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# Cones

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*Dedicated to the Memory of Maurice Auslander*

ABSTRACT. A cone is the full translation subquiver of $\mathbb{Z}A_\infty$ given by the predecessors of some fixed vertex on the boundary. We consider an abelian category $\mathcal{A}$ with Auslander-Reiten sequences and we denote its Auslander-Reiten quiver by $\Gamma$. We are going to present a criterion for an indecomposable object $X$ in $\mathcal{A}$ in order that the predecessors of $X$ in $\Gamma$ form a cone. As a consequence, we will see that for suitable graded algebras, any stable Auslander-Reiten component which contains a Koszul module is of the form $\mathbb{Z}A_\infty$.

Let $k$ be a field. Let $\mathcal{A}$ be an abelian $k$-category (with finite dimensional Hom-sets) such that every object has finite length. Assume that $\mathcal{A}$ has projective covers and injective envelopes, thus $\mathcal{A}$ has Auslander-Reiten sequences. We denote by $\tau$ the corresponding Auslander-Reiten translation.

We are interested in Auslander-Reiten components of the form $\mathbb{Z}A_\infty$. The full translation subquiver of $\mathbb{Z}A_\infty$ given by all predecessors of a fixed vertex on the boundary is said to be a *cone*.

THEOREM. *Let $X$ be an indecomposable object in $\mathcal{A}$ such that for all $t \in \mathbb{N}_0$, the object $\tau^t X$ is defined and that none of the objects $\tau^t X$ with $t \geq 1$ is simple. Let $\Gamma$ be the Auslander-Reiten component of $\mathcal{A}$ containing $[X]$. Assume that*

$$\operatorname{Hom}(\tau^t X, \tau^s X) = 0 \quad \textit{for all} \quad 0 \leq s < t.$$

*Then the set of predecessors of $[X]$ in $\Gamma$ is a cone.*

We note the following: if we assume, in addition, that $\Gamma$ is a stable translation quiver, then $\Gamma$ is of the form $\mathbb{Z}A_\infty$.

**Proof.** Let $X_{ii} = \tau^i X$ for alle $i \in \mathbb{N}_0$. We are going to construct indecomposable modules $X_{st}$ with $0 \leq s \leq t$ such that the modules $X_{ij}$ with $s \leq i \leq j \leq t$ form a wing in $\Gamma$.

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$$\begin{array}{ccccccccccccc}
 & & & & & & X_{st} & & & & & & \\
 & & & & & \nearrow & & \searrow & & & & & \\
 & & & & & X_{s+1,t} & & X_{s,t-1} & & & & & \\
 & & & & \nearrow & & \searrow\ \nearrow & & \searrow & & & & \\
 & & & & X_{s+2,t} & & X_{s+1,t-1} & & X_{s,t-2} & & & & \\
 & & & \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow & & & \\
 & & & \cdots & & & & & & \cdots & & & \\
 & & \nearrow & & & & & & & & \searrow & & \\
 & X_{t-1,t} & & & & & \cdots & & & & & X_{s,s+1} & \\
\nearrow & & \searrow & & & & & & & & \nearrow & & \searrow \\
X_{tt} & & X_{t-1,t-1} & & & & \cdots & & & & X_{s+1,s+1} & & X_{ss}
\end{array}$$

This of course implies that there exists an irreducible monomorphism

$$u_{s+1,t}\colon X_{s+1,t} \to X_{st}$$

with cokernel $X_{ss}$. Consequently $X_{st}$ has a filtration with factors (going down) of the form $X_{ss}, X_{s+1,s+1}, \dots, X_{tt}$.

Let us assume that we have constructed the modules $X_{st}$ for $0 \le t-s \le n$, as well as irreducible maps $u_{s+1,t}\colon X_{s+1,t} \to X_{st}$ with cokernel $X_{ss}$, for $1 \le t-s \le n$. First, let us show that $\tau X_{st}$ is isomorphic to $X_{s+1,t+1}$.

By definition, this is true for $s=t$. Thus, let us assume $s<t$. First of all, we claim that $X_{st}$ cannot be indecomposable projective. For the proof, we consider the length $d_{st}$ of the top of $X_{st}$ and we show that $d_{st} \ge 2$ for $s<t$. Consider the Auslander-Reiten sequence

$$0 \to X_{s+1,t} \to X_{st} \oplus X_{s+1,t-1} \to X_{s,t-1} \to 0$$

(where $X_{s+1,t-1} = 0$ for $t = s+1$). Since $X_{s+1,t}$ is not simple, the top of the middle term is isomorphic to the direct sum of the tops of the outer terms, thus

$$d_{st} = d_{s+1,t} + d_{s,t-1} - d_{s+1,t-1}.$$

Using induction, we see that $d_{st} = \sum_{i=s}^{t} d_{ii}$. But this implies $d_{st} \ge t-s+1$.

There is an irreducible map $X_{s+1,t} \to X_{st}$, thus there also has to be an irreducible map $\tau X_{st} \to X_{s+1,t}$. We know the Auslander-Reiten sequence ending in $X_{s+1,t}$; there are at most two arrows ending in $X_{s+1,t}$, one starting in $X_{s+1,t+1}$. There is a second such arrow in case $t \ge s+2$, and it starts in $X_{s+2,t}$. However, by induction we know that $\tau^{-}X_{s+2,t}$ is isomorphic to $X_{s+1,t-1}$, and the objects $X_{st}$ and $X_{s+1,t-1}$ are not isomorphic. Thus the only possible choice for $\tau X_{st}$ is $X_{s+1,t+1}$.

We consider the corresponding Auslander-Reiten sequence

$$0 \to X_{s+1,t+1} \xrightarrow{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]} Z \oplus X_{s+1,t} \xrightarrow{[-c\ \ d]} X_{st} \to 0,$$

and we can assume that $d = u_{s+1,t}$. Since this sequence is exact, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & X_{s+1,t+1} \arrow[r, "a"] \arrow[d, "b"'] & Z \arrow[r, "qc"] \arrow[d, "c"'] & X_{ss} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_{s+1,t} \arrow[r, "u_{s+1,t}"'] & X_{st} \arrow[r, "q"'] & X_{ss} \arrow[r] & 0
\end{tikzcd}$$

where $q\colon X_{st} \to X_{ss}$ is the cokernel map for $u_{s+1,t}$. In particular, we see that $Z \neq 0$. Since $a$ occurs in an Auslander-Reiten sequence, we see that $a$ is irreducible. In particular, $a$ is not a split monomorphism.

As we have noted above, the object $X_{s+1,t+1}$ has a filtration with factors $X_{s+1,s+1}, \ldots, X_{t+1,t+1}$. By assumption, $\mathrm{Hom}(X_{ii}, X_{ss}) = 0$ for $s+1 \leq i$, and therefore also $\mathrm{Hom}(X_{s+1,t+1}, X_{ss}) = 0$. Since the given sequence

$$0 \to X_{s+1,t+1} \to Z \to X_{ss} \to 0$$

does not split and $\mathrm{Hom}(X_{s+1,t+1}, X_{ss}) = 0$, we see that $Z$ has to be indecomposable.

Thus, we have constructed an indecomposable module $X_{s,t+1} = Z$ and an irreducible monomorphismus

$$u_{s+1,t+1} = a\colon\ X_{s+1,t+1} \to X_{s,t+1}$$

with cokernel $X_{ss}$. This completes the proof.

## An application

Let $A = \bigoplus_{i \geq 0} A_i$ be a connected finite dimensional $\mathbb{Z}$-graded $k$-algebra, let $A_0$ be semisimple and assume that $A$ is generated by $A_0 \oplus A_1$. We assume in addition that $A$ is self-injective and of Loewy length at least 4.

Let $\mathcal{A}$ be the category of finite dimensional graded $A$-modules. Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a graded $A$-module. We say that $M$ is *generated in degree* $t$ provided $M$ is generated as an $A$-module by the subspace $M_t$. We recall that an $A$-module $M$ is said to be a *Koszul* module provided that there exists a projective resolution of the form

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where $P_i$ is a projective $A$-module and generated in degree $i$, for all $i \geq 0$.

Theorem. *Let $M$ be an indecomposable non-projective Koszul module. Then*

$$\mathrm{Hom}(\tau^t M, \tau^s M) = 0 \quad \textit{for all} \quad 0 \leq s < t,$$

*and $\tau^t M$ is not simple, for $0 < t$. As a consequence, the set of predecessors of $[M]$ in the Auslander-Reiten quiver of the category of graded $A$-modules is a cone.*

**Proof.** Let $d+1$ be the Loewy length of $A$. Since $A_0$ is semisimple and $A$ is connected, generated by $A_0 \oplus A_1$, and self-injective, it follows easily that $A_d$ is the socle of $A$. Also, the Auslander-Reiten translation $\tau$ is given by $\tau = \nu\Omega^2$, where $\nu$ is the Nakayama functor and $\Omega$ Heller's loop functor (which attaches to any module $M$ its first syzygy).

Let $M$ be an indecomposable non-projective Koszul module. By definition, $M$ is generated in degree 0, and $\Omega^2 M$ is generated in degree 2. It follows that $\tau M = \nu\Omega^2 M$ is generated in degree $2-d$. By induction, we see that $\tau^s M$ is generated in degree $(2-d)s$. Recall that we assume that $d > 2$. If we assume that $0 \leq s < t$, then we have $(2-d)t < (2-d)s$ and therefore

$$\mathrm{Hom}(\tau^t M, \tau^s M) = 0 \quad \text{for all} \quad 0 \leq s < t.$$

Also, we need to know that the modules $\tau^t M$ with $t \geq 1$ are not simple. Let us assume that the module $\tau^t M = \nu^t\Omega^{2t} M$ is simple, where $M$ is a Koszul module and $t \geq 1$. Then also $S = \Omega^{2t}M$ is simple. Since $M$ is a Koszul module, $\Omega^i M$ is generated in degree $i$, for all $i \geq 0$. Clearly, if $S$ is simple and generated in degree $i$, then $\Omega^{-1}S$ is generated in degree $i-d$. Altogether, we see that $\Omega^{2t-1}M$ is generated in degree $2t-1$, but $\Omega^{-1}S$ is generated in degree $2t-d$, a contradiction.

Remark. If the Loewy length of $A$ is at least 5, the same method can be applied for additional modules. Let $S(1), \dots, S(n)$ be the simple $A$-modules generated in degree 0 (one from each isomorphism class), let $S = \bigoplus_{i=1}^n S(i)$.

Given a graded $A$-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$, we denote by $M[n]$ the graded module obtained from $M$ by shifting the degree: $M[n]_i = M_{i-n}$. Note that a graded $A$-module $M$ is a Koszul module if and only if given $i \in \mathbb{N}_0$, and $j \in \mathbb{Z}$, then we have $\mathrm{Ext}^i(M, S[j]) \neq 0$ only in case $i = j$.

*Let $d+1$ be the Loewy length of $A$. Let $M$ be a graded $A$-module with the following property: given an arbitrary integer $j$ and an even integer $n \geq 0$ with*

$$\mathrm{Ext}^n(M, S[j]) \neq 0,$$

*then $n \leq j \leq n+d-3$. Then*

$$\mathrm{Hom}(\tau^t M, \tau^s M) = 0 \quad \textit{for all} \quad 0 \leq s < t.$$

Proof. Note that the Nakayama functor $\nu$ satisfies $\nu S = S[-d]$. We have for $n \geq 1$ and any $j$

$$\begin{aligned} \mathrm{Hom}(\tau^n M, S[j]) &= \mathrm{Hom}(\nu^n\Omega^{2n}M, S[j]) \\ &= \mathrm{Hom}(\Omega^{2n}M, \nu^{-n}S[j]) \\ &= \mathrm{Hom}(\Omega^{2n}M, S[j+dn]) \\ &= \mathrm{Ext}^{2n}(M, S[j+dn]) = 0. \end{aligned}$$

Thus, if $\mathrm{Hom}(\tau^n M, S[j]) \neq 0$, then it follows that

$$2n \leq j + dn \leq (2-d)n + d - 3,$$

and therefore

$$(2-d)n \leq j \leq (2-d)(n-1) - 1.$$

In particular, it follows from $j \geq (2-d)n$ that $(\tau^n M)_i = 0$ for $i < (2-d)n$.

Now assume, we consider modules $\tau^s M, \tau^t M$, where $0 \leq s < t$. On the one hand, we know that $(\tau^s M)_i = 0$ for $i < (2-d)s$. (For $0 < s$, this has been shown above. For $s = 0$, we use the assumption that $\mathrm{Hom}(M, S[j]) \neq 0$ implies $0 \leq j$.) On the other hand, let $T$ be the top of $\tau^t M$. Since $\mathrm{Hom}(T, S[j]) = \mathrm{Hom}(\tau^t M, S[j])$, we see that $\mathrm{Hom}(T, S[j]) \neq 0$ implies $j \leq (2-d)(t-1) - 1$. But $s \leq t-1$ and $2-d \leq 0$ yields

$$j \leq (2-d)(t-1) - 1 \leq (2-d)s - 1 < (2-d)s.$$

This shows that $T_j \neq 0$ implies $j < (2-d)s$.

As a consequence, $\mathrm{Hom}(\tau^t M, \tau^s M) = 0$, since under any homomorphism $\tau^t M \to \tau^s M$, the top $T$ of $\tau^t M$ is sent to zero.

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