$ be the category of all (finite dimensional) representations of Q over k, this is a hereditary abelian category. Consider the full subcategory $\operatorname{add} M$ of $\mathcal{A}$. Since we assume that $\mathrm{Ext}^1_{kQ}(M,M) = 0$, it follows that $\operatorname{add} M$ is extension closed. According to Lemma 1.1 in [R3], the class $\mathcal{C}$ of subfactors of objects in $\operatorname{add} M$ is closed under extensions.

Since M is of finite length, it has finite Loewy length, say $L(M) = t$. It follows that all the objects in $\mathcal{C}$ have Loewy length at most t.

Let S be a composition factor of M. We claim that S is nilpotent. Otherwise, according to Proposition 1, we know that $\mathrm{Ext}^1_{kQ}(S,S) \neq 0$. Since $\mathcal{A}$ is hereditary, there are objects in $\mathcal{A}$ which are of finite length, but of arbitrarily large Loewy length, such that all the composition factors are of the form S. These objects belong to the extension closure of any subcategory of $\mathcal{A}$ containing S. However, any object in $\mathcal{C}$ has Loewy length at most t.

Since all the composition factors of M are nilpotent, it follows that M itself is nilpotent. Since M is sincere, all the simple modules $S(x)$ occur as composition factors (thus as subfactors) of M. Altogether we see: On the one hand, the subcategory $\mathcal{C}$ is extension closed and contains all the representations $S(x)$, on the other hand, all the objects in $\mathcal{C}$ have Loewy length at most t. It follows that the length of the paths in Q is bounded by $t-1$. In particular, Q is a directed quiver. Also, $\mathcal{C}$ contains the indecomposable projective kQ-modules, therefore M is faithful. This completes the proof.

Remark 1. We should stress that the sincere representations of Q are usually not “sincere” kQ-modules in the sense of [R3]: Let M be a representation of Q. In order that M is a “sincere” kQ-module in the sense of [R3], any simple kQ-module has to occur as a subfactor of M, a much stronger assertion than our assumption that $M_x \neq 0$ for any vertex x of Q.

Remark 2. Recall that representations without self-extensions are the basic incredients in tilting theory. Let Q be a directed quiver with s vertices and let M be a representation of Q without self-extensions. If we denote by $m(M)$ the number of isomorphism classes of indecomposable direct summands of M, then always $m(M) \leq s$, and the tilting modules are those representations with $m(M) = s$. Since any representation without self-extensions occurs as a direct summand of a tilting module, such representations often are called “partial tilting modules”.

4. Directed quivers with 2 vertices.

Let us assume that we deal with the quiver $\Theta(n)$. A representation of $\Theta(n)$ is of the form $(U,V;\alpha_i)_i$ where U,V are vector spaces and $\alpha_i\colon U\to V$ with $1\le i\le n$ is a sequence of n linear transformations.

For $n\le 1$, all the indecomposable representations are exceptional and also (rather trivial) tree modules, thus we may assume that $n\ge 2$. The exceptional modules are the preprojective and the preinjective modules [R1]. Since the dual of a tree module again is a tree module, it is sufficient to deal with the preprojective modules.

Here we use the reflection functor σ as introduced by Bernstein, Gelfand and Ponomarev [BGP]: Start with $\alpha=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}$, this is a linear transformation $U\to V^n$ and let $\beta\colon V^n\to W$ be its cokernel. Thus, we deal with an exact sequence of the form

$$U \xrightarrow{\ \alpha\ } V^n \xrightarrow{\ \beta\ } W \longrightarrow 0.$$

We can write $\beta=[\,\beta_1\ \ \cdots\ \ \beta_n\,]$ with linear transformations $\beta_i\colon V\to W$. Thus we obtain a representation $(V,W;\beta_i)_i$ of the quiver. Let $P(0)$ be the simple projective representation (its dimension vector is $[0,1]$), let $P(t)=\sigma^tP(0)$. These representations $P(t)$ with $t=0,1,2,\dots$ are the preprojective representations of Q. Since these representations $P(t)$ are exceptional, they are uniquely determined by their dimension vectors.

Fix some $t\ge 0$ and let $P(t)=(U,V;\alpha_i)_i$ with dimension vector $[u,v]$. Let us assume that there exists a basis $\mathcal{B}_U$ of U and a basis $\mathcal{B}_V$ of V such that the coefficient quiver of M with respect to $\mathcal{B}_U\cup\mathcal{B}_V$ is a tree and such that one of the maps α_i is a monomorphism mapping $\mathcal{B}_U$ into $\mathcal{B}_V$. Of course, these assertions are obvious for $t=0$ and also for $t=1$.

We can assume that α_1 is a monomorphism mapping $\mathcal{B}_U$ into $\mathcal{B}_V$. Let U' be the subspace of V generated by the set $\mathcal{B}_V\setminus\alpha_1(\mathcal{B}_U)$ and denote this set by $\mathcal{B}_{U'}$. Then $V=\alpha_1(U)\oplus U'$. We denote by $\varepsilon\colon V\to U$ the projection with kernel U', thus $\varepsilon\alpha_1=1_U$. Similarly, we may consider the projection $\pi\colon V\to U'$ with kernel $\alpha_1(U)$. Consider the map $\beta\colon V^n\to V^{n-1}\oplus U'=W$, given by the following matrix:

$$\beta=\begin{bmatrix}-\alpha_2\varepsilon & 1 & & \\ \vdots & & \ddots & \\ -\alpha_n\varepsilon & & & 1\\ \pi & 0 & \cdots & 0\end{bmatrix}.$$

One easily checks that $\beta\alpha=0$. Since β is surjective and $\dim U+\dim W=\dim V^n$, it follows that the sequence

$$0\to U\overset{\alpha}{\to}V^n\overset{\beta}{\to}W\to 0$$

is exact. Thus β is the cokernel of α, and $P(t+1) = \sigma P(t)$ is given by

$$(V, W; \beta_1, \beta_2, \dots, \beta_n),$$

where β_i are the columns of β.

Let $\mathcal{B}_W$ be the union of the $n-1$ canonical copies of $\mathcal{B}_V$ inside $W = V^{n-1} \oplus U'$ together with the copy of $\mathcal{B}_{U'}$ considered as a basis of $U' = 0 \oplus U'$. In this way, we see that the linear transformations $\beta_2, \dots, \beta_n$ are injective maps sending $\mathcal{B}_V$ into $\mathcal{B}_W$.

By assumption, the coefficient quiver of $P(t)$ with respect to $\mathcal{B}_U \cup \mathcal{B}_V$ is a tree, thus there are precisely $u+v-1$ non-zero coefficients. Also, the matrix corresponding to α_1 has precisely u non-zero entries, thus the sum of the number of non-zero entries of the matrices corresponding to $\alpha_2, \dots, \alpha_n$ is $v-1$.

Consider now the matrices corresponding to the linear transformations β_i with respect to the bases $\mathcal{B}_V, \mathcal{B}_W$. For $2 \le i \le n$, there are precisely v non-zero entries, whereas the number of non-zero entries for β_1 is $(v-u)+(v-1)$ (the number $v-u$ is the dimension of U'). Altogether, the number of non-zero entries is

$$(n-1)\cdot v + (v-u) + (v-1) = (n+1)\cdot v - u - 1 = v + w - 1,$$

where $w = n \cdot v - u$ is the dimension of W. Thus, we see that $P(t+1)$ has a coefficient quiver with $v+w$ vertices and $v+w-1$ arrows. Since $P(t+1)$ is indecomposable, this coefficient quiver has to be connected, thus it has to be a tree. Also, we know that all the maps β_i with $2 \le i \le n$ are monomorphisms mapping $\mathcal{B}_V$ into $\mathcal{B}_W$. This completes the induction step.

Explicit matrix presentations for the preprojective representations $P(t)$ will be exhibited in section 7.

5. Proof of Theorem.

We recall the following result due to Schofield: *Let M be an exceptional representation of Q, and assume that M is not simple. Then there are exceptional modules X, Y with the properties*

$$\operatorname{Hom}_{kQ}(X,Y) = \operatorname{Hom}_{kQ}(Y,X) = \operatorname{Ext}^1_{kQ}(Y,X) = 0,$$

and an exact sequence

$$0 \to Y^v \to M \to X^u \to 0,$$

where $[u,v]$ is the dimension vector of an exceptional representation of $\Theta(n)$ and $n = \dim \operatorname{Ext}^1_{kQ}(X,Y)$.

The proof of Schofield [S] (see also [CB3]) assumes that k is algebraically closed, for a proof dealing with arbitrary fields k, we refer to [R2].

Thus, let us assume there are given exceptional representations X, Y with

$$\operatorname{Hom}_{kQ}(X,Y) = \operatorname{Hom}_{kQ}(Y,X) = \operatorname{Ext}^1_{kQ}(Y,X) = 0, \qquad \dim \operatorname{Ext}^1_{kQ}(X,Y) = n,$$

and a non-trivial exact sequence

$$(*) \qquad 0 \to Y^v \to M \to X^u \to 0,$$

where $[u,v]$ is the dimension vector of an exceptional representation of $\Theta(n)$. By induction, we can assume that both X and Y are tree modules, thus there are bases $\mathcal{B}_X$, $\mathcal{B}_Y$ such that the coefficient quivers of X with respect to $\mathcal{B}_X$ and of Y with respect to $\mathcal{B}_Y$ are trees. It x is the dimension of X and y the dimension of Y, then the coefficient quiver of X with respect to $\mathcal{B}_X$ has $x-1$ arrows, that of Y with respect to $\mathcal{B}_Y$ has $y-1$ arrows.

Consider now any exact sequence of the form $(*)$. Such an exact sequence corresponds to a representation of the quiver $\Theta(n)$ as follows: there are v diagonal blocks of the form Y, there are u diagonal blocks of the form X; we consider those additional blocks which are responsable for obtaining an extension of the form $(*)$. We choose matrix units $E_{XY}(\alpha_i, s_i, t_i)$ with $1 \le i \le n$ whose residue classes modulo the image I of $\delta_{X,Y}$ form a basis of $\operatorname{Ext}^1_{kQ}(X,Y)$. Let us denote $\epsilon_i = E_{XY}(\alpha_i, s_i, t_i) + I$. If we deal with an extension in $\operatorname{Ext}^1_{kQ}(X^u, Y^v)$ involving only scalar multiples of ϵ_i, we consider a $(u \times v)$-matrix Z_i with entries in k. Since the dimension of $\operatorname{Ext}^1_{kQ}(X,Y)$ is $n+1$, we obtain in this way $n+1$ $(u \times v)$-matrices $Z_0, \dots, Z_n$.

The given exact sequence $(*)$ corresponds to an exceptional representations Z of $\Theta(n)$ with dimension vector $[u,v]$. This representation Z can be written using precisely $u+v-1$ non-zero coefficients (all being equal to 1). This pattern of coefficients can be used in order to write down explicitely the extension $(*)$, altogether we obtain a representation M which is given by matrices which are distributed into blocks. There are the v diagonal blocks of the form Y, there are u diagonal blocks of the form X; the blocks of the form Y involve $y-1$ non-zero coefficients, the blocks of the form X invlove $x-1$ non-zero coefficients: altogether we obtain in this way $v \cdot (y-1) + u \cdot (x-1) = ux + vy - u - v$ non-zero coefficients. Inserting the coefficients of Z, we obtain in addition $u+v-1$ non-zero coefficients. Thus the total number is $(ux+vy-u-v)+(u+v-1) = ux + vy - 1 = \dim M - 1$.

Of course, we know that M is indecomposable, thus the coefficient quiver has to be connected. Since the number of arrows of this quiver is $\dim M - 1$, whereas the number of vertices is $\dim M$, we see that the coefficient quiver is a tree, thus M is a tree module.

6. Examples: Quivers of finite representation type.

Let us consider the representation finite quivers, thus the quivers of type A_n, D_n, E_6, E_7, E_8. The indecomposable representations of these quivers have been determined by Gabriel [G] in 1972; for the non-trivial representations of a quiver with "subspace orientation", his paper has exhibited corresponding matrix presentations using 0-1-matrices. For most of the cases, he used bases so that the coefficient quivers are trees. It seems to be worthwhile to exhibit tree presentations also in the remaining cases. The existence of such presentations is mentioned already by Crawley-Boevey in [CB2]. We will use the notations of Gabriel's paper [G].

Case E_7. The representation N_1. It is given by k^4 with subspaces

$$k\times 0^3 \subset k^2\times 0^2 \subset k^3\times 0\ , \quad 0^2\times k^2 \subset 0\times k^3 \quad U$$

where U is generated by $(1,1,1,0)$ and $(1,0,0,1)$.

Case E_8. The representation P_3. It is given by k^5 with subspaces

$$k\times 0^4 \subset k^2\times 0^3 \subset k^3\times 0^2 \subset k^4\times 0\ , \quad 0^4\times k \subset 0^2\times k^3\ , \quad U$$

where U is generated by $(1,1,1,0,0),\ (1,0,0,1,0),\ (0,1,0,0,1)$.
The representation Q_2. It is given by k^6 with subspaces

$$k\times 0^5 \subset k^2\times 0^4 \subset k^3\times 0^3 \subset k^5\times 0\ , \quad 0^4\times k^2 \subset 0^2\times k^4\ , \quad U$$

where U is generated by $(1,1,1,0,1,0),\ (0,1,0,1,0,0),\ (0,0,0,1,0,1)$.
The representation Q_3. It is given by k^6 with subspaces

$$k\times 0^5 \subset k^2\times 0^4 \subset k^4\times 0^2 \subset k^5\times 0\ , \quad 0^4\times k^2 \subset 0^2\times k^4\ , \quad U$$

where U is generated by $(1,0,0,1,1,0),\ (0,1,1,0,0,0),\ (1,0,1,0,0,1)$.

7. Directed quivers with 2 vertices: A recipe

As we have mentioned above, we are going to write down explicitly the preprojective representations $P(t)$ of the directed quiver $\Theta(n)$ with 2 vertices and n arrows. The special cases $n \leq 2$ are well-known (in the case $n = 2$, we deal with matrix pencils as considered already by Kronecker), thus let us assume that $n \geq 3$.

First, let us define inductively the numbers a_t as follows:

$$a_0 = 0, \quad a_1 = 1, \quad a_{t+1} = n\cdot a_t - a_{t-1} \ \text{ for } \ t \geq 1.$$

Clearly, the dimension vector of $P(t)$ is $[a_t, a_{t+1}]$, therefore we have to exhibit n matrices of size $a_{t+1} \times a_t$, or, as we prefer, their transposes, thus $(a_t \times a_{t+1})$-matrices.

Given two matrices A_1, A_2 with the same number of rows, let $A_1 \big| A_2$ be the matrix obtained by *concatenation*: if A_1 has b_1 columns, and A_2 has b_2 columns, then the first b_1 columns of $A_1 \big| A_2$ are those of A_1, whereas the column of $A_1 \big| A_2$ with index $b_1 + i$ is equal to the column of A_2 with index i, for $1 \le i \le b_2$.

For any pair of non-negative numbers a, b, let $Z(a;b)$ be the zero $(a \times b)$-matrix; also, let $E(a;b)$ be the $(a \times b)$-matrix with diagonal entries $E(a;b)_{ii} = 1$, and all other entries equal to zero; we will consider $E(a;b)$ only in case $b \le a$, so that $E(a;b)$ has rank b and has the following shape:

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & \\ & 0 & \\ & & \end{bmatrix}$$

The matrices which we will write down will be concatenations of matrices of the form $Z(a;b)$ and $E(a;b)$, where $a = a_t$ is fixed and any b belongs to the set $\{a_1, a_2, \dots, a_t\}$.

Let us denote by $E'(a;b,c)$ the alternating concatenation of $n-2$ copies of $E(a;b)$ and $n-3$ copies of $Z(a;c)$, thus

$$E'(a;b,c) = E(a;b)\big|Z(a;c)\big|E(a;b)\big|\cdots\big|Z(a;c)\big|E(a;b)\big|,$$

of course, it is an $(a \times e')$-matrix, with $e' = (n-2)b + (n-3)c$. Let

$$E(a;b,c) = E'(a;b,c)\big|Z(a;b),$$

this is an $(a \times e)$-matrix, with $e = (n-1)b + (n-3)c$.

¿From now on, we fix some $t \ge 1$. Let $a = a_t$.

Proposition. *The representation $P(t)$ of $\Theta(n)$ is given by the transposes of the following n matrices $M_1, \dots, M_n$*

$$M_i = Z(a;(i-1)a)\big|E(a;a)\big|Z(a;a_{t+1}-ia)$$

for $1 \le i < n$, and

$$M_n = Z(a;a_{t-1})\big|E(a;a_{t-1},a_t)\big|\cdots\big|E(a;a_1,a_2)\big|Z(a;n-1)\big|E(a;a).$$

Here, the dots refer to the concatenation of all the matrices $E(a;a_{i-1},a_i)$ with $t \geq i \geq 2$. The matrix $Z(a;n-1)$ may be considered as the concatenation of $E(a;a_0,a_1)\big|Z(a;2)$, thus an alternativ description of M_n is

$$M_n = Z(a;a_{t-1})\big|E(a;a_{t-1},a_t)\big|\cdots\big|E(a;a_0,a_1)\big|Z(a;2)\big|E(a;a).$$

Note that for $t = 1$, this concatenation is reduced to the $(1\times n)$-matrix $M_n = Z(a;n-1)\big|E(a;a) = [\,0 \quad 0 \quad \cdots \quad 0 \quad 1\,]$. Thus, for $t = 1$ the matrices $M_1,\dots,M_n$ are just all possible $(1\times n)$-matrices with one entry 1 and the remaining entries 0.

As a less trivial example, consider the case $n = 4$ and $t = 2$. In this case, the matrix M_4 is of the form

$$\begin{aligned} M_4 &= Z(4;1)\big|E(4;1,4)\big|Z(4;3)\big|E(4;4) \\ &= Z(4;1)\big|E(4;1)\big|Z(4;4)\big|E(4;1)\big|Z(4;1)\big|Z(4;3)\big|E(4;4) \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & & & \\ & & & & & & & & & & & & 1 & & \\ & & & & & & & & & & & & & 1 & \\ & & & & & & & & & & & & & & 1 \end{bmatrix} \end{aligned}$$

(as usual, all the remaining entries are supposed to be zero).

Proof of Proposition: We use the notation introduced in section 4, but we change the inductive construction of $P(t{+}1)$ of $\Theta(n)$ slightly: First of all, instead of $P(t{+}1) = (V,W;\beta_1,\dots,\beta_n)$, we consider the isomorphic representation

$$P'(t{+}1) = (V,W;\beta_2,\dots,\beta_n,\beta_1)$$

(it is isomorphic to $P(t{+}1)$ since it is indecomposable and has the same dimension vector). Second, instead of starting with $M = (U,V,\alpha_1,\dots,\alpha_n)$, we construct $P(t{+}1)$ and $P'(t{+}1)$ as cokernels of a corresponding map α' which is based on the representation $M' = (U,V,\alpha_1,-\alpha_2,\dots,-\alpha_n)$ (note that M' is isomorphic to M, again using the argument that M' is indecomposable and has the same dimension vector as M). In this way, we remove all the minus signs in the matrix presentations of $P(t{+}1)$ and $P'(t{+}1)$.

It remains to be seen that the matrices corresponding to the maps

$$\beta_2,\dots,\beta_n,\beta_1$$

are as exhibited above. This should be clear for the first $n-1$ matrices. Thus let us consider the matrix corresponding to β_1, or better its transpose; we denote this transpose by M_n', it as an $(a_{t+1}\times a_{t+2})$-matrix. As abbreviation, let $a' = a_{t+1}$, and let $d = a' - a$. By induction we assume that the transpose of the matrices describing $P'(t)$ are the matrices $M_1, \dots, M_n$ as described above. Let $\underline{M_i}$ be the $(a' \times a')$-matrix obtained from the $(a \times a')$-matrix M_i by adding d zero rows as last rows. Let F be the $(a \times d)$-matrix with coefficients $F_{a+i,i} = 1$, for $1 \le i \le d$, and $F_{ij} = 0$ otherwise (thus, the first a rows are zero rows, the remaining rows of F form a unit matrix); the transpose of this matrix just describes the projection π. Note that M_n' is the following concatenation

$$M_n' = \underline{M_2}\big|\cdots\big|\underline{M_n}\big|F.$$

In order to rewrite this matrix, note that for $1 \le i < n$, we have

$$\underline{M_i} = Z(a';(i-1)a)\big|E(a';a)\big|Z(a';a'-ia),$$

therefore

$$\begin{aligned}
&\underline{M_2}\big|\cdots\big|\underline{M_{n-1}}\\
&\quad = Z(a';a)\big|E(a';a)\big|Z(a';a'-2a)\big|Z(a';2a)\big|E(a';a)\big|Z(a';a'-3a)\big|\cdots\\
&\qquad\qquad \cdots\big|Z(a';(n-2)a)\big|E(a';a)\big|Z(a';a'-(n-1)a)\\
&\quad = Z(a';a)\big|E'(a';a,a')\big|Z(a';a'-(n-1)a).
\end{aligned}$$

Let us stress that the last matrix mentioned here is $Z(a';a'-(n-1)a)$, whereas the first one of

$$\underline{M_n} = Z(a';a_{t-1})\big|E(a';a_{t-1},a_t)\big|\cdots\big|E(a';a_1,a_2)\big|Z(a';n-1)\big|E(a';a).$$

is $Z(a';a_{t-1})$. These two matrices $Z(a';a'-(n-1)a)$ and $Z(a';a_{t-1})$ concatenate to $Z(a';a)$, since

$$a'-(n-1)a + a_{t-1} = a.$$

It remains to concatenate $\underline{M_n}$ and F. The last matrix involved in the concatenation which yields $\underline{M_n}$ is $E(a';a)$. Of course, $E(a';a)\big|F = E(a',a')$. Altogether we see that

$$M_n' = Z(a';a)\big|E(a';a_t,a_{t+1})\big|\cdots\big|E(a';a_1,a_2)\big|Z(a';n-1)\big|E(a';a'),$$

as required.

The case $n = 3$. For $t = 1$, the matrices M_1, M_2, M_3 are as follows:

$$\begin{matrix}[1 & 0 & 0]\\ [0 & 1 & 0]\\ [0 & 0 & 1]\end{matrix}$$

For $t = 2$, they are:

$$\begin{bmatrix} 1 & & & 0 & 0 & 0 & 0 & 0 \\ & 1 & & & & & & \\ & & 1 & & & & & \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & & & 0 & 0 \\ & & & & 1 & & & \\ & & & & & 1 & & \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

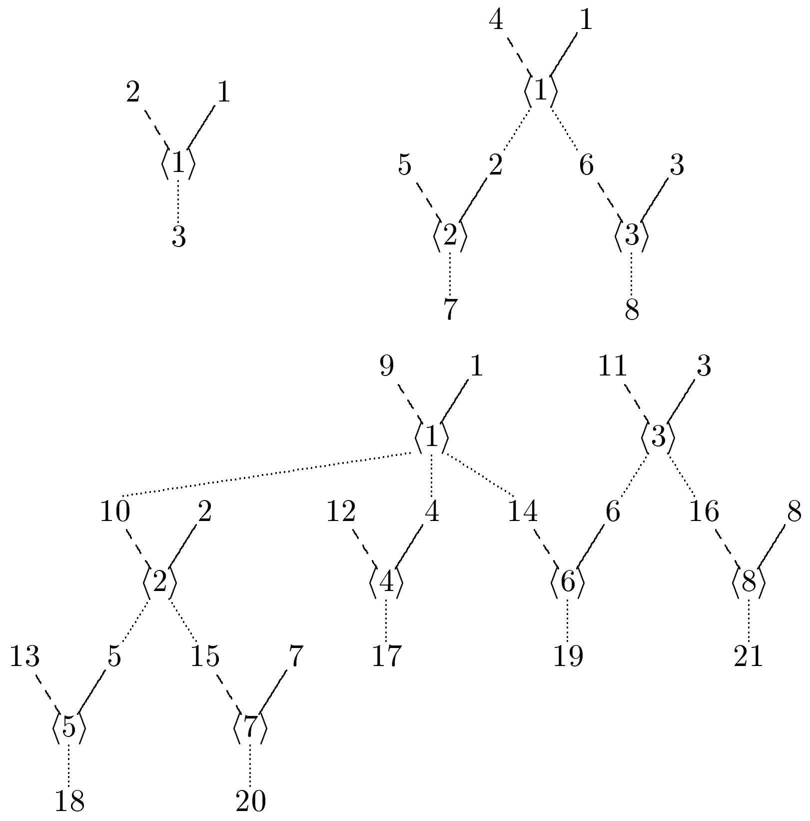
For $t = 3$, we deal with the matrices

$$\begin{bmatrix} 1 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & & & & & & & & & & & & & & & & & & & \\ & & 1 & & & & & & & & & & & & & & & & & & \\ & & & 1 & & & & & & & & & & & & & & & & & \\ & & & & 1 & & & & & & & & & & & & & & & & \\ & & & & & 1 & & & & & & & & & & & & & & & \\ & & & & & & 1 & & & & & & & & & & & & & & \\ & & & & & & & 1 & & & & & & & & & & & & & \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & 1 & & & & & & & & & & & \\ & & & & & & & & & & 1 & & & & & & & & & & \\ & & & & & & & & & & & 1 & & & & & & & & & \\ & & & & & & & & & & & & 1 & & & & & & & & \\ & & & & & & & & & & & & & 1 & & & & & & & \\ & & & & & & & & & & & & & & 1 & & & & & & \\ & & & & & & & & & & & & & & & 1 & & & & & \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & & & & & & & \\ & & & & 1 & & & & & & & & & & 1 & & & & & & \\ & & & & & 1 & & & & & & & & & & 1 & & & & & \\ & & & & & & & & & & & & & & & & 1 & & & & \\ & & & & & & & & & & & & & & & & & 1 & & & \\ & & & & & & & & & & & & & & & & & & 1 & & \\ & & & & & & & & & & & & & & & & & & & 1 & \\ & & & & & & & & & & & & & & & & & & & & 1 \end{bmatrix}$$

Here are the corresponding coefficient quivers: we label the vertices using numbers, they refer to the usual row and the column indices, and we add a bracket to mark the row indices. Of course, there is no need to label the vertices, but it may be helpful for the reader in order to be able to compare the quivers with the matrices as exhibited above. This distinction between the row and the column indices allows us to drop the orientation of the arrows; for the preprojective representations, the arrows point from the vertices with brackets to those without (for the preinjective representations, one has to take the opposite direction). In order to distinguish the different colours of the arrows, we use solid, dashed and dotted ones (the solid arrows correspond to the matrix M_1, the dashed ones to M_2, and the dotted ones to the matrix M_3).



8. An open question

The aim of this note was to show that any exceptional representation M of a quiver over a field k is a tree module, thus is given by a coloured tree $c : \Gamma \to Q$. Note that our inductive construction of $c\colon \Gamma \to Q$ seems to depend on the base field k, or at least its characteristic. Our procedure uses two ingredients: First of all, a tree presentation of the preprojective and the preinjetive representations of a directed quiver with two vertices, and these presentations will work over any field. Second, given two orthogonal exceptional representations M, M', we need a basis of $\operatorname{Ext}^1_{kQ}(M, M')$, or better: a subset $\Phi_{MM'}$ of $\mathcal{E}_{MM'}$ such that the residue classes modulo the image of $\delta_{M,M'}$ form a basis of $\operatorname{Ext}^1_{kQ}(M, M')$. Our choice of $\Phi_{MM'}$ was based on the trivial fact that for any vector space over k, any generating set contains as a subset a basis, but such a choice may depend on the k.

In general, we deal with the following problem: Assume there is given an $(n \times m)$-matrix Δ with coefficients 0 and 1. For any field k, this matrix Δ describes a linear map $\delta = \delta_\Delta \colon k^m \to k^n$, and we are looking for a subset $I = I(\Delta, k)$ of $\{1, 2, \ldots, n\}$ such that the subspace $\langle I \rangle$ of k^n generated by the basis vectors with index in I is a direct complement for the image of δ. In general, the choice of such a subset I will depend on the characteristic of k. For example, the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

is regular in characteristic different from 2 (thus I has to be empty), but singular in characteristic 2 (take for I any singleton).

Problem 1. Let M be an exceptional representation of Q over k. Does there exist a (coloured) tree T which occurs as the coefficient quiver of M with respect to some basis $\mathcal{B}$, such that for any field k' the representation of Q with coefficient quiver T and coefficients 0 and 1 is indecomposable?

Actually, we may ask the following stronger queation:

Problem 2. Let M be an exceptional representation of Q over k and let T be any (coloured) tree which occurs as the coefficient quiver of M with respect to some basis $\mathcal{B}$. Let k' be a field. Is the representation of Q over k' with coefficient quiver T and coefficients 0 and 1 indecomposable?

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