A Construction of Endofinite Modules

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Let k be a field, let Λ be a finite dimensional k-algebra. Krause [K] has considered Λ-modules which are obtained as direct limits of an increasing chain of finite dimensional Λ-modules and which are similar to the Prüfer groups occurring in abelian group theory. He has shown that the Ziegler closure of such a module X always contains an indecomposable endofinite module of infinite length, a socalled "generic" module. This is of importance since the "generic" modules seem to parametrize the families of finite length modules, and families of finite length modules are one of the main objects of present concern [C]. An essential condition used by Krause is the existence of a locally nilpotent endomorphism of X with kernel of finite length. In his proof, Krause uses functor categories as well as model theoretical considerations. The aim of the present note is to present a direct approach (using only modules and their elements) in order to recover and strengthen his result. The author is endebted to Krause for many stimulating remarks.

Let φ be an endomorphism of the Λ-module X. Then φ induces an injection of $\operatorname{Ker}\varphi^{t+1}/\operatorname{Ker}\varphi^{t}$ into $\operatorname{Ker}\varphi^{t}/\operatorname{Ker}\varphi^{t-1}$, for $t \geq 1$. In case $\operatorname{Ker}\varphi$ is finite dimensional, it follows that the vector spaces $\operatorname{Ker}\varphi^{t}/\operatorname{Ker}\varphi^{t-1}$ have the same dimension, for almost all t.

Given a module M and any set I, we denote by M^I the product of copies of M which are indexed by the elements of I, and $M^{(I)}$ denotes the corresponding direct sum of these copies (the module M^I may be constructed as the set of functions $I \to M$, and $M^{(I)}$ is the submodule of all functions with finite support).

Theorem 1. *Let X be an infinite dimensional Λ-module, let φ be a locally nilpotent endomorphism of X with kernel of finite length. Let I be some infinite set. Then the product X^I is the direct sum of a non-zero module $\overline{P}$ of finite endolength and of copies of X.*

The endolength of $\overline{P}$ is bounded by the minimum of the dimension of the vector spaces $\operatorname{Ker}\varphi^{t}/\operatorname{Ker}\varphi^{t-1}$, *with $t \geq 1$.*

Note that if X is a Λ-module and φ is a locally nilpotent endomorphism of X with kernel of finite length, then X satisfies the descending chain condition when considered as a $k[\varphi]$-module and thus also when considered as a module over $\mathrm{End}_\Lambda(X)$. As a consequence, X is Σ-algebraically compact and therefore X as well as all the powers X^I can be written as direct sums of modules with local endomorphism rings [JL, 8.1, 8.2]. Also, if we fix a direct decomposition of a Σ-algebraically compact module $M = \bigoplus_i M_i$ where the M_i are modules with local endomorphism rings, then Azumaya's Theorem (which generalizes the Theorem of Krull-Remak-Schmidt) shows that any indecomposable direct summand U is isomorphic to one of the modules M_i (note that any direct complement of U is also Σ-algebraically compact and therefore again a direct sum of indecomposables, thus U is one of the direct summands of a decomposition of M into indecomposables).

For any countably generated Λ-module M, let $\mathcal{G}(M)$ be the set of isomorphism classes of those indecomposable direct summands of $M^{\mathbb{N}}$ which are infinite dimensional and of finite endolength (thus "generic" modules in the terminology introduced in [C]).

Note that $\mathcal{G}(M)$ may be empty: take any finite dimensional module N and let $M = N^{(\mathbb{N})}$ be the direct sum of countably many copies of N. Then any product of copies of M is again a direct sum of copies of N, thus $M^{\mathbb{N}}$ has no indecomposable direct summand which is infinite dimensional. Let us stress that the examples of this kind satisfy the usual requirements considered in this paper, provided N is non-zero: the module M is infinite dimensional, and there is the shift endomorphism φ (sending the summand with index i to that with index $i-1$ and mapping the first one to zero); this is a locally nilpotent endomorphism of M and its kernel is N, thus of finite length.

Theorem 2. *Let X be an infinite dimensional Λ-module, and assume that X has a locally nilpotent endomorphism φ with kernel of finite length. Then:*

(a) *Let I be any infinite set. The isomorphism class of a module U belongs to $\mathcal{G}(X)$ if and only if U is indecomposable, isomorphic to a direct summand of X^I, but not isomorphic to a direct summand of X.*

(b) *If X has an indecomposable direct summand which is infinite dimensional, then $\mathcal{G}(X)$ is not empty.*

(c) *The number of isomorphism classes in $\mathcal{G}(X)$ is bounded by the minimum of the dimension of the various vector spaces $\mathrm{Ker}\,\varphi^t / \mathrm{Ker}\,\varphi^{t-1}$, with $t \geq 1$.*

(d) *Let U be a finitely generated submodule of X with $\varphi(U) \subseteq U$. Then $\mathcal{G}(X) = \mathcal{G}(X/U)$.*

The last assertion has the following consequence:

Let X be an infinite dimensional Λ-module, let φ be a locally nilpotent endomorphism with kernel of finite length. Then there exists an infinite dimensional Λ-module X', and a locally nilpotent endomorphism φ' of X' with kernel of finite length such that φ' is, in addition, surjective and $\mathcal{G}(X) = \mathcal{G}(X')$.

Namely, let $U = \operatorname{Ker} \varphi^t$ with sufficiently large t. Then U is of finite length and $\varphi(U) \subseteq U$. Let $X' = X/U$, and let φ' be the endomorphism of X' induced by φ. Then φ' is locally nilpotent and has a kernel of finite length. In addition, φ' will be surjective, for large t.

1. Filtered modules. Let us start with a module M which is endowed with a filtration

$$0 = M[0] \subseteq M[1] \subseteq \cdots \subseteq M[t] \subseteq \cdots \subseteq \bigcup_{t\in\mathbb{N}} M[t] = M,$$

using finite dimensional submodules $M[t]$ indexed by integers $t \geq 0$.

Let I be any set. Let $P = M^I$. We construct submodules of P as follows: For $t \geq 0$, let $P'[t] = (M[t])^I$, and let $P' = \bigcup_{t\in\mathbb{N}} P'[t]$, thus there are the inclusions:

$$0 = P'[0] \subseteq P'[1] \subseteq \cdots \subseteq P'[t] \subseteq \cdots \subseteq P' \subseteq P.$$

(Whenever it is useful, we will write P_M instead of P and P'_M instead of P'; note however, that P'_M not only depends on M but on the given filtration.)

We claim that P' *is a pure submodule of P and isomorphic to a direct sum of copies of M.* Since k is a field, the module k^I is a free k-module. Thus there is an index set J and an isomorphism $\varepsilon\colon k^{(J)} \to k^I$. Let us fix such an isomorphism ε.

Tensoring with $M[t]$ (over k) we obtain an isomorphism

$$1 \otimes \varepsilon\colon M[t] \otimes k^{(J)} \to M[t] \otimes k^I,$$

for any t. The left hand side is $M[t] \otimes k^{(J)} \simeq (M[t])^{(J)}$. Since $M[t]$ is a finite dimensional k-space, the right hand side is just $M[t] \otimes k^I \simeq (M[t])^I$. Also, there is the following commutative diagram

$$\begin{array}{ccccc}
M[t-1] \otimes k^{(J)} & \xrightarrow{1\otimes\varepsilon} & M[t-1] \otimes k^I & \xrightarrow{\sim} & (M[t-1])^I \\
{\scriptstyle \iota\otimes 1}\big\downarrow & & {\scriptstyle \iota\otimes 1}\big\downarrow & & {\scriptstyle \iota^I}\big\downarrow \\
M[t] \otimes k^{(J)} & \xrightarrow{1\otimes\varepsilon} & M[t] \otimes k^I & \xrightarrow{\sim} & (M[t])^I
\end{array}$$

where $\iota : M[t-1] \to M[t]$ is the inclusion map. Altogether, we see that

$$M \otimes k^{(J)} = \bigcup_{t\in\mathbb{N}} \left(M[t] \otimes k^{(J)}\right) \simeq \bigcup_{t\in\mathbb{N}} (M[t])^I = P'.$$

This shows that P' is the direct sum of copies of M.

The direct sum decomposition of P' exhibited here is indexed by the set J. Let us consider a finite subset J' of J and the subspace $k^{J'} = k^{(J')}$ of $k^{(J)}$. Clearly, there exists a cofinite subset I' of I such that $k^{I'}$ is a direct complement for $\varepsilon(k^{(J')})$ in k^I. But this implies that also $M^{I'}$ is a direct complement for $\varepsilon(M^{(J')})$ in M^I. In particular, we see that the submodules $\varepsilon(M^{(J')})$ are direct summands of $P = M^I$. The set of submodules of the form $\varepsilon(M^{(J')})$ is filtered and its direct limit is P'. It follows that P' is a pure submodule of P.

Let us assume in addition that all the submodules $M[t]$ are proper submodules of M and that I is an infinite set. Then P' *is a proper submodule of* P. For, we may assume that I contains $\mathbb{N}$ as a subset. If we consider any submodule $M[t]$, there is t' with $M[t]$ a proper subset of $M[t']$. Thus, there exists a sequence $0 = t_0 < t_1 < t_2 < \dots$ of non-negative integers with proper inclusions $M[t_{i-1}] \subset M[t_i]$ for $i \in \mathbb{N}$. Take an element $z_i \in M[t_i] \setminus M[t_{i-1}]$, for $i \in \mathbb{N}$, and let $z_i = 0$ for $i \in I \setminus \mathbb{N}$; then the element $z = (z_i)_i \in P$ will not belong to any $P'[t]$, thus not to P'.

2. Proof of Theorem 1. Now assume that X is an infinite dimensional Λ-module, and that φ is a locally nilpotent endomorphism of X with kernel of finite length. For $t \geq 0$, let $X[t]$ be the kernel of φ^t. We have

$$0 = X[0] \subset X[1] \subset \cdots \subset X[t] \subset \cdots \subset \bigcup_{t\in\mathbb{N}} X[t] = X,$$

the last equality comes from the fact that φ is locally nilpotent.

This shows that we can apply all the considerations of section 1. Let I be an infinite set, let $P = X^I$. Let $P'[t] = (X[t])^I$, and $P' = \bigcup_{t\in\mathbb{N}} P'[t]$. Then P' is a proper submodule of P, it is a pure submodule and isomorphic to a direct sum of copies of X.

Let n_t be the dimension of $X[t]/X[t-1]$. As we have mentioned above, the map φ induces a monomorphism $X[t+1]/X[t] \to X[t]/X[t-1]$, for $t \geq 1$; thus we see that $n_1 \geq n_2 \geq \dots$, and there is some t' such that $n_t = n$ for all $t \geq t'$. Since X is infinite dimensional, we must have proper inclusions, thus $n \geq 1$.

Next, we claim that P' is a direct summand of P. We have noted that X is Σ-algebraically compact, therefore P' is algebraically compact. On the other hand, P' is a pure submodule of P. This shows that P' is a direct summand of P, thus $P \simeq P' \oplus P/P'$.

We are going to study $\overline{P} = P/P'$. Let $x_{1t'}, \dots, x_{st'}, \dots, x_{nt'}$ be elements of $X[t']$ whose residue classes in $X[t']/X[t'-1]$ form a basis. For $t > t'$, we choose inductively elements $x_{st} \in X[t]$ such that $\varphi x_{st} = x_{s,t-1}$. We can add finitely many elements $x_{st} \in X[t]$ where $t < t'$, and $1 \le s \le n_t$, such that we obtain a basis of X with the property $\varphi x_{st} = x_{s,t-1}$ for all $t \geq 2$.

For $1 \le s \le n$, let X_s be the subspace of X generated by the elements $x_{s't}$, with $1 \le s' \le s$ and arbitrary t. Note that X_s is a $k[\varphi]$-submodule of X, and we have the chain

$$0 = X_0 \subset X_1 \subset \cdots \subset X_s \subset \cdots \subset X_n \subseteq X$$

(of course, X_n is of finite codimension in X).

Also, let $P_s = (X_s)^I$. Thus, there is the chain

$$0 = P_0 \subset P_1 \subset \cdots \subset P_s \subset \cdots \subset P_n \subseteq P,$$

and clearly we have

$$P_n + P' = P.$$

Let us consider now special endomorphisms of P. The elements of P are of the form $y = (y_i)_i$ with $y_i \in X$. Let E be a subset of $\operatorname{End}_\Lambda(X)$. Let $\gamma\colon I \to I \times E$ be a (set) map, the image of i under γ will be denoted $(\gamma(i), \gamma_i)$. Given such a map γ, we define a corresponding endomorphism

$$\tilde{\gamma}\colon P \to P \quad \text{by} \quad (\tilde{\gamma} y)_i = \gamma_i y_{\gamma(i)};$$

these endomorphisms of P will be said to be *tiled, with coordinates in E*. Of course, such an endomorphism $\tilde{\gamma}$ is a Λ-endomorphism of P. We denote by C_E the k-subalgebra of $\operatorname{End}_\Lambda(P)$ generated by the tiled endomorphisms with coordinates in E.

We are interested in $C_\varphi = C_{\{\varphi\}}$. The subspaces P_s are C_φ-submodules. Also, P' is a C_φ-submodule.

Lemma. *Let y be an element of P_s which does not belong to $P_{s-1} + P'$. Then $P_s \subseteq P_{s-1} + P' + C_\varphi y$.*

Proof: Let Φ be the k-subalgebra of $\operatorname{End}(X)$ generated by φ. Given an element $x = \sum_{t=1}^{b} \lambda_{st} x_{st}$ with $\lambda_{sb} \neq 0$, then

$$\langle x_{s1}, \dots, x_{sb} \rangle \subseteq \Phi x.$$

Given $x, x' \in X$, such that $x' \in \Phi x$, let us fix some element $\varphi(x', x)$ in Φ such that $\varphi(x', x)(x) = x'$.

By assumption, there is given an element $y \in P_s$ which does not belong to $P_{s-1} + P' = \bigcup_t (P_{s-1} + P'[t])$. We write $y = y' + y''$, where $y' \in P_{s-1}$ and where all components y''_j are linear combinations of the basis elements x_{st} with arbitrary t, say

$$y''_j = \sum_{t=1}^{b(j)} \lambda_{st}^{(j)} x_{st},$$

and we choose $b(j) \geq 0$ minimal. Thus, either $b(j) = 0$ or else $\lambda_{s,b(j)}^{(b(j))} \neq 0$. Consider now some $t \in \mathbb{N}$. Since y does not belong to $P_{s-1} + P'[t-1]$, we see that there exists $j(t) \in I$ such that $b(j(t)) \geq t$. For $t = 0$, we choose $j(0) = j(1)$; thus $j(t)$ is defined for all integers $t \geq 0$, and we have $b(j(t)) \geq t$.

Let z be an arbitrary element of P_s, say $z = z' + z''$, where $z' \in P_{s-1}$ and where all components z''_j are linear combinations of the basis elements x_{st} with arbitrary t. Thus

$$z''_i = \sum_{t=1}^{c(i)} \mu_{st}^{(i)} x_{st}.$$

Note that the number $c(i)$ determines some index $j(c(i)) \in I$ so that $b(j(c(i))) \geq c(i)$, thus we know that $z''_i \in \Phi y''_{j(c(i))}$. In particular, the endomorphism $\varphi(z''_i, y''_{j(c(i))})$ is defined and we have $\varphi(z''_i, y''_{j(c(i))})(y''_{j(c(i))}) = z''_i$.

We define a set map $\gamma\colon I \to I \times \operatorname{End}(X)$ as follows: the image of i shall have the first coordinate $\gamma(i) = j(c(i))$, the second one should be $\gamma_i = \varphi(z''_i, y''_{j(c(i))})$. We have

$$(\tilde{\gamma} y'')_i = \gamma_i y_{\gamma(i)} = \varphi(z''_i, y''_{j(c(i))})(y''_{j(c(i))}) = z''_i,$$

thus $\tilde{\gamma} y'' = z''$. It follows that

$$\tilde{\gamma} y - z = \tilde{\gamma} y' + \tilde{\gamma} y'' - z' - z'' = \tilde{\gamma} y' - z'$$

belongs to P_{s-1}. This completes the proof of the lemma.

Let $\overline{P}_s = (P_s + P')/P'$, thus we obtain a chain of C_φ-submodules

$$0 = \overline{P}_0 \subset \overline{P}_1 \subset \ldots \subset \overline{P}_n = \overline{P}.$$

Corollary. *The C_φ-modules $\overline{P}_s/\overline{P}_{s-1}$ are simple.*

The corollary shows that $\overline{P}$ has length n as a C_φ-module, and C_φ is a subalgebra of $\operatorname{End}_\Lambda P$. Thus $\overline{P}$ has length at most n when considered as an

$\operatorname{End}_\Lambda P$-module: we see that $\overline{P}$ is of finite endolength. This completes the proof of Theorem 1.

3. The set $\mathcal{G}(X)$. First, let us consider possible indecomposable direct summands of X.

Proposition 1. *Let X be a Λ-module, let φ be a locally nilpotent endomorphism of X with kernel of finite length. Let W be an indecomposable direct summand of X which has finite endolength. Then W is finite dimensional.*

Proof: Let us assume that W is infinite dimensional. Let E be its endomorphism ring. Since the E-module W has finite length, we can choose a simple E-submodule S of W. Let R denote the radical of E and note that $D = E/R$ is a division ring. Of course, S is annihilated by R, thus is a D-module (and the D-modules ${}_DS$ and ${}_DD$ are isomorphic). In particular, S is an infinite dimensional k-space.

We can write $X = Y \oplus Z$ where $Y = W^{(J)}$ is a direct sum of copies of W and Z has no direct summand isomorphic to W. We write the endomorphism φ of X in matrix form $\begin{bmatrix} \varphi_{YY} & \varphi_{YZ} \\ \varphi_{ZY} & \varphi_{ZZ} \end{bmatrix}$, with homomorphisms indexed in the following way $\varphi_{MN}\colon N \to M$. Let us denote the canonical inclusion maps of the direct sum decomposition of Y by $\mu_j\colon W \to Y$, the corresponding projections by $\pi_j\colon Y \to W$. If one of the maps $\pi_i\varphi_{YZ}\varphi_{ZY}\mu_j$ would not belong to the radical R of E, then this map would be invertible. But it factors through Z and Z has no direct summand of the form W. This shows that all the maps $\pi_i\varphi_{YZ}\varphi_{ZY}\mu_j$ belong to R and therefore vanish on S. As a consequence, $\varphi_{YZ}\varphi_{ZY}$ vanishes on the subspace $S^{(J)}$ of $W^{(J)} = Y$. Of course, $S^{(J)}$ is also an $\operatorname{End}(Y)$-submodule of Y. It follows that the subspace $S^{(J)}$ of X is a φ-submodule, and that the action of φ on $S^{(J)}$ is given by φ_{YY}.

Since $S^{(J)}$ is a φ-submodule of X and φ is locally nilpotent, φ_{YY} yields a locally nilpotent endomorphism of $S^{(J)}$. But the action of φ_{YY} on $S^{(J)}$ is given by a large matrix with entries in D, and the kernel of this linear transformation will be a D^{op}-subspace of $D^{(J)}$. Thus the kernel of the restriction $\overline{\varphi}_{YY}$ of φ_{YY} to $S^{(J)}$ is a direct sum of copies of S. Since $\overline{\varphi}_{YY}$ has non-zero kernel, it follows that this kernel contains a direct summand of the form S, thus is infinite dimensional over k. As a consequence, the kernel of φ is infinite dimensional, contrary to the assumption.

Proposition 1 has the following consequence.

Corollary. *Let X be a Λ-module, let φ be a locally nilpotent endomorphism of X with kernel of finite length. If the isomorphism class of a module Y belongs to $\mathcal{G}(X)$, then Y is not isomorphic to a direct summand of X itself.*

As we will see later, also the converse is true: the isomorphism class of any indecomposable direct summand of $X^{\mathbb{N}}$ which is not isomorphic to a direct summand of X belongs to $\mathcal{G}(X)$.

Proof of Corollary: Let U be a module and assume that its isomorphism class belongs to $\mathcal{G}(X)$. By definition, U is indecomposable and has finite endolength. Thus Proposition 1 asserts that U cannot be isomorphic to a direct summand of X.

For the proof of assertion (a) of Theorem 2, the following general result will be useful:

Proposition 2. *Let M be a module of k-dimension c. Let I be any index set and J a set of cardinality c. If N is a direct summand of M^I and has local endomorphism ring, then N is isomorphic to a direct summand of M^J.*

Proof. Let y be a non-zero element of N. As an element of M^I, we write y in the form $y = (y_i)_i$ with coordinates $y_i \in M$. Choose a subset I' of I of cardinality at most c such that the elements y_i with $i \in I'$ generate the k-subspace of M generated by all the elements y_i with $i \in I$; this is possible since the k-dimension of M is c. Every element y_i can be written as finite linear combination

$$y_i = \sum_{j \in I(i)} \alpha_{ij} y_j,$$

where $I(i)$ is a finite subset of I'. For $i \in I$, we define a Λ-homomorphism $\psi_i \colon M^{I'} \to M$ by

$$\psi_i(z) = \sum_{j \in I(i)} \alpha_{ij} z_j, \qquad \text{where} \quad z = (z_i)_{i \in I'}.$$

These maps combine to a Λ-homomorphism $\psi \colon M^{I'} \to M^I$, with $(\psi(x))_i = \psi_i(x)$. Also, let $\pi \colon M^I \to M^{I'}$ be the canonical projection, thus $\pi\big((z_i)_{i \in I}\big) = (z_i)_{i \in I'}$. The calculation

$$\psi_i \pi(y) = \psi_i\big((y_i)_{i \in I'}\big) = \sum_{j \in I(i)} \alpha_{ij} y_j = y_i$$

shows that $\psi\pi(y) = y$. Let $\mu \colon N \to M^I$ be the inclusion map and $\varepsilon \colon M^I \to N$ a projection map (so that $\varepsilon\mu = 1$). Consider the composition $\gamma = \varepsilon\psi\pi\mu \colon N \to N$. Since $\mu(y) = y$ and $\psi\pi(y) = y$, we see that $\gamma(y) = y$. If we assume that γ belongs to the radical R of the endomorphism ring of N, then $\gamma - 1$ is invertible, therefore $(\gamma - 1)(y) = 0$ will imply that $y = 0$, a contradiction. This shows that γ does not belong to R, therefore γ is invertible (since the endomorphism ring of N is a local ring). As a consequence, $\pi\mu$ is a split monomorphism, and therefore N is isomorphic to a direct summand of $M^{I'}$, thus also of M^J.

Now let us assume again that X is infinite dimensional and has a locally nilpotent endomorphism φ with kernel of finite length. Let I be any set. Let U be an indecomposable direct summand of X^I and assume that U is not isomorphic to a direct summand of X.

There is a result of Auslander [A, Corollary 3.2] which asserts that for any index set I, an indecomposable direct summand of X^I of finite length is isomorphic to a direct summand of X. Thus, we see that U is infinite dimensional.

Since U is algebraically compact and indecomposable, it has a local endomorphism ring. Also note that X has countable k-dimension, thus we can apply Proposition 2 in order to see that U is isomorphic to a direct summand of $X^{\mathbb{N}}$. Note that there is a direct decomposition $X^{\mathbb{N}} \simeq P' \oplus \overline{P}$ as given in Theorem 1, where P' is a direct sum of copies of X and $\overline{P}$ is of finite endolength. Since we assume that U is not isomorphic to a direct summand of X, it follows that U is isomorphic to a direct summand of $\overline{P}$.

As we know, $\overline{P}$ is a module of finite endolength, thus also U has finite endolength [C, 4.5]. It follows that the isomorphism class of U belongs to $\mathcal{G}(X)$. This yields one implication of the assertion (a), the other one has already been verified.

A module M of finite endolength e is always a direct sum of indecomposable modules of finite endolength: there are finitely many pairwise non-isomorphic modules $M_1, \dots, M_m$ which are indecomposable and of finite endolength such that M is isomorphic to $\bigoplus_{i=1}^m M_i^{(I(i))}$, with suitable sets $I(1), \dots, I(m)$, and we have $m \le e$ (see [C, 4.5]). We apply this to the module $M = \overline{P}$. Here, $e \le n$, where n is the minimum of the dimension of the vector spaces $\operatorname{Ker}\varphi^t / \operatorname{Ker}\varphi^{t-1}$, with $t \ge 1$. We see that $\overline{P} = \bigoplus_{i=1}^m M_i^{(I(i))}$, with indecomposable modules M_i and suitable sets $I(i)$, and $m \le n$. Using again Azumaya's theorem, it follows that a module which is not isomorphic to a direct summand of X and whose isomorphism class belongs to $\mathcal{G}(X)$ is isomorphic to one of the modules M_i. This establishes (c).

Let us state explicitly the following criterion which we have established:

Proposition 3. *Let X be an infinite dimensional module, let φ be a locally nilpotent endomorphism with kernel of finite length. Let $X^{\mathbb{N}} = P_1 \oplus P_2$ be a direct decomposition where P_1 is a direct sum of copies of X and P_2 is of finite endolength. Then the isomorphism classes in $\mathcal{G}(X)$ are precisely those of the indecomposable direct summands of $\overline{P}_2$ which are infinite dimensional.*

Proof: The decomposition $X^{\mathbb{N}} = P_1 \oplus P_2$ can differ from the decomposition $X^{\mathbb{N}} = P' \oplus \overline{P}$ given in Theorem 1 only in the following way: finite dimensional summands of the module $\overline{P}$ may have been shifted to P_1. We know that X has no infinite dimensional indecomposable direct summand of finite endolength, thus the sets of isomorphism classes of infinite dimensional indecomposable direct summands of P_2 and of $\overline{P}$ are the same.

Consider a direct summand Y of X. Let $Y[t] = Y \cap \operatorname{Ker} \varphi^t$ for $t \geq 0$. We have

$$0 = Y[0] \subseteq Y[1] \subseteq \cdots \subseteq Y[t] \subseteq \cdots \subseteq \bigcup_{t \in \mathbb{N}} Y[t] = Y,$$

using again that φ is locally nilpotent. Thus, we can apply the considerations of section 1 also in this case: Let I be an infinite set, let $P_Y = Y^I$ and $P'_Y = \bigcup_{t\in\mathbb{N}} (Y[t])^I$. Then P'_Y is a pure submodule of P_Y and isomorphic to a direct sum of copies of Y. If Y is infinite dimensional, then all the submodules $Y[t]$ are proper submodules of Y, and then P'_Y is a proper submodule of P_Y. Since Y is a direct summand of X, it is Σ-algebraically compact, thus P'_Y is algebraically compact and therefore P'_Y is a direct summand of P_Y, thus $P_Y \simeq P'_Y \oplus \overline{P}_Y$, where $\overline{P}_Y = P_Y/P'_Y$. Let $X = Y \oplus Z$, where Z is some complement. Then

$$P_X = P_Y \oplus P_Z.$$

Let us show that

$$P'_X = P'_Y \oplus P'_Z.$$

On the one hand, we have $Y[t] \oplus Z[t] \subseteq X[t]$. On the other hand, $X = Y \oplus Z = \bigcup_{t\in\mathbb{N}} Y[t] \oplus Z[t]$, thus for any t, there exists $t' \geq t$ with $X[t] \subseteq Y[t'] \oplus Z[t']$. It follows that

$$Y[t]^I \oplus Z[t]^I \subseteq X[t]^I \subseteq Y[t']^I \oplus Z[t']^I,$$

and therefore

$$P'_X = \bigcup_{t\in\mathbb{N}} X[t]^I = \bigcup_{t\in\mathbb{N}} Y[t]^I \oplus Z[t]^I = P'_Y \oplus P'_Z.$$

As a consequence, $\overline{P}_X = \overline{P}_Y \oplus \overline{P}_Z$.

Let us now assume that X has an indecomposable direct summand Y which is infinite dimensional, say $X = Y \oplus Z$. As we have seen, $\overline{P}_Y$ is non-zero. Since $\overline{P}_X = \overline{P}_Y \oplus \overline{P}_Z$, we know that $\overline{P}_Y$ is a direct sum of indecomposable modules which are endofinite. Let U be an indecomposable direct summand of $\overline{P}_Y$. Note that U cannot be finite dimensional, since otherwise it would be a finite dimensional direct summand of Y^I, and therefore isomorphic to a direct summand of Y itself, using again Auslander's theorem; but this is impossible. This shows that $\mathcal{G}(Y)$ is non-empty. Since the isomorphism classes in $\mathcal{G}(Y)$ also belong to $\mathcal{G}(X)$, we see that $\mathcal{G}(X)$ is non-empty. This establishes (b).

Let U be a submodule of X of finite length with $\varphi(U) \subseteq U$. Then there exists $s \in \mathbb{N}$ such that $U \subseteq X[s]$. Let us show that $\overline{P}_X$ may be identified with $\overline{P}_{X/U}$.

By definition, $\overline{P}_X = P_X/P'_X$, with $P_X = X^{\mathbb{N}}$ and $P'_X = \bigcup_{t\geq 0}\big(\varphi^{-t}(0)\big)^{\mathbb{N}}$. Similarly, $\overline{P}_{X/U} = P_{X/U}/P'_{X/U}$, where $P_{X/U} = (X/U)^{\mathbb{N}}$ and $P'_{X/U} = \bigcup_{t\geq 0}\big(\varphi^{-t}(U)\big)^{\mathbb{N}}$. We have the following inclusions

$$\varphi^{-t}(0) \subseteq \varphi^{-t}(U) \subseteq \varphi^{-(s+t)}(U).$$

and therefore

$$\big(\varphi^{-t}(0)\big)^{\mathbb{N}} \subseteq \big(\varphi^{-t}(U)\big)^{\mathbb{N}} \subseteq \big(\varphi^{-(s+t)}(U)\big)^{\mathbb{N}}.$$

As a consequence,

$$P'_X = \bigcup_{t\geq 0}\big(\varphi^{t}(0)\big)^{\mathbb{N}} = \bigcup_{t\geq 0}\big(\varphi^{t}(U)\big)^{\mathbb{N}}.$$

Of course, P'_X has the submodule $U^{\mathbb{N}}$, and we have

$$\bigcup_{t\geq s}\big(\varphi^{t}(0)/U\big)^{\mathbb{N}} = \bigcup_{t\geq 0}\big(\varphi^{t}(U)/U\big)^{\mathbb{N}}.$$

The left hand side is canonically isomorphic to $P'_X/U^{\mathbb{N}}$, whereas the right hand side is just $P'_{X/U}$. It follows that

$$\overline{P}_X = P_X/P'_X \simeq \big(P_X/U^{\mathbb{N}}\big)/\big(P'_X/U^{\mathbb{N}}\big) \simeq P_{X/U}/P'_{X/U} = \overline{P}_{X/U}.$$

As we have seen above, the isomorphism classes in $\mathcal{G}(X)$ are those of the infinite dimensional, indecomposable direct summands of $\overline{P}_X$. Similarly, the isomorphism classes in $\mathcal{G}(X/U)$ are those of the infinite dimensional, indecomposable direct summands of $\overline{P}_{X/U}$. This establishes (d).

Even in case X is indecomposable, the set $\mathcal{G}(X)$ may contain more than one isomorphism class, as the following examples show.

Example 1. In the paper [R], we have presented various infinite dimensional indecomposable modules. The last example exhibited there is the one we are interested in: we deal with a contracting $\mathbb{Z}$-word x which gives rise to an indecomposable Λ-module $C(x) = M(x)$, where Λ is a certain special biserial algebra. Observe that $C(x)$ has a unique monogenic submodule K of dimension 5, and note that the factor module $C(x)/K$ can be embedded into $C(x)$ with codimension 1. The composition of the projection map $C(x) \to C(x)/K$ and the inclusion map $C(x)/K \to C(x)$ is a locally nilpotent endomorphism φ of $C(x)$ with kernel K. Note that $C(x)/K$ is the direct sum of two indecomposable modules Y_1, Y_2 with disjoint supports. If we apply Theorem 1 to the module $C(x)$, we obtain an endofinite module $\overline{P}$ which is the direct sum $\overline{P} = \overline{P}_1 \oplus \overline{P}_2$ of two non-zero modules $\overline{P}_1, \overline{P}_2$ such that $\operatorname{Hom}(\overline{P}_1, \overline{P}_2) = 0 = \operatorname{Hom}(\overline{P}_2, \overline{P}_1)$ (the module $\overline{P}_i$ is a direct sums of copies of the "generic" module having the same support as Y_i).

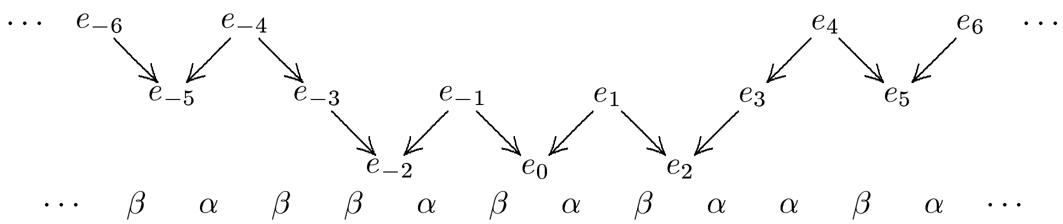
Example 2. Again, we want to exhibit an example of an indecomposable module X with a locally nilpotent endomorphism φ having a finite length kernel, such that $\mathcal{G}(X)$ contains two isomorphism classes. But this time, φ will be in addition surjective. Let Λ be the factor algebra of the polynomial ring $k[T_1, T_2]$ modulo the ideal generated by T_1T_2, T_1^3, T_2^3. We denote the residue class of T_1 by α, that of T_2 by β. Note that Λ is a five dimensional algebra (with basis $1, \alpha, \alpha^2, \beta, \beta^2$), and again it is a special biserial algebra. In order to deal with Λ-modules, we use notations as in [R].

We start with the $\mathbb{N}$-words $x = \big(\beta\alpha^{-1}\beta\big)^{\infty}$, and $y = \big(\alpha\beta^{-1}\alpha\big)^{\infty}$, and form the $\mathbb{Z}$-word

$$z = x^{-1}y = \cdots \big(\beta^{-1}\alpha\beta^{-1}\big)\big(\beta^{-1}\alpha\beta^{-1}\big) \cdot \big(\alpha\beta^{-1}\alpha\big)\big(\alpha\beta^{-1}\alpha\big) \cdots,$$

thus $z = \cdots l_{-2}l_{-1} \cdot l_0 l_1 l_2 \cdots$, where we have $l_i = \alpha$ if either $i < 0$ and $i \equiv 1 \pmod 3$ or $i \geq 0$ and $i \not\equiv 1 \pmod 3$; otherwise $l_i = \beta^{-1}$.

Let $X = M(z)$, this module is an infinite dimensional k-space with basis $e_i, i \in \mathbb{Z}$, and the action of α, β on X is given by the following rules: If either $i < 0$ and $i \equiv 2 \pmod 3$ or else $i > 0$ and $i \not\equiv 2 \pmod 3$, then $\alpha(e_i) = e_{i-1}$. If either $i < 0$ and $i \not\equiv 1 \pmod 3$ or else $i \geq 0$ and $i \equiv 1 \pmod 3$, then $\beta(e_i) = e_{i+1}$.



Let V be the submodule of X generated by the vectors e_{-1}, e_1 and $e_{-3} + e_3$. Then V is of the form $F_w(k, 1)$, where

$$w = \big(\beta^{-1}\alpha\beta^{-1}\big)\big(\alpha\beta^{-1}\alpha\big).$$

It is obvious that X/V is isomorphic to X, thus there exists an epimorphism $\varphi\colon X \to X$ with kernel V. This is the endomorphism of X we are interested in: it is locally nilpotent, the kernel is V, thus of finite length, and it is surjectiv.

On the other hand, consider the simple submodule W with basis e_0. We have $\varphi(W) = 0 \subset W$, thus $\mathcal{G}(X) = \mathcal{G}(X/W)$. The module X/W is the direct sum of two indecomposable modules $M\big((\alpha^{-1}\beta^2)^\infty\big)$, and $M\big((\beta^{-1}\alpha^2)^\infty\big)$. It is easy to see that $\mathcal{G}(X/W)$ consists of the isomorphism classes of the modules $F_v(k(T), T\cdot), F_{v'}(k(T), T\cdot)$, where $v = \alpha^{-1}\beta^2$ and $v' = \beta^{-1}\alpha^2$.

We should add the following remark. As we have seen, the kernel $X[1]$ of φ is $V = F_w(k, 1)$. Since φ is surjective, all the factors of the filtration $X[t]$ of X are of the form $F_w(k, 1)$. There also exists the Prüfer module X' with quasi-socle $F_w(k, 1)$; it is an indecomposable module with a surjective locally nilpotent endomorphism and with kernel $F_w(k, 1)$, however $\mathcal{G}(X')$ consists of a single isomorphism class, namely $F_w(k(T), T\cdot)$.

References

[A] M. Auslander: Large modules over artin algebras. In: Algebra, Topology, and Category Theory. New York, San Francisco, London. 1976. 1-17.

[C] W. Crawley-Boevey: Modules of finite length over their endomorphism rings. In: Representations of Algebras (ed. H. Tachikawa and S. Brenner). London Math. Soc. Lecture Note Series 168. Cambridge (1992), 127–184.

[K] H. Krause: Generic modules over artin algebras. SFB 343 Bielefeld. Preprint 95-121.

[JL] C. U. Jensen, H. Lenzing: Model Theoretic Algebra. Gordon and Breach. New York (1989).

[R] C. M. Ringel: Algebraically compact modules I. In: Abelian Groups and Modules (ed. A. Facchini and C. Menini). Kluwer. Dordrecht (1995), 419-439.