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The Liu-Schulz Example

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ABSTRACT. Liu Shiping and Rainer Schulz have constructed a symmetric algebra $\Lambda=\Lambda(q)$ of dimension 8 and an indecomposable Λ-module M such that all the syzygy modules $\Omega^t M$ are 4-dimensional and pairwise non-isomorphic. In this way, they have exhibited an Auslander-Reiten component which contains infinitely many isomorphism classes of modules of dimension 4. The algebra $\Lambda(q)$ depends on a non-zero parameter $q\in k$, and the interesting behaviour occurs when q is not a root of unity. On the other hand, the case $q=1$ yields the exterior algebra $\Lambda(1)$ on the 3-dimensional vector space k^3, thus we may consider $\Lambda(q)$ as a sort of *quantization* of $\Lambda(1)$. The purpose of this note is to present a more detailed study of properties of this algebra $\Lambda=\Lambda(q)$ and of some of the Λ-modules.

Let k be a field, let R be a finite-dimensional k-algebra. We denote by τ (= $D\,\mathrm{Tr}$) the Auslander-Reiten translation in the category of finite-dimensional R-modules. Liu Shiping and Rainer Schulz have exhibited in [LS] an 8-dimensional algebra Λ and an indecomposable Λ-module M such that all the modules $\tau^t M$ (with $t\in\mathbb{Z}$) are 4-dimensional cyclic right ideals of Λ and pairwise non-isomorphic. In this way, they have answered a question raised in [R].

The algebra Λ is symmetric, thus $\tau=\Omega^2$, where Ω is Heller's loop functor, and even all the modules $\Omega^t M$ (with $t\in\mathbb{Z}$) are 4-dimensional cyclic right ideals and pairwise non-isomorphic. Such a behaviour would be impossible for some other classes of algebras such as group algebras of finite groups. We recall the following: let G be a finite group and kG its group algebra, let X be an indecomposable non-projective kG-module. Eisenbud [E] has shown: if the modules $\Omega^t X$ (with $t\in\mathbb{N}$) are of bounded dimension, then $\Omega^t X\simeq X$ for some $t\geq 1$ ("the modules of complexity 1 are just the periodic ones").

The algebra $\Lambda=\Lambda(q)$ constructed by Liu-Schulz depends on a non-zero parameter $q\in k$, and the interesting behaviour mentioned above occurs when q is not a root of unity. The purpose of this note is to present a detailed study of properties of the algebra $\Lambda(q)$ and of some of the $\Lambda(q)$-modules. The case $q=1$ yields the

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exterior algebra $\Lambda(1)$ on the 3-dimensional vector space k^3, and we may consider $\Lambda(q)$ as a sort of "quantization" of $\Lambda(1)$. Also, over a field of characteristic 2, the algebra $\Lambda(1)$ is just the group algebra kE, where E is the elementary abelian group of order 8.

The example exhibited by Liu and Schulz is very interesting. The reader should observe that the features mentioned below should be considered not as pathological ones; in contrast, it seems that they are rather natural, and that it will be easy to construct many other algebras with similar properties. We may take these considerations as an indication that the representation theory of finite-dimensional algebras still is at its beginning: there may be many other basic features which still have to be discovered.

We are going to consider the class $\mathcal{M}$ of 4-dimensional cyclic right ideals of $\Lambda(q)$ in detail. We want to show the following: First, if M belongs to $\mathcal{M}$, then also the syzygy module ΩM belongs to $\mathcal{M}$. Second, the Auslander-Reiten component containing a module in $\mathcal{M}$ is of the form $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\tau^n$ for some n, and all the modules on the boundary of the component belong to $\mathcal{M}$.

The algebras $A(q)$ and $\Lambda(q)$ considered below are special cases of Sklyanin algebras and their Koszul duals, as studied by Artin, Tate, van den Bergh [ATB], by Paul Smith [Sm2,Sm3,Sm4] and others. We recommend the report [Sm4] by Smith which outlines the relevant concepts; the Liu-Schulz example is mentioned there in Remark 10.9. The present note may be considered as an elementary introduction. The author is endebted to Paul Smith for helpful comments.

1. The algebra $\Lambda = \Lambda(q)$

Let q be a non-zero element of k. We define the following algebra $\Lambda = \Lambda(q)$ using generators and relations.

As **generators** we take $x_0,\ x_1,\ x_2$; the **relations** are

$$x_i^2 = 0, \quad \text{and} \quad x_{i+1}x_i + q\cdot x_ix_{i+1} = 0 \quad \text{for} \quad i = 0,1,2$$

(here, and in the following, $x_3 = x_0$; in fact, we may take as indices residue classes of integers modulo 3).

Remark 1. Note that $\Lambda(q)$ may be considered as a q-deformation of $\Lambda(1)$ and $\Lambda(1)$ is just the exterior algebra on k^3 (another quantization of the exterior algebra $\Lambda(1)$ on k^3 has been considered in [Sm1]: there, one takes the relations $x_jx_i = q\cdot x_ix_j$ for $i < j$, and $x_i^2 = 0$). For k a field of characteristic 2, we may interprete $\Lambda(1)$ also as the group algebra of the elementary abelian group of order 8.

Remark 2. The considerations below will show that the ideal I generated by x_0 is 4-dimensional, and it is easy to see that Λ is isomorphic to the trivial extension of Λ/I by I. This is the way the algebra Λ is exhibited in the Liu-Schulz paper [LS]. Note however that in this presentation the 3-fold symmetry which will be of interest in all our considerations, remains hidden.

Since the relations are homogeneous, we may consider Λ as a graded algebra

$$\Lambda = \bigoplus_{i \geq 0} \Lambda_i,$$

where Λ_i is the k-subspace of Λ generated by the monomials of degree i. Clearly, $\Lambda_0 = k$, and x_0, x_1, x_2 is a basis of Λ_1. Also, it is easy to see that x_0x_1, x_1x_2 and x_2x_0 is a basis of Λ_2.

1.1. *The subspace Λ_2 is contained in the center of Λ.*

Proof. It is obvious that

$$x_0(x_0x_1) = 0 = (x_0x_1)x_0,$$
$$x_1(x_0x_1) = 0 = (x_0x_1)x_1.$$

Also, we have

$$x_2(x_0x_1) = (x_2x_0)x_1 = -q^{-1}x_0x_2x_1 = x_0(x_1x_2) = (x_0x_1)x_2.$$

This shows that x_0x_1 belongs to the center of Λ. Similarly, also x_1x_2 and x_2x_0 belong to the center of Λ.

1.2. *The space Λ_3 is one-dimensional with non-zero element*

$$x_0x_1x_2 = x_1x_2x_0 = x_2x_0x_1;$$

and we have $\Lambda_i = 0$ for $i \geq 4$.

Proof: Since x_0x_1 and x_1x_2 belong to the center of Λ, we have

$$x_1x_2x_0 = x_0x_1x_2 = x_2x_0x_1.$$

It follows that this element $z = x_1x_2x_0$ generates Λ_3. In order to see that z is non-zero, we only have to show that $\Lambda_3 \neq 0$.

Let F be the free k-algebra generated by X_0, X_1, X_2, let F_i be the subspace of F generated by all monomials of degree i, thus $F = \bigoplus_{i \geq 0} F_i$ is a graded algebra. We denote by I the k-subspace in F generated by the elements X_i^2 and $X_{i+1}X_i + q \cdot X_iX_{i+1}$ for $i = 0, 1, 2$, thus $\Lambda = F/FIF$, with $x_i = X_i + FIF$.

We note the following: the subspaces $I' = F_1I$ and $I'' = IF_1$ are at most 18-dimensional, they are subspaces of F_3, and F_3 has dimension 27. The intersection $I' \cap I''$ contains the elements X_i^3, the elements

$$X_i^2X_{i+1} + qX_iX_{i+1}X_i + q^2X_{i+1}X_i^2, \quad \text{and} \quad X_iX_{i+1}^2 + qX_{i+1}X_iX_{i+1} + q^2X_{i+1}^2X_i,$$

where $i = 0, 1, 2$, and also

$$\begin{aligned} X_0(X_2X_1 + qX_1X_2) + X_1(X_0X_2 + qX_2X_0) + X_2(X_1X_0 + qX_0X_1) \\ = (X_2X_1 + qX_1X_2)X_0 + (X_0X_2 + qX_2X_0)X_1 + (X_1X_0 + qX_0X_1)X_2. \end{aligned}$$

Clearly, these ten elements are linearly independent. Thus the dimension of $I'+I''$ is at most $2\cdot 18-10=26$. This shows that $\Lambda_3 = F_3/(I'+I'')\neq 0$.

In order to see that Λ_4 (and therefore Λ_i for all $i\geq 4$) is zero, it is sufficient to observe that $x_iz=0$ for $i=0,1,2$. But this follows from the different presentations $z=x_ix_{i+1}x_{i+2}$ and the relation $x_i^2=0$.

As a consequence, we have:

1.3. *The algebra $\Lambda=\bigoplus_{i=0}^3\Lambda_i$ is 8-dimensional.*

Since Λ_3 is one-dimensional with basis element z, the multiplication map yields a bilinear map $\sigma\colon \Lambda_1\times\Lambda_2\to k$ with

$$\sigma(x,y)z = xy.$$

1.4. *The bilinear map σ is non-degenerate.*

Proof: Consider the basis x_0,x_1,x_2 of Λ_1 and the basis $y_0=x_1x_2$, $y_1=x_2x_0$, $y_2=x_0x_1$ of Λ_2. We have $x_iy_i=x_0x_1x_2$, and $x_iy_j=0$ for $i\neq j$.

There is the following corollary:

1.5. *The algebra Λ is symmetric with socle Λ_3.*

Proof: We have to construct a linear form $\phi\colon \Lambda\to k$ such that, on the one hand, the only proper right ideal contained in the kernel of ϕ is 0, whereas, on the other hand, we have $\phi(xy)=\phi(yx)$ for all $x,y\in\Lambda$.

We take as ϕ the projection of $\Lambda=\bigoplus\Lambda_i$ onto $\Lambda_3=k$. Since we have $xy=yx$ for $x\in\Lambda_1, y\in\Lambda_2$, we see easily that $\phi(xy)=\phi(yx)$ for all $x,y\in\Lambda$. Since we know in addition that the bilinear map $\sigma\colon \Lambda_1\times\Lambda_2\to k$ is non-degenerate, it follows that Λ_3 is the right socle of Λ. If U is a non-zero right ideal of Λ, then U intersects the right socle non-trivially and thus $\Lambda_3\subseteq U$. It follows that U is not contained in the kernel of ϕ.

2. The right ideals generated by elements of Λ_1

We now deal with the right ideals generated by non-zero elements of Λ_1. Thus we consider $M_\alpha = x_\alpha\Lambda$, where

$$x_\alpha = \alpha_0x_0+\alpha_1x_1+\alpha_2x_2,$$

with $\alpha=(\alpha_0,\alpha_1,\alpha_2)$ a non-zero element of k^3.

For $i=0,1,2$, we denote by L_i the set of non-zero elements $\alpha=(\alpha_0,\alpha_1,\alpha_2)$ of k^3 with $\alpha_i=0$. Note that we obtain in this way just three lines in $\mathbb{P}_2$.

2.1 *Let α be a non-zero element of k^3. Then*

$$\dim_k M_\alpha = \begin{cases} 4 & \textit{in case } \alpha\in L_0\cup L_1\cup L_2 \textit{ or } q^3=1,\\ 5 & \textit{otherwise.}\end{cases}$$

If M_α is 5-dimensional, then $M_\alpha = kx_\alpha \oplus \Lambda_2 \oplus \Lambda_3$.

Proof. First, consider an element $\alpha \in L_0$. Let $y_\alpha = -q\alpha_1 x_0 x_1 + \alpha_2 x_2 x_0$. Note that

$$x_\alpha x_0 = \alpha_1 x_1 x_0 + \alpha_2 x_2 x_0 = -q\alpha_1 x_0 x_1 + \alpha_2 x_2 x_0 = y_\alpha.$$

We show that

$$x_\alpha,\ y_\alpha,\ x_1x_2,\ x_0x_1x_2$$

is a basis of M_α. We have $x_\alpha x_1 = q\alpha_2 x_1 x_2,\ x_\alpha x_2 = \alpha_1 x_1 x_2$. Since at least one of α_1, α_2 is non-zero, it follows that $x_1x_2 \in M_\alpha$. Also, $y_\alpha = x_\alpha x_0$ belongs to M_α. The elements y_α and x_1x_2 are linearly independent and they generate $M_\alpha \cap \Lambda_2$.

In this way, we have seen that M_α is 4-dimensional for $\alpha \in L_0$. The same argument can be used in case α belongs to L_1 or to L_2.

Assume now that α does not belong to one of the lines L_0, L_1, L_2. Let us multiply x_α by x_0, x_1, x_2 from the right, we obtain the following elements:

$$\begin{aligned}
x_\alpha x_0 &= -q\alpha_1 x_0 x_1 && && +\alpha_2 x_2 x_0 \\
x_\alpha x_1 &= && \alpha_0 x_0 x_1 - q\alpha_2 x_1 x_2 && \\
x_\alpha x_2 &= && && \alpha_1 x_1 x_2 - q\alpha_0 x_2 x_0
\end{aligned}$$

The determinant is

$$\det \begin{bmatrix} -q\alpha_1 & 0 & \alpha_2 \\ \alpha_0 & -q\alpha_2 & 0 \\ 0 & \alpha_1 & -q\alpha_0 \end{bmatrix} = (1-q^3)\alpha_1\alpha_2\alpha_3.$$

Since all $\alpha_i \neq 0$, we see that the determinant is zero precisely if $q^3 = 1$. Thus, for $q^3 = 1$, the (linearly independent) elements $x_\alpha x_0$ and $x_\alpha x_1$ generate $M_\alpha \cap \Lambda_2$, and therefore a basis of M_α is given by $x_\alpha, x_\alpha x_0, x_\alpha x_1, x_0x_1x_2$.

On the other hand, if $q^3 \neq 1$, then M_α contains Λ_2, thus $M_\alpha = kx_\alpha \oplus \Lambda_2 \oplus \Lambda_3$. This completes the proof.

Note that $kx_\alpha \oplus \Lambda_2 \oplus \Lambda_3$ is always a twosided ideal. We consider now a 4-dimensional right ideal M_α and want to see under what conditions this is a twosided ideal.

2.2 *Assume that α is a non-zero element of k^3 such that M_α is 4-dimensional. The right ideal M_α is a twosided ideal, if one of the following conditions is satisfied:*
(a) $\alpha \in (L_0 \cap L_1) \cup (L_1 \cap L_2) \cup (L_2 \cap L_0)$, *or*
(b) $\alpha \in L_0 \cup L_1 \cup L_2$ *and* $q^2 = 1$, *or*
(c) $q = 1$.
Otherwise, the twosided ideal generated by M_α is $kx_\alpha \oplus \Lambda_2 \oplus \Lambda_3$.

Proof: First, let us assume that α belongs to L_0. We have $x_1 x_\alpha = \alpha_2 x_1 x_2$ and $x_2 x_\alpha = -q\alpha_1 x_1 x_2$; these two elements belong to M_α. We consider now

$$x_0 x_\alpha = x_0(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 x_0 x_1 - q\alpha_2 x_2 x_0.$$

The two elements $y_\alpha = -q\alpha_1x_0x_1 + \alpha_2x_2x_0$ and $\alpha_1x_0x_1 - q\alpha_2x_2x_0$ are linearly independent if and only if the determinant

$$\det\begin{bmatrix} -q\alpha_1 & \alpha_2 \\ \alpha_1 & -q\alpha_2 \end{bmatrix} = (q^2-1)\alpha_1\alpha_2$$

is non-zero. This means: If $q^2 = 1$ or $\alpha_1 = 0$ or $\alpha_2 = 0$, then x_0x_α is a multiple of the element y_α and therefore belongs to M_α. In this case, we see that $\Lambda x_\alpha \subseteq x_\alpha\Lambda$, and clearly, we also have the opposite inclusion. On the other hand, if none of these three conditions is satisfied, then the k-subspace generated by x_0x_α and $y_\alpha = x_\alpha x_0$ contains the elements x_0x_1 and x_2x_0, thus the twosided ideal generated by x_α contains Λ_2. It follows that $\Lambda x_\alpha\Lambda = kx_\alpha \oplus \Lambda_2 \oplus \Lambda_3$.

Now assume that α does not belong to one of the lines L_0, L_1, L_2, and that $q^3 = 1$. In this case, $M_\alpha \cap \Lambda_2$ is generated by $x_\alpha x_0$ and $x_\alpha x_1$. Consider now in addition the element x_0x_α. We write down these three elements explicitly:

$$\begin{aligned} x_\alpha x_0 &= -q\alpha_1x_0x_1 && && +\alpha_2x_2x_0 \\ x_\alpha x_1 &= \alpha_0x_0x_1 && -q\alpha_2x_1x_2 && \\ x_0x_\alpha &= \alpha_1x_0x_1 && && -q\alpha_2x_2x_0 \end{aligned}$$

The determinant is

$$\det\begin{bmatrix} -q\alpha_1 & 0 & \alpha_2 \\ \alpha_0 & -q\alpha_2 & 0 \\ \alpha_1 & 0 & -q\alpha_2 \end{bmatrix} = (q-q^3)\alpha_1\alpha_2^2.$$

If $q \neq 1$, then also $q^2 \neq 1$ (since we assume that $q^3 = 1$), and then this determinant is non-zero, thus the elements $x_\alpha x_0,\ x_\alpha x_1,\ x_0x_\alpha$ generate Λ_2.

On the other hand, for $q = 1$, the determinant is zero, thus x_0x_α belongs to the subspace generated by $x_\alpha x_0,\ x_\alpha x_1$ (since these two vectors are linearly independent), thus to M_α. By symmetry, also x_1x_α and x_2x_α belong to M_α. It follows that M_α is a twosided ideal.

2.3 *Let α, β be non-zero elements of k^3. If the right modules M_α and M_β are isomorphic then they are equal, and this happens if and only if $k\alpha = k\beta$.*

Proof: If $k\alpha = k\beta$, then clearly M_α and M_β are isomorphic. Conversely, assume that M_α and M_β are isomorphic. Then $\Lambda M_\alpha = \Lambda M_\beta$, and therefore $k\alpha = k\beta$.

The last assertion allows to take the projective plane $\mathbb{P}_2$ as index set for the set of modules $\{M_\alpha |\, \alpha\}$ (given a non-zero element $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, we denote the corresponding point in $\mathbb{P}_2$ by $(\alpha_0 : \alpha_1 : \alpha_2)$; of course, this point is just the linear subspace $k\alpha$).

3. The corresponding syzygies

Let k^* be the set of non-zero elements of k. We define an action of k^* on the subset $L_0 \cup L_1 \cup L_2$ of $\mathbb{P}_2$, as follows: For $q \in k^*$ and $(\alpha_0 : \alpha_1 : \alpha_2) \in \mathbb{P}_2$, let

$$\begin{aligned} q * (0 : \alpha_1 : \alpha_2) &= (0 : \alpha_1 : q\alpha_2), \\ q * (\alpha_0 : 0 : \alpha_2) &= (q\alpha_0 : 0 : \alpha_2), \\ q * (\alpha_0 : \alpha_1 : 0) &= (\alpha_0 : q\alpha_1 : 0). \end{aligned}$$

3.1 *Let $\alpha \in L_0 \cup L_1 \cup L_2$. Then*

$$\Omega M_\alpha = M_{q*\alpha}.$$

Proof: We consider the case of an element of L_0, thus let $\alpha = (0, \alpha_1, \alpha_2)$. We use the following obvious equality:

$$(\alpha_1 x_1 + \alpha_2 x_2)(\alpha_1 x_1 + q\alpha_2 x_2) = 0.$$

The canonical map $\Lambda_\Lambda \to M_\alpha$ which sends $1 \in \Lambda$ to x_α has $x_{q*\alpha}$ in its kernel, thus we see that $M_{q*\alpha}$ is a submodule of ΩM_α. Since both modules $M_{q*\alpha}$ and ΩM_α are 4-dimensional, these modules are equal.

Consider now the case $q^3 = 1$. We define an action of k^* on the whole projective plane $\mathbb{P}_2$ as follows:

$$q * (\alpha_0 : \alpha_1 : \alpha_2) = (q^2\alpha_0 : \alpha_1 : q\alpha_2).$$

The restriction of this action to the lines L_0 and L_2 is the action considered above. Note that the assumption $q^3 = 1$ implies that the restriction also to L_1 coincides with the action considered above.

3.2. *Let $q^3 = 1$, and consider a nonzero triple $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in k^3$. Then*

$$\Omega M_\alpha = M_{q*\alpha}.$$

We have to verify that

$$(\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2)(q^2\alpha_0 x_0 + \alpha_1 x_1 + q\alpha_2 x_2) = 0,$$

but this is straight forward.

4. Graded modules

As we have noted above, the algebra $\Lambda = \Lambda(q)$ is a $\mathbb{Z}$-graded algebra, and it will be of interest to consider the category of graded Λ-modules. Graded modules will always be $\mathbb{Z}$-graded.

In general, given a graded algebra $R = \bigoplus_{i\in\mathbb{Z}} R_i$, and a graded R-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$, we denote by $M[n]$ the graded module obtained from M by shifting the degree: $M[n]_i = M_{i-n}$. Also, we say that M is *generated in degree* t provided M is generated as an R-module by the subspace M_t. We recall that an R-module M is said to be a *Koszul module* provided there exists a projective resolution of the form

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_i is a projective R-module and generated in degree i, for all $i \geq 0$.

The algebra Λ will be considered as a graded Λ-module (generated in degree 0). Of course, homogeneous right ideals are graded submodules. For example, the right ideal M_α with non-zero $\alpha \in k^3$, is a graded Λ-module; it is generated in degree 1.

4.1 *Let α be a non-zero element of k^3. Assume that α belongs to $L_0 \cup L_1 \cup L_2$ or that $q^3 = 1$. The module $M_\alpha[-1]$ is a Koszul module.*

Proof: We have an exact sequence of graded Λ-modules

$$0 \to M_{q*\alpha} \to P_0 \to M_\alpha[-1] \to 0,$$

where $P_0 = \Lambda_\Lambda$. Using induction, we can construct a projective resolution of $M_\alpha[-1]$, as required.

We denote by $\tilde{\tau}$ the Auslander-Reiten translation in the category of graded Λ-modules.

4.2 *The class of graded Λ-modules of the form $M_\alpha[n]$, where $\alpha \in \mathbb{P}_2$, and $n \in \mathbb{Z}$, such that M_α is 4-dimensional, is closed under $\tilde{\tau}$ and $\tilde{\tau}^{-1}$.*

Proof. Let us denote by $\tilde{\Omega} M$ the first syzygy module of M category of graded Λ-modules. For any indecomposable non-projective graded Λ-module, we have $\tilde{\tau} M = \tilde{\Omega}^2 M[-3]$, since $M \mapsto M[-3]$ is the Nakayama functor in the category of graded Λ-modules. We have seen above that $\tilde{\Omega} M_\alpha[-1] = M_{q*\alpha}$. This yields the assertion.

4.3. Theorem. *Let α be a non-zero element of k^3. Assume that α belongs to $L_0 \cup L_1 \cup L_2$ or that $q^3 = 1$. The Auslander-Reiten component of the category of graded Λ-modules which contains the module M_α is of the form $\mathbb{Z}A_\infty$ and all the modules on the boundary of the component are 4-dimensional with simple top and simple socle.*

Proof: This is an immediate consequence of [R2]. Starting with $M = M_\alpha[-1]$, we see that its predecessors in the Auslander-Reiten quiver of the category of graded Λ-modules form a cone. Of course, the same assertion is true for $M[n]$ with arbitrary $n \in \mathbb{Z}$, thus also for $\tilde{\tau}^{-1}M$. This shows that we obtain a component of the form $\mathbb{Z}A_\infty$, and that the modules on the boundary of the component are 4-dimensional with simple top and simple socle.

Given a Koszul module M, then also the modules $\Omega^n M[-n]$ are Koszul modules. A Koszul module M is always generated in degree 0. As a consequence, $M_i = 0$ for $i < 0$ and for $i > 3$. Actually, if $M_3 \neq 0$, then M has a non-zero projective direct summand. Let us assume that M has no indecomposable projective direct summand. In this case, we define the *dimension vector* of M as the triple $\mathbf{dim}\, M = [\dim M_0, \dim M_1, \dim M_2]$. We consider the linear transformation $\omega\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ given by the matrix

$$\begin{bmatrix} 3 & 3 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

4.4. *Let M be a Koszul Λ-module, and assume that M has no indecomposable projective direct summand. Then, for all $n \geq 0$,*

$$\mathbf{dim}\, \Omega^n M[-n] = (\mathbf{dim}\, M)\omega^n.$$

Proof: It is sufficient to consider the case $n=1$, the general case follows by induction. Let $a_i = \dim M_i$. Then the projective cover of M is of the form $a_0\Lambda_\Lambda$, and we have the following exact sequence

$$0 \to \Omega M \to a_0\Lambda_\Lambda \to M \to 0.$$

It follows that $\mathbf{dim}\,\Omega M[-1] = [3a_0, 3a_0, a_0] - [a_1, a_2, 0]$.

4.5. *Let M be an indecomposable non-projective Koszul Λ-module. Then either $\mathbf{dim}\,M$ is a multiple of $[1,2,1]$, or else $\mathbf{dim}\,\Omega^n M[-n] \neq \mathbf{dim}\,M$ for all $n \geq 1$.*

Proof: The only eigenvalue of the linear transformation ω is 1, and the only eigenvectors are the non-zero multiples of $[1,2,1]$.

5. The remaining 4-dimensional cyclic right ideals

5.1. *Let α be a non-zero element of k^3 and assume that M_α is not a twosided ideal. Let u be an element of ΛM_α which does not belong to $\Lambda_2 \oplus \Lambda_3$. Then $u\Lambda$ is isomorphic to M_α as a right Λ-module.*

Proof. Since M_α is not a twosided ideal, there exists some x_i (with $i = 0, 1$ or 2) such that $x_i x_\alpha$ does not belong to M_α. Since ΛM_α is 5-dimensional, we know that $\Lambda M_\alpha = M_\alpha + k x_i x_\alpha$, and therefore $u = x_\alpha a + c x_i x_\alpha$ for some $a \in \Lambda$ and some $c \in k$. Since u does not belong to $\Lambda_2 \oplus \Lambda_3$, we see that a has to be invertible. Thus u and ua^{-1} generate the same right ideal. We have

$$\begin{aligned} ua^{-1} &= x_\alpha + c x_i x_\alpha a^{-1} \\ &= x_\alpha + c a^{-1} x_i x_\alpha \\ &= (1 + c a^{-1} x_i) x_\alpha, \end{aligned}$$

since $x_i x_\alpha$ belongs to the center of Λ. It follows that the left multiplication by the (invertible) element $1 + ca^{-1}x_i$ yields an isomorphism $M_\alpha \to ua^{-1}\Lambda = u\Lambda$.

Given a right ideal U of Λ, we denote by $\operatorname{gr} U$ the corresponding graded right ideal; by definition, $\operatorname{gr} U = \bigoplus_i U_{(i)}$, where $U_{(i)} = \big(U \oplus \bigoplus_{j>i} \Lambda_j\big) \cap \Lambda_i$.

Let α be a non-zero element of k^3 and assume that M_α is 4-dimensional and a twosided ideal. We fix some element u_α in Λ_2 which does not belong to M_α. For $c \in k$, let

$$M_{\alpha,c} = (x_\alpha + c u_\alpha)\Lambda$$

(note that $M_{\alpha,c}$ will depend on the choice of u_α).

5.2. *Let α be a non-zero element of k^3 and assume that M_α is 4-dimensional and a twosided ideal. The modules $M_{\alpha,c}$ with $c \in k$ are pairwise non-isomorphic, and we have $\operatorname{gr} M_{\alpha,c} = M_\alpha$. Conversely, given a right ideal U such that $\operatorname{gr} U = M_\alpha$, then $U = M_{\alpha,c}$ for some $c \in k$.*

Proof. Clearly, $\operatorname{gr} M_{\alpha,c} = M_\alpha$. Conversely, let U be a right ideal such that $\operatorname{gr} U = M_\alpha$. Since $U_{(1)} = kx_\alpha$, there is $a \in \Lambda_2 \oplus \Lambda_3$ such that $x_\alpha + a$ belongs to U. The element $x_\alpha + a$ generates a right ideal of dimension at least 4, on the other hand,

U is 4-dimensional, thus $U = (x_\alpha + a)\Lambda$. Since $\Lambda_2 \oplus \Lambda_3$ is contained in $M_\alpha \oplus ku_\alpha$, we can write $a = x_\alpha b + cu_\alpha$ for some $b \in \Lambda$ and $c \in k$. Clearly, b has to belong to the radical of Λ, thus $1+b$ is invertible. We have $cu_\alpha(1+b)^{-1} = cu_\alpha + a'$ for some $a' \in \Lambda_3$, thus $a' = x_\alpha b'$ with $b' \in \Lambda_2$. Note that $u_\alpha b' = 0$, thus

$$x_\alpha + a = \big(x_\alpha + cu_\alpha\big)(1+b')(1+b),$$

and $(1+b')(1+b)$ is some invertible element of Λ. It follows that $x_\alpha + cu_\alpha$ generates U, thus $U = M_{\alpha,c}$.

An isomorphism $M_{\alpha,c} \to M_{\alpha,c'}$ is given by some left multiplication, thus there are invertible elements $a, b \in \Lambda$ such that

$$a(x_\alpha + cu_\alpha) = (x_\alpha + c'u_\alpha)b.$$

We write $a = \sum a_i,\ b = \sum b_i$, with $a_i, b_i \in \Lambda_i$, and we can assume that $a_0 = 1$. We must have

$$x_\alpha + cu_\alpha + a_1x_\alpha = x_\alpha b_0 + c'u_\alpha b_0 + x_\alpha b_1,$$

thus $b_0 = 1$ and

$$(c - c')u_\alpha = -a_1x_\alpha + x_\alpha b_1.$$

Since M_α is a twosided ideal, the right hand side belongs to M_α, whereas u_α does not belong to M_α. Thus we conclude that $c = c'$.

5.3. *Let α be a non-zero element of k^3 and assume that M_α is 4-dimensional and a twosided ideal. Let $c \in k$. If $q = 1$ or if α belongs to $(L_0 \cap L_1) \cup (L_1 \cap L_2) \cup (L_2 \cap L_0)$, then*

$$\Omega M_{\alpha,c} = M_{\alpha,-c}.$$

Otherwise, we can assume that $q = -1$, and $x_\alpha = x_i + \alpha_{i+1}x_{i+1}$ and $u_\alpha = x_{i+1}x_{i+2}$. Then

$$\Omega M_{\alpha,c} = M_{q*\alpha,-c}.$$

Proof. According to 2.2, we have to consider three cases. First, let us assume that $q = 1$. We have the equality

$$(x_\alpha + cu_\alpha)(x_\alpha - cu_\alpha) = 0,$$

since on the one hand u_α belongs to the center of Λ, and, on the other hand, $x_\alpha^2 = 0$; for the latter, we use the condition $q = 1$.

Similarly, if x_α is of the form x_i, then again we have the equality

$$(x_\alpha + cu_\alpha)(x_\alpha - cu_\alpha) = 0.$$

This shows that $\Omega M_{\alpha,c} = M_{\alpha,-c}$ in these cases.

Finally, consider the case $q = -1$, and $x_\alpha = x_i + \alpha_{i+1}x_{i+1}$. Let $u_\alpha = x_{i+1}x_{i+2}$. Then

$$\begin{aligned}(x_\alpha + &cx_{i+1}x_{i+2})(x_{q*\alpha} - cx_{i+1}x_{i+2}) \\ &= (x_i + \alpha_{i+1}x_{i+1} + cx_{i+1}x_{i+2})(x_i - \alpha_{i+1}x_{i+1} - cx_{i+1}x_{i+2}) \\ &= 0,\end{aligned}$$

since $x_\alpha x_{q*\alpha} = 0$ and $x_{i+1}x_{i+2}$ is in the center. Thus, we see that

$$\Omega M_{\alpha,c} = M_{q*\alpha,-c}.$$

Remark. Note that we have to specify for $q = -1$ the element u_α. Namely, if we work for $x_\alpha = x_i + x_{i+1}$ with the element $u'_\alpha = x_{i+2}x_i$, then we deal with the equality

$$(x_i + x_{i+1} + cx_{i+2}x_i)(x_i - x_{i+1} + cx_{i+2}x_i) = 0,$$

thus, for this choice u'_α, the syzygy module $\Omega M_{\alpha,c}$ has to be written in the form $M_{q*\alpha,c}$.

5.4. *Let α be a non-zero element of k^3 and assume that M_α is 4-dimensional and a twosided ideal. Let $c \in k$. Then*

$$\Omega^2 M_{\alpha,c} = M_{\alpha,c}.$$

We have denoted by $\mathcal{M}$ the class of 4-dimensional cyclic right ideal of Λ.

5.5. Theorem. *The class $\mathcal{M}$ is closed under Ω. If M is a non-homogeneous right ideal in $\mathcal{M}$, then $\Omega^2 M$ is isomorphic to M.*

Finally, let us consider arbitrary modules with simple top and simple socle.

5.6. *If U is a Λ-module with simple top and simple socle, then its dimension is $1, 2, 4, 5,$ or 8.*

Proof: Of course, U is isomorphic to a right ideal of Λ, thus we can assume that $U = x\Lambda$ for some non-zero element $x \in \Lambda$. If x belongs to Λ_3, then U is simple, if x belongs to $\Lambda_2 \oplus \Lambda_3$, but not to Λ_3, then U has length 2. If x belongs to the radical of Λ, but not to $\Lambda_2 \oplus \Lambda_3$, then the considerations above show that the dimension of U is 4 or 5. Of course, if x is not in the radical of Λ, the U is of dimension 8.

6. Auslander-Reiten components

6.1. Theorem. *Let M belong to $\mathcal{M}$. In the category of Λ-modules, the Auslander-Reiten component containing M is of the form $\mathbb{Z}A_\infty$ or of the form $\mathbb{Z}A_\infty/\tau^n$ for some n. All the modules on the boundary of the component belong to $\mathcal{M}$.*

Proof. If M is a homogeneous right ideal, then we use the covering theory of Gabriel and Green. The assertion follows from the corresponding assertions dealing with graded Λ-modules.

Consider now a non-homogeneous right ideal M in $\mathcal{M}$. Since the algebra Λ is symmetric, the Auslander-Reiten translation is given by $\tau = \Omega^2$. According to 5.5, we know that $\tau M = M$. It is well-known (see [HPR] or [H]) that this implies (for connected non-uniserial algebras) that M belongs to a homogeneous tube $\mathbb{Z}A_\infty/\tau$.

It remains to be seen that also in this case M is on the boundary of the component. Consider the Auslander-Reiten sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow M \longrightarrow 0.$$

Note that both the socle as well as the top of N has length 2. Thus, if N would be indecomposable, then $N = N_1 \oplus N_2$, where both modules N_1, N_2 have simple top and simple socle. Since N is of length 8, it follows from 4.6 that both modules N_1, N_2 are of length 4. However, there is an irreducible map $M \to N_1$, and this map is a proper monomorphism or a proper epimorphism, thus M and N_1 cannot have the same length.

Remark. We are grateful to the referee for pointing out the following: It is a direct consequence of Theorem 3.12 of [L] that a regular Auslander-Reiten component containing a τ-orbit which consists of modules of a fixed length is a stable tube or of the form $\mathbb{Z}A_\infty$.

7. The Koszul property

Let us consider finitely generated graded algebras $R = \bigoplus_{i\geq 0} R_i$ with $R_0 = k$ such that R_1 is a finite dimensional k-space which generates R as an algebra. A typical example is constructed as follows: start with a finite dimensional k-space V, and consider the tensor algebra $T(V) = \bigoplus_{i\geq 0} V^{\otimes i}$, where $V^{\otimes i}$ is the i-fold tensor power of V (with $V^{\otimes 0} = k$). The algebra R is said to be *quadratic*, provided R is the factor algebra of the tensor algebra $T(R_1)$ modulo an ideal generated by elements in $R_1^{\otimes 2}$. For any quadratic algebra R, there is defined its quadratic dual $R^!$, see [M]. Since R is finitely generated, we have $R^{!!} = R$.

The algebra R is said to be a *Koszul algebra* provided the simple R-module concentrated in degree 0 is a Koszul module. It is well-known that Koszul algebras always are quadratic, and if R is a Koszul algebra, then also $R^!$ is a Koszul algebra.

As before, q is a non-zero element of k. Let us consider the algebra $A = A(q)$ with generators a_0, a_1, a_2 and relations

$$a_i a_{i+1} - q \cdot a_{i+1} a_i = 0 \quad \text{for} \quad i = 0, 1, 2.$$

We consider A as a graded algebra $A = \bigoplus_{i\geq 0} A_i$, with $A_0 = k$ and A_1 the subspace of A with basis a_0, a_1, a_2. Note that A is a ring without zero divisors.

The algebras $\Lambda(q)$ and $A(q)$ both are quadratic.

7.1. *We have $A(q)^! = \Lambda(q)$.*

Proof: We consider the elements x_i as those of a dual basis to the basis a_i of A_1.

7.2. *The algebra $A(q)$ is a Koszul algebra.*

Proof. Consider the following sequence of maps

$$0 \longrightarrow A[3] \xrightarrow{\ f\ } A[2]^3 \xrightarrow{\ g\ } A[1]^3 \xrightarrow{\ h\ } A[0] \longrightarrow 0$$

where the maps f, g, h are given by the following matrices

$$f = \begin{bmatrix} x_2 \\ x_0 \\ x_1 \end{bmatrix}, \quad g = \begin{bmatrix} x_1 & 0 & -qx_2 \\ -qx_0 & x_2 & 0 \\ 0 & -qx_1 & x_0 \end{bmatrix}, \quad h = [\, x_0 \quad x_1 \quad x_2 \,].$$

It is easy to see that the compositions gf and hg are zero.

We want to determine the kernel of the map g. Thus, consider a triple (a_0, a_1, a_2) such that

$$x_1 a_0 = q x_2 a_2, \qquad q x_0 a_0 = x_2 a_1, \qquad q x_1 a_1 = x_0 a_2.$$

The first equality shows that a_0 is divisible (on the left) by x_2, say $a_0 = x_2 a$ for some $a \in A$. Then we see

$$x_2 a_1 = q x_0 a_0 = q x_0 x_2 a = x_2 x_0 a,$$

division by x_2 yields $a_1 = x_0 a$. Similarly, we get

$$q x_2 a_2 = x_1 a_0 = x_1 x_2 a = q x_2 x_1 a,$$

thus $a_2 = x_1 a$. This shows that the tripel (a_0, a_1, a_2) is the image of the element $a \in A$ under the map f.

On the other hand, the map h maps onto the maximal ideal of A generated by x_0, x_1, x_2.

It follows without difficulty that the above complex is a resolution of the trivial A-module k (consider homogeneous parts, and note that we only have to see that the dimensions are correct; but the dimensions are just those as in the special case $q = 1$).

As a consequence, we obtain:

7.3. *The algebra $\Lambda(q)$ is a Koszul algebra.*

8. More Koszul modules

We consider again graded Λ-modules. We have seen in the previous section that the simple Λ-module S generated in degree 0 is a Koszul module.

8.1. *Let J be the radical of Λ. Then $J[-1]$ is a Koszul module.*

Proof. Given a Koszul module M, then $\Omega M[-1]$ is a Koszul module, again. Of course, we have $J = \Omega S$.

There is the following general criterion:

8.2. *Let $\mathcal{N}$ be a class of graded R-modules, where R is a $\mathbb{Z}$-graded algebra. Assume for any module N in $\mathcal{N}$, the module $\Omega N[-1]$ has a finite filtration such that any factor is a Koszul module or belongs to $\mathcal{N}$. Then any module in $\mathcal{N}$ is a Koszul module.*

We can use this criterion in order to verify that many Λ-modules are Koszul modules.

8.3. *Let N be a 2-dimensional graded module, generated in degree 0. Then N is a Koszul module.*

Proof. If N is semisimple, then N is the direct sum of two copies of S, thus a Koszul module. Thus, we may assume that N is indecomposable. It follows that $N = \Lambda/U$, where U is a homogeneous right ideal of Λ of dimension 6. Of course, $U = \Omega N$. There is a 2-dimensional subspace $U' \subset \Lambda_1$ such that $U = U'\Lambda$. Since U' intersects non-trivially the subspace $kx_1 + kx_2$, there is an element $\alpha \in L_0$ such that U contains the right ideal M_α. As we know, $M_\alpha[-1]$ is a Koszul module, and $U[-1]/M_\alpha[-1]$ is again of the form Λ/V where V is a homogeneours right ideal of Λ of dimension 6.

8.4. *Let α be a non-zero element of k^3, and assume that the module M_α is 5-dimensional. Then $\Omega M_\alpha[-2]$ is a Koszul module.*

Proof. Clearly, $\Omega M_\alpha[-2]$ is an indecomposable graded module of dimension 2 and generated in degree 0.

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