SOME ALGEBRAICALLY COMPACT MODULES. I

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Abstract. Given a finite dimensional monomial algebra, one knows that some finite dimensional indecomposable modules may be described by words (finite sequences of letters) using as letters the arrows of the quiver and their formal inverses. To every word w, one can attach a so-called string module $M(w)$. Here, we are going to construct certain infinite dimensional modules: We will consider $\mathbb{N}$-words and $\mathbb{Z}$-words (thus infinite sequences of letters) satisfying suitable periodicity conditions. To every such $\mathbb{N}$-word or $\mathbb{Z}$-word x, we describe an algebraically compact module $C(x)$. This module $C(x)$ is obtained from the corresponding string module $M(x)$ as a kind of completion.

Keywords. Algebraically compact modules, pure injective modules. Linearly compact modules. Quivers. Words, $\mathbb{N}$-words, $\mathbb{Z}$-words. String modules, Prüfer modules, $\mathbf{p}$-adic modules. Special biserial algebras. Kronecker modules. Shift endomorphism.

1. Finite Words and Finite Dimensional Modules

Let k be a field. Let A be a finite dimensional monomial algebra, thus $A = kQ/\mathcal{I}$ where $Q = (Q_0, Q_1)$ is a finite quiver and $\mathcal{I}$ an admissible ideal generated by monomials. (We recall that an *admissable ideal generated by monomials* is an ideal $\mathcal{I}$ of the path algebra kQ which is generated by paths ("monomials") of length at least 2 such that all paths of length n are contained in $\mathcal{I}$, for some n.)

Given an arrow α with starting point $s(\alpha)$ and terminal point $t(\alpha)$, we denote by α^{-1} a formal inverse of α, with starting point $s(\alpha^{-1}) = t(\alpha)$ and terminal point $t(\alpha^{-1}) = s(\alpha)$. Given such a formal inverse $l = \alpha^{-1}$, one writes $l^{-1} = \alpha$. Let $\overline{Q}_1$ be the set of all arrows and their formal inverses. We consider words using the elements of $\overline{Q}_1$ as letters (the arrows will be said to be *direct* letters, their formal inverses will be said to be *inverse* letters); a *word* of *length* $n \geq 1$ is a sequence of the form

$$w = l_1 l_2 \cdots l_n \qquad \text{with} \qquad s(l_i) = t(l_{i+1}) \qquad \text{for all} \qquad 1 \leq i < n$$

(one may consider w just as the sequence $(l_1, l_2, \ldots, l_n)$, but it will be convenient, to delete the brackets and the colons), and one requires the following:

(W1) We have $l_i^{-1} \neq l_{i+1}$, for all $1 \leq i < n$.

(W2) No proper subsequence of w or its inverse belongs to $\mathcal{I}$.

(By definition, the *inverse* of $w = l_1 \ldots l_n$ is $w^{-1} = l_n^{-1} \ldots l_1^{-1}$; a proper subsequence of w is of the form $l_i l_{i+1} \ldots l_{j-1} l_j$ (with $1 \leq i < j \leq n$; if w is a word, the subsequences of w are called *subwords.* Usually, one also introduces words of length 0, corresponding to the vertices of the quiver; but we will not need them). Given a word $w = l_1l_2\cdots l_n$, the vertex $s(w) = s(l_n)$ is called the starting point of w, the vertex $t(w) = t(l_1)$ is called the terminal point of w. (In order to avoid any irritation, let us stress: since we deal with left modules, we write $\beta\alpha$ for the composition of the arrows $\alpha\colon a \to b$ and $\beta\colon b \to c$; thus, given a module M, and a word $w = l_1l_2\cdots l_n$, the subspace wM of M will be obtained by first applying l_n to M, then l_{n-1}, and so on, and finally l_1. Nevertheless we usually will index the letters of the word $w = l_1l_2\cdots l_n$ from left to right; and we will call l_1 the *first* letter of w, and l_n the *last* letter.)

Given words $w = l_1l_2\cdots l_n$ and $w' = l'_1l'_2\cdots l'_m$, the *concatenation* of w and w' is $ww' = l_1l_2\cdots l_nl'_1l'_2\cdots l'_m$, provided this again is a word (thus we need that $s(w) = t(w')$, that $l_n^{-1} \neq l'_1$, and that also the condition (W2) is satisfied for ww') A word w containing both direct and inverse letters and such that also $w^2 = ww$ is a word is said to be a *cyclic* word. Of course, if w is a cyclic word, we must have $t(w) = s(w)$ (in addition, the first and the last letter of w cannot be inverse to each other, and finally, the condition (W2) has to be satisfied for w^2). Given a cyclic word w, we can form the powers $w^m = w\cdots w$, for $m \geq 1$. The powers w^m with $m \geq 2$ are said to be *proper* powers. The cyclic word w is said to be *primitive* provided it is not a proper power of some other word.

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To every word w, one knows how to attach a finite dimensional indecomposable A-module $M(w)$, or, equivalently, a representation of the quiver Q satisfying suitable relations. Let us recall the construction. Write $w = l_1l_2\cdots l_t$, and let $c(i) = s(l_i)$ for $1 \leq i \leq t$ and $c(0) = t(l_1)$. Given two vertices a, b, let

$$k_{ab} = \begin{cases} k & \text{in case } a = b, \\ 0 & \text{otherwise;} \end{cases}$$

in case $a = b$, we denote $e_{ab} = 1_k \in k_{ab}$, otherwise we put $e_{ab} = 0$.

For any vertex a, let

$$M(w)_a = \bigoplus_{i=0}^{t} k_{ac(i)}\ ;$$

an element in $M(w)_a$ is of the form $(\lambda_i)_{0\leq i\leq t}$ with $\lambda_i \in k_{ac(i)}$. The module $M(w)$ itself is the direct sum of these vector spaces $M(w)_a$, where $a \in Q_0$, thus $M(w) = \bigoplus_{i=0}^{t} k$.

For any arrow $\alpha\colon a \to b$, let $M(w)_\alpha\colon M(w)_a \to M(w)_b$ be defined by

$$M(w)_\alpha\big((\lambda_i)_i\big) = (\mu_i)_i$$

where

$$\begin{aligned} \mu_0 &= \delta(\alpha, l_1)\lambda_1, \\ \mu_i &= \delta(\alpha, l_i^{-1})\lambda_{i-1} + \delta(\alpha, l_{i+1})\lambda_{i+1} \qquad \text{for} \qquad 1 \leq i < t, \\ \mu_t &= \delta(\alpha, l_t^{-1})\lambda_{t-1}; \end{aligned}$$

here, $\delta(\alpha,\beta)$ is the Kronecker function (taking values 1 or 0, depending whether $\alpha = \beta$ or not). Note that $l_i^{-1} \neq l_{i+1}$, therefore at most one of the summands $\delta(\alpha, l_i^{-1})\lambda_{i-1}$ and $\delta(\alpha, l_{i+1})\lambda_{i+1}$ is non-zero. One calls $M(w)$ the *string module* attached to w. (A first extensive study of such string modules was done by Gelfand and Ponomarev, see [GP]).

Let us consider one example: Take the quiver with three vertices a, b, c and arrows $\alpha\colon a \to a,\ \beta\colon a \to b,\ \gamma\colon b \to c,\ \delta\colon b \to c,\ \epsilon\colon c \to a$; and let $\mathcal{I}$ be the ideal generated by all monomials of length 3.

$$\begin{array}{ccccccc} & & & \beta & & b & \\ \alpha & & a & & \gamma & & \delta \\ & & & \epsilon & & c & \end{array}$$

Let $w = \alpha\beta^{-1}\gamma^{-1}\delta\gamma^{-1}$; for the convenience of the reader, we will display such a word in the following way:

$$\begin{array}{cccccc} & \bullet & & & & \\ \bullet & & \bullet & & \bullet & \\ & & & \bullet & & \bullet \\ \alpha & \beta & \gamma & \delta & \gamma & \end{array}$$

Note that $c(0) = c(1) = a$, $c(2) = c(4) = b$, and $c(3) = c(5) = c$. We see that all three vector spaces $M(w)_a, M(w)_b, M(w)_c$ are two-dimensional, and the matrices to be used are

$$M(w)_\alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ M(w)_\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ M(w)_\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$M(w)_\delta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ M(w)_\epsilon = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

2. $\mathbb{N}$-Words and $\mathbb{Z}$-Words

The set of positive integers will be denoted by $\mathbb{N}$, and we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

An $\mathbb{N}$-*word* is of the form $l_1 l_2 \cdots l_n \cdots$ such that all finite subsequences $l_1 l_2 \cdots l_t$ are words (again, such an $\mathbb{N}$-word is just a sequence $(l_1, l_2, \ldots)$).

Given a cyclic (of course finite) word w, we may consider the $\mathbb{N}$-word $w^\infty = ww\cdots$. The $\mathbb{N}$-words of the form w^∞ are said to be *periodic.* If w is a primitive cyclic word of length n, then w^∞ is said to be *of period* n. Note that an $\mathbb{N}$-word $w = l_1 l_2 \cdots l_n \cdots$ is periodic provided $l_{i+p} = l_i$ for all $i \geq 1$ and some $p \geq 2$.

Given an $\mathbb{N}$-word $x = l_1 l_2 \cdots l_n \cdots$, the $\mathbb{N}$-words $l_s l_{s+1} \cdots$ with $s \geq 1$ will be said to be $\mathbb{N}$-*subwords of* w.

An $\mathbb{N}$-word is said to be *almost periodic* provided there exists a periodic $\mathbb{N}$-subword. In case $y = l_{s+1} l_{s+2} \cdots$ is a periodic $\mathbb{N}$-subword of the $\mathbb{N}$-word $x = l_1 l_2 \cdots l_n \cdots$, and either $s = 0$ or else $l_s l_{s+1} l_{s+2} \cdots$ is not a periodic $\mathbb{N}$-word, one calls y a *maximal periodic* $\mathbb{N}$-*subword*. For any almost periodic $\mathbb{N}$-word, there exists a unique maximal periodic $\mathbb{N}$-subword.

A $\mathbb{Z}$-*word* x is of the form $x = \cdots l_{-2}l_{-1}l_0l_1l_2\cdots$, with letters l_i, for all $i \in \mathbb{Z}$; again, we require that all finite subsequences $l_{-t}\cdots l_t$ (with $t \geq 0$) are words. Of course, the inverse x^{-1} of the $\mathbb{Z}$-word $x = \cdots l_{-2}l_{-1}l_0l_1l_2\cdots$ is the $\mathbb{Z}$-word $x^{-1} = \cdots l_2^{-1}l_1^{-1}l_0^{-1}l_{-1}^{-1}l_{-2}^{-1}\cdots$.

Assume that there is given a $\mathbb{Z}$-word $x = \cdots l_{-2}l_{-1}l_0l_1l_2\cdots$. The $\mathbb{N}$-words $l_sl_{s+1}\cdots$ with $s \in \mathbb{Z}$ will be said to be the $\mathbb{N}$-*subwords of* x. If we fix $s \in \mathbb{Z}$, we may consider the $\mathbb{N}$-subword $y = l_sl_{s+1}\cdots$ of x and the $\mathbb{N}$-subword $z = l_{s-1}^{-1}l_{s-2}^{-1}\cdots$ of x^{-1}; we will write in this case $x = z^{-1}y$. Conversely, given two $\mathbb{N}$-words $z = l_1l_2\cdots$ and $y = l'_1l'_2\cdots$, we may consider $\cdots l_2^{-1}l_1^{-1}l'_1l'_2\cdots$; in case this is a $\mathbb{Z}$-word, it will be denoted by $z^{-1}y$; of course, y is an $\mathbb{N}$-subword of $z^{-1}y$, and z is an $\mathbb{N}$-subword of $(z^{-1}y)^{-1} = y^{-1}z$.

Given a cyclic word w, we may consider the $\mathbb{Z}$-word ${}^\infty w^\infty = \cdots ww\cdots$; the $\mathbb{Z}$-words of the form ${}^\infty w^\infty$ are said to be *periodic.* A $\mathbb{Z}$-word x is called *right periodic* provided it is of the form $z^{-1}y$ where y is a periodic $\mathbb{N}$-word. Similarly, the $\mathbb{Z}$-word x may be called *left periodic* provided x^{-1} is right periodic. Finally, the $\mathbb{Z}$-word x is called *biperiodic* provided x is both left periodic and right periodic, but not periodic.

If the $\mathbb{Z}$-word $x = \cdots l_{-2}l_{-1}l_0l_1l_2\cdots$ is right periodic, but not periodic, there exists an $s \in \mathbb{Z}$ such that the $\mathbb{N}$-words $y = l_{s+1}l_{s+2}\cdots$ is a periodic $\mathbb{N}$-word, whereas the $\mathbb{N}$-word $l_sl_{s+1}\cdots$ is not periodic. Then y is said to be the *maximal periodic* $\mathbb{N}$-*subword of* x.

3. Expanding and Contracting $\mathbb{N}$-Subwords

We want to single out two kinds of maximal periodic $\mathbb{N}$-subwords: the expanding ones and the contracting ones. A periodic $\mathbb{N}$-word w^∞ is said to be *expanding* provided the last letter of w is inverse, and *contracting* provided the last letter of w is direct.

Consider now an $\mathbb{N}$-word or $\mathbb{Z}$-word x with a maximal periodic $\mathbb{N}$-subword y, and let us assume that x is not a periodic $\mathbb{N}$-word. Thus $x = u^{-1}ly$, where l is a letter and y is a periodic $\mathbb{N}$-subword, whereas ly no longer is periodic (here, u is either a (finite) word of length ≥ 0 or an $\mathbb{N}$-word). The periodic $\mathbb{N}$-word y is called *expanding as an* $\mathbb{N}$-*subword*, provided l is a direct letter and y is an expanding $\mathbb{N}$-word. Correspondingly, y is called *contracting as an* $\mathbb{N}$-*subword*, provided l is an inverse letter and y is a contracting $\mathbb{N}$-word.

An almost periodic $\mathbb{N}$-word x will be said to be *expanding* or *contracting,* provided the maximal periodic $\mathbb{N}$-subword is expanding, or contracting as an $\mathbb{N}$-subword, respectively.

Assume now that x is biperiodic. We call x *expanding*, provided the maximal periodic $\mathbb{N}$-subwords of both x and x^{-1} are expanding as $\mathbb{N}$-subwords. We call x a *mixed* $\mathbb{Z}$-word, provided the maximal periodic $\mathbb{N}$-subword of x is expanding as an $\mathbb{N}$-subword, and the maximal periodic $\mathbb{N}$-subword of x^{-1} is contracting as an $\mathbb{N}$-subword. Finally, we call x *contracting,* provided the maximal periodic $\mathbb{N}$-subwords of x and x^{-1} are both contracting as $\mathbb{N}$-subwords.

4. Some Infinite Dimensional Modules

In the same way as the construction of $M(w)$, we attach an infinite dimensional module $M(x)$ to every $\mathbb{N}$-word x, and to every $\mathbb{Z}$-word x. We call the modules $M(x)$ *string modules.* We write down the formula for x a $\mathbb{Z}$-word.

Thus, let $x = \cdots l_{-2}l_{-1}l_0l_1l_2\cdots$ with $c(i) = s(l_i)$. For any vertex a, we put

$$M(x)_a = \bigoplus_{i\in\mathbb{Z}} k_{ac(i)} \ ;$$

an element in $M(x)_a$ is of the form $(\lambda_i)_{i\in\mathbb{Z}}$ with $\lambda_i \in k_{ac(i)}$ such that only finitely many λ_i are non-zero. Again, the module $M(x)$ itself is the direct sum of these vector spaces $M(x)_a$, where $a \in Q_0$, thus $M(x) = \bigoplus_{i\in\mathbb{Z}} k$.

For any arrow $\alpha\colon a \to b$, let $M(x)_\alpha\colon M(x)_a \to M(x)_b$ be defined by

$$M(x)_\alpha\big((\lambda_i)_i\big) = (\mu_i)_i$$

where

$$\mu_i = \delta(\alpha, l_i^{-1})\lambda_{i-1} + \delta(\alpha, l_{i+1})\lambda_{i+1}$$

for all $i \in \mathbb{Z}$.

We also may define a module $\overline{M}(x)$, it is obtained by replacing the infinite direct sum by the corresponding product. Again, it may be sufficient to consider the case of the $\mathbb{Z}$-word x. For any vertex a, we put

$$\overline{M}(x)_a = \prod_{i\in\mathbb{Z}} k_{ac(i)} \ ;$$

an element in $\overline{M}(x)_a$ is of the form $(\lambda_i)_{i\in\mathbb{Z}}$, with arbitrary $\lambda_i \in k_{ac(i)}$. The module $\overline{M}(x)$ is the direct sum of these vector spaces $M(x)_a$, therefore $M(x) = \prod_{i\in\mathbb{Z}} k$ (here we use that the vertex set Q_0 of the quiver Q is finite). As before, for any arrow $\alpha\colon a \to b$, let $\overline{M}(x)_\alpha\colon \overline{M}(x)_a \to \overline{M}(x)_b$ be defined by

$$\overline{M}(x)_\alpha\big((\lambda_i)_i\big) = (\mu_i)_i$$

where

$$\mu_i = \delta(\alpha, l_i^{-1})\lambda_{i-1} + \delta(\alpha, l_{i+1})\lambda_{i+1},$$

for all $i \in \mathbb{Z}$.

In addition, we also may consider the submodule $M^+(x)$ of $M(x)$, where $M^+(x)_a$ is the set of elements $(\lambda_i)_{i\in\mathbb{Z}}$ with $\lambda_i \in k_{ac(i)}$, such that $\lambda_i = 0$ for $i \ll 0$.

5. The Main Theorem.

Consider now an almost periodic $\mathbb{N}$-word x which is expanding or contracting, or a biperiodic $\mathbb{Z}$-word x which is expanding, mixed or contracting. We define an infinite dimensional module $C(x)$ as follows:

$$C(x) = \begin{cases} \overline{M}(x) & \text{in case } x \text{ is expanding,} \\ M^+(x) & \text{in case } x \text{ is mixed, and} \\ M(x) & \text{in case } x \text{ is contracting.} \end{cases}$$

Theorem. *Let x be either an almost periodic $\mathbb{N}$-word x which is expanding or contracting, or else a biperiodic $\mathbb{Z}$-word x which is expanding, mixed or contracting. Then: The module $C(x)$ is algebraically compact. If x is contracting, then $C(x)$ is even Σ-algebraically compact.*

In the next two sections, we will recall the notion of an algebraically compact module and the related notion of a linearly compact module: we are going to prove that the modules $C(x)$ are linearly compact as modules over the endomorphism ring E. Actually, we will consider a certain subring of E, the shift ring, and show that $C(x)$ is linearly compact even as a module over its shift ring.

This is the first of a series of papers devoted to algebraically compact modules. In part II, we will show that the modules $C(x)$ presented here are also indecomposable. This has been known only in case x is a contracting $\mathbb{N}$-word or a contracting $\mathbb{Z}$-word: we recall that Krause [K] has shown that all the string modules $M(x)$ are indecomposable. In [R2], the modules $C(x)$ will be used in order to sew together various components of the Auslander-Reiten quiver of a string algebra.

Remark. Our restriction of dealing with finite dimensional monomial algebras only, is not too essential. Let B be an arbitrary finite dimensional algebra, let $\mathcal{J}$ be an ideal of B, and $A = B/\mathcal{J}$. We may consider the A-modules just as those B-modules which are annihilated by $\mathcal{J}$. Clearly, a B-module M annihilated by $\mathcal{J}$ is algebraically compact or Σ-algebraically compact if and only if the same holds for M considered as an A-module. Thus, any ideal $\mathcal{J}$ of B with $A = B/\mathcal{J}$ a monomial algebra will allow to look for A-modules (thus B-modules) of the form $C(x)$. In particular, let $\mathcal{R}$ be the radical of B. If B is any basic k-algebra, where k is an algebraically closed field, then $B/\mathcal{R}^2$ is a monomial algebra.

6. Linearly Compact Modules

Linearly compact modules were introduced by Zelinski [Z]. We recall the definition, some examples and the important fact that the class of linearly compact modules is closed under extensions.

Let R be a ring, let V be a (left) R-module. Recall that the module V is said to be *linearly compact* provided the following condition is satisfied: assume that there is given an index set I and for any $i \in I$ a submodule V_i of V as well as an element $x_i \in V$; if for every finite subset J of I there exists x_J in V with $x_J - x_j \in V_j$ for $j \in J$, then there exists $x \in V$ with $x - x_i \in V_i$, for all $i \in I$. (The usual short formulation is as follows: Any finitely solvable system of congruences $x \equiv x_i$ (mod V_i) has a simultaneous solution.)

Let us mention some reformulation of the condition. A set $\mathcal{U}$ of submodules of V will be called *codirected*, provided for every pair U_1, U_2 in $\mathcal{U}$, there exists $U \in \mathcal{U}$ with $U \subseteq U_1$, and $U \subseteq U_2$. Given submodules $U \subseteq U' \subseteq V$, we denote by $\pi_{UU'}\colon V/U \to V/U'$ the canonical projection (given by $(v+U)\pi_{UU'} = v + U'$ for $v \in V$). Given a codirected set $\mathcal{U}$ of submodules U of V, the *inverse limit* $\varprojlim V/U$ is the submodule of all elements $(x_U)_U \in \prod_{U\in\mathcal{U}} V/U$ such that for $U \subseteq U'$, the canonical projection $\pi_{UU'}$ maps x_U onto $x_{U'}$. Of course, there is a canonical map $V \to \varprojlim V/U$ (given by $v \mapsto (v+U)_U$ for $v \in V$ and $U \in \mathcal{U}$).

Lemma. *Let V be an R-module. The following conditions are equivalent:*

(i) *The module V is linearly compact.*

(ii) *For every codirected set $\mathcal{U}$ of R-submodules U of V, the canonical map $V \longrightarrow \varprojlim V/U$ is surjective.*

Examples. **(1)** Any artinian R-module is linearly compact. Namely, let $\mathcal{U}$ be a codirected set of submodules of V. Since V is artinian, there is a minimal element in $\mathcal{U}$, say U_0, and since $\mathcal{U}$ is codirected, we have $U_0 \le U$ for any $U \in \mathcal{U}$. But this means that $\varprojlim V/U$ can be identified with V/U_0.

(1′) If R is a k-algebra, where k is a field, and V is an R-module which is finite dimensional as a k-space, then V is linearly compact. This follows from (1), since finite dimensional modules are artinian.

(2) Let k be a field and $R = k[[T]]$ the ring of formal power series in one variable T over k. Then the regular representation ${}_RR$ is linearly compact.

Proof: The non-zero submodules of ${}_RR$ are the ideals RT^n with $n \in \mathbb{N}_0$. The only interesting inverse limits to be considered occur when $\mathcal{U}$ is an infinite set of non-zero ideals, and then we obtain as inverse limit just ${}_RR$ itself.

(3) Let $\eta\colon S \to R$ be a ring homomorphism. Let V be an R-module; we may consider V also as an S-module via η. If V considered as an S-module is linearly compact, then also as an R-module.

Proof: If U is an R-submodule of V, then it is also an S-submodule.

An important result which we will need is the following one:

Proposition (Zelinski). *Let V be an R-module, let U be a submodule. If both U and V/U are linearly compact, then also V is linearly compact.*

7. Algebraically Compact Modules

Let M be a (left) A-module. Let $a = (a_{ij})_{ij}$ be an $(m \times n)$-matrix, and $b = (b_i)_i$ a vector of length m, with coefficients a_{ij}, b_i in A. Let $U(a,b)$ be the set of elements $y \in M$ such that there are elements $x_1, \dots, x_n \in M$ with

$$\sum_{j=1}^{n} a_{ij} x_j = b_i y \qquad \text{for} \qquad 1 \le i \le m.$$

This is a subgroup (but usually not a submodule!) of M. A subgroup of the form $U(a,b)$ is called a *finitely definable subgroup* of M.

Definition: A module M is called *algebraically compact* (or *pure injective)*, provided for every codirected system of finitely definable subgroups M_i of M, the canonical map

$$M \longrightarrow \varprojlim M/M_i$$

is surjective.

Lemma. *Let E be the endomorphism ring of M. If U is a finitely definable subgroup of M, then U is a submodule of M_E.*

Proof: Let $U = U(a,b)$. Let f be an endomorphism of M. Let $y \in U$, thus there are elements $x_1, \ldots, x_n \in M$ such that $\sum_{j=1}^n a_{ij}x_j = b_i y$ for $1 \le i \le m$. Then $\sum_{j=1}^n a_{ij}(x_j f) = \left(\sum_{j=1}^n a_{ij}x_j\right) f = (b_i y)f = b_i(yf)$, thus yf belongs to $U(a,b)$. This shows that $Uf \subseteq U$.

Corollary. *Let M be an A-module, let E be its endomorphism ring. If M_E is linearly compact, then M (as an A-module) is algebraically compact.*

This follows immediately from the fact that finitely definable subgroups of an A-module M are E-submodules.

Remark. Note that the class of algebraically compact modules is not closed under extensions; examples do exist already for the Kronecker algebra (see section 12).

Also we recall that a module M is said to be Σ-*algebraically compact* provided all direct sums of copies of M are algebraically compact. The module M is Σ-algebraically compact, if and only if it satisfies the descending chain condition for finitely definable subgroups. Thus, let ${}_AM$ be an A-module with endomorphism ring E. If M_E is artinian, then ${}_AM$ is Σ-algebraically compact.

8. The Shift Endomorphism.

Let x be an almost periodic $\mathbb{N}$-word or a biperiodic $\mathbb{Z}$-word, let $y = l_{s+1}l_{s+2}\cdots$ be the maximal periodic $\mathbb{N}$-subword of x.

Let $C = C(x)$. In case x is an $\mathbb{N}$-word, $C = \bigoplus_{i\ge 0} k$ or $C = \prod_{i\ge 0} k$, thus always we may consider C as a subspace of $\prod_{i\ge 0} k$. Similarly, for x a $\mathbb{Z}$-word, C is a subspace of $\prod_{i\in\mathbb{Z}} k$. In both cases, we may write the elements of C as sequences $(\lambda_i)_i$ with index set $\mathbb{N}_0$ or $\mathbb{Z}$, the coefficients λ_i being elements of k.

Given an element $u \in \mathbb{Z}$, we denote by $C^{\ge u}$ the set of elements $(\lambda_i)_i$ of C with $\lambda_i \ne 0$ only for $i \ge u$. Similarly, one may define $C^{>u}, C^{<u}, C^{\le u}$. Also, for $u \le v$, let $C^{(u,v)} = C^{>u} \cap C^{<v}$. More generally, if I is a subset of $\mathbb{Z}$, let C^I be the set of elements $(\lambda_i)_i$ of C with $\lambda_i \ne 0$ only for $i \in I$.

If l_u is a direct letter, we may consider $C^{<u}$ as a submodule and the corresponding factor module will be identified with $C^{\geq u}$, thus we have the following exact sequence:

$$0 \to C^{<u} \to C \to C^{\geq u} \to 0.$$

Similarly, in case l_u is an inverse letter, we have the following exact sequence with canonical maps:

$$0 \to C^{\geq u} \to C \to C^{<u} \to 0.$$

Remark. The modules $C^{\geq u}$ and $C(l_{u+1}l_{u+2}\cdots)$ should not be confused. Namely, let y be the maximal periodic $\mathbb{N}$-subword of x. Then one of the $\mathbb{N}$-words y and $l_{u+1}l_{u+2}\cdots$ may be contracting, whereas the other is expanding (again, one may construct an example looking at the Kronecker algebra).

We are going to define a special endomorphism $\Phi = \Phi_x$ of $C(x)$, called the (canonical) *shift.* We consider the two cases of y being expanding or contracting. We can write $y = w^\infty$, where w is some primitive cyclic word, say of length n.

First, consider the case where y is expanding. This means that the last letter of w is inverse. Since $y = l_{s+1}l_{s+2}\cdots$, and w is of length n, we see that l_{s+n} is inverse, whereas l_s is direct (or else $s = 0$ and x is a periodic $\mathbb{N}$-word). In any case, we see that $C^{<s}$ is a submodule with factor module $C^{\geq s}$, and also $C^{\geq s+n}$ is a submodule. Also, the modules $C^{\geq s}$ and $C^{\geq s+n}$ are canonically isomorphic: they are of the form $C(w^\infty) = \overline{M}(w^\infty)$. The canonical maps

$$C \longrightarrow C^{\geq s} \simeq C^{\geq s+n} \longrightarrow C$$

combine to an endomorphism of C which will be called the *shift map* $\Phi = \Phi_x$ of $C = C(x)$; note that for $(\lambda_i)_i \in C$, we have

$$(\lambda_i)_i\Phi = (\mu_i)_i, \qquad \text{where} \qquad \mu_i = \begin{cases} \lambda_{i-n} & \text{for} \quad i \geq s+n, \\ 0 & \text{for} \quad i < s+n. \end{cases}$$

Let $p(T) = \sum_{t\geq 0} a_t T^t$ be a power series in the variable T with coefficients a_t in k. Then $p(\Phi)$ is defined by

$$(\lambda_i)_i p(\Phi) = (\nu_i)_i, \qquad \text{where} \qquad \nu_i = \sum_{0\leq tn\leq i-s} a_t\lambda_{i-tn}$$

Clearly, $p(\Phi)$ is an endomorphism of C, and we obtain in this way an embedding of the power series ring $k[[T]]$ into the endomorphism ring E of C (the algebra homomorphismus $k[[T]] \to E$ which sends T to Φ is injective, since Φ is not nilpotent and the only non-zero ideals of $k[[T]]$ are those generated by the powers of T).

Similarly, we deal with the case where y is contracting, so that the last letter of w is direct. Thus, now, the letter l_{s+n} is direct, whereas the letter l_s is inverse (or else $s = 0$ and x is a periodic $\mathbb{N}$-word). Consequently, $C^{\geq s+n}$ is a factor module, and $C^{\geq s}$ is a submodule of C. The modules $C^{\geq s}$ and $C^{\geq s+n}$ both are of the form $C(w^\infty) = M(w^\infty)$. Again, we consider a composition of canonical maps, namely

$$C \longrightarrow C^{\geq s+n} \simeq C^{\geq s} \longrightarrow C$$

and call it the *shift* $\Phi = \Phi_x$ of $C = C(x)$. We may define Φ directly as follows:

$$(\lambda_i)_i \Phi = (\mu_i)_i, \qquad \text{where} \qquad \mu_i = \begin{cases} \lambda_{i+n} & \text{for} \quad i \geq s, \\ 0 & \text{for} \quad i < s. \end{cases}$$

Also in this case, we define $p(\Phi)$ for any power series $p(T) = \sum_{t \geq 0} a_t T^t$; here, we set

$$(\lambda_i)_i p(\Phi) = (\nu_i)_i, \qquad \text{where} \qquad \nu_i = \sum_{t \geq 0} a_t \lambda_{i+tn};$$

note that the sum $\sum_{t \geq 0} a_t \lambda_{i+tn}$ is finite, since $\lambda_j = 0$ for $j \gg 0$. Again, $p(\Phi)$ is an endomorphism of C, and we obtain in this way an embedding of the power series ring $k[[T]]$ into the endomorphism ring of C.

We denote by E the endomorphism ring of $C = C(x)$. If x is an almost periodic $\mathbb{N}$-word, let S be the subring of all power series $p(\Phi)$, with $\Phi = \Phi_x$. If x is a biperiodic $\mathbb{Z}$-word, the shift $\Phi_{x^{-1}}$ yields a second interesting endomorphism of C which we denote by Ψ (the precise formula will be written down, below). We also form the power series $p(\Psi)$. We denote by S the subring of E generated by the power series in Φ and the power series in Ψ. The subring S of E will be called the *shift ring* of C.

Remark. Let x be an $\mathbb{N}$-word. For the further considerations, it will be essential that we consider not only the shift Φ, but all power series $p(\Phi)$, in case x is expanding. In case x is contracting, it would be sufficient to work with the shift alone, and we include all the power series in the shift ring S only for symmetry. Similarly, dealing with a mixed or a contracting $\mathbb{Z}$-word, one may work with a corresponding proper subring of S avoiding power series of a "contracting" operator.

9. The Module $C(x)$ Considered as a Module over its Shift Ring.

Theorem. *Let x be either an almost periodic $\mathbb{N}$-word which is expanding or contracting, or else a biperiodic $\mathbb{Z}$-word which is expanding, mixed or contracting. Let $C = C(x)$, and let S be its shift ring. Then C_S is linearly compact.*

Proof. First, consider the case of an almost periodic $\mathbb{N}$-word $x = l_1 l_2 \cdots$ with maximal periodic $\mathbb{N}$-subword $w^\infty = l_{s+1} l_{s+2} \cdots$, where w is a primitive cyclic word of length n. For $0 \leq i < n$, let $I(i)$ be the set of all integers of the form $s + i + tn$ with $t \in \mathbb{N}_0$. Then $\mathbb{N}_0$ is the disjoint union of the set $\{ i \mid 0 \leq i < s \}$ and the sets $I(i)$ with $0 \leq i < n$, therefore, as a vector space, we have the following decomposition

$$C = C^{<s} \oplus \bigoplus_{i=0}^{n-1} C^{I(i)}.$$

Obviously, this is a direct decomposition of C as an S-module. Assume now that x is either expanding or contracting, so that $\Phi = \Phi_x$ is defined. Then, the structure of th S-module C_S is as follows: Φ maps $C^{<s}$ onto zero, thus $C^{<s}$ is semisimple. If x is expanding, then $C^{I(i)}$ is just the regular representation of $S = k[[T]]$, whereas if x is contracting, then $C^{I(i)}$ is an artinian S-module. This shows that C_S is linearly compact.

Next, consider the case of a biperiodic $\mathbb{Z}$-word x, and assume that x is expanding, or mixed, or contracting. Without loss of generality, we also can assume that $z = l_0^{-1}l_{-1}^{-1}l_{-2}^{-1}\cdots$ is the maximal periodic $\mathbb{N}$-subword of x^{-1}, say of period m. In case z is expanding, the shift Ψ is given by

$$(\lambda_i)_i\Psi = (\mu_i)_i, \qquad \text{where} \qquad \mu_i = \begin{cases} \lambda_{i+m} & \text{for} \quad i \leq -m, \\ 0 & \text{for} \quad i > -m. \end{cases}$$

In case z is contracting, the shift Ψ is given by

$$(\lambda_i)_i\Psi = (\mu_i)_i, \qquad \text{where} \qquad \mu_i = \begin{cases} \lambda_{i-m} & \text{for} \quad i \leq 0, \\ 0 & \text{for} \quad i > 0. \end{cases}$$

Let $y = l_{s+1}l_{s+2}\cdots$ be the maximal periodic $\mathbb{N}$-subword of x, let y be of period n. Choose $t \in \mathbb{N}$ such that both $tm > -s$ and $tn > -s$. We consider

$$C_- = C^{\leq -tm}, \qquad C_0 = C^{(-tm, s+tn)}, \qquad \text{and} \qquad C_+ = C^{\geq s+tn}.$$

The direct sum of these vector spaces is just C

$$C = C_- \oplus C_0 \oplus C_+.$$

Since $tm < -s$, we see that Φ operates trivially on C_- and maps C into $C_0 \oplus C_+ = C^{>-tm}$. Similarly, since $tn > -s$, we see that Ψ operates trivially on C_+ and maps C into $C_- \oplus C_0 = C^{<s+tn}$

Consider first the case when x is expanding. Obviously, C_- is just the image of Ψ^t; in particular, C_- is linearly compact as a $k[[\Psi]]$-module. Since Φ operates trivially on C_-, we see that C_- is an S-submodule of C and also linearly compact as an S-module.

Similarly, in case x is expanding or mixed, C_+ is just the image of Φ^t, and therefore linearly compact as a $k[[\Phi]]$-module. Since Ψ operates trivially on C_+, we see that C_+ is an S-submodule of C and also linearly compact as an S-module.

Next, let x be mixed or contracting. Then $C^{>-tm}$ is just the kernel of Ψ^t, and the corresponding factor module C_- is an artinian $k[[\Psi]]$-module. Also, Φ maps C into $C^{>-tm}$, thus $C^{>-tm}$ is an S-submodule (and Φ operates trivially on C_-). As an artinian S-module, C_- is linearly compact.

Similarly, in case x is contracting, $C^{<s+tn}$ is the kernel of Φ^t, and the correspon-ding factor module C_+ is an artinian $k[[\Phi]]$-module. Also, Ψ maps C into $C^{<s+tn}$, thus $C^{<s+tn}$ is an S-submodule (and Ψ operates trivially on C_+). As an artinian S-module, C_+ is linearly compact.

If x is expanding, $C_- \oplus C_+$ is a linearly compact S-submodule of C with finite dimensional (and therefore also linearly compact) factor module. Thus C_S is linearly compact.

If x is mixed, C has the following filtration by S-submodules: $0 \subset C_+ \subset C^{>-tm} \subset C$, and all three factors are linearly compact: this has been shown above for C_+ and for $C/C^{-tm} \simeq C_-$, and $C^{>-tm}/C_+ \simeq C_0$ is finite dimensional.

If x is contracting, then C_0 is an S-submodule, and $C/C_0 \simeq C_- \oplus C_+$. Again, we see that C_S is linearly compact. This completes the proof.

Remark 1. We have considered above the following filtrations: For x expanding:

$$C_- \oplus C_+ \;\subset\; C\,,$$

for x mixed:

$$C_+ \;\subset\; C^{>-tm} \;\subset\; C\,,$$

and for x contracting:

$$C_0 \;\subset\; C\,.$$

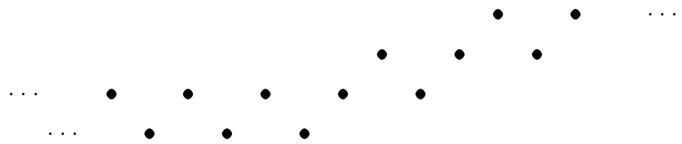
Note that all these submodules are not only S-submodules, but also A-submodules.

Remark 2. In case $s > 0$, the situation is much easier. Then, we can write C_S as the direct sum

$$C = C^{\leq 0} \oplus C^{(0,s)} \oplus C^{\geq s};$$

the shift Φ operates trivially on the first two summands, whereas Ψ operates trivially on the last two summands.

However, there are many examples with $s \leq 0$. Here is such an example: Consider the free algebra in two generators α, β, with relations $\alpha\beta = \beta\alpha = \alpha^3 = \beta^2 = 0$. The $\mathbb{Z}$-word $x = z^{-1}y$ with $y = (\alpha\beta^{-1}\alpha)^\infty$ and $z = (\alpha^{-1}\beta)^\infty$, thus



$$\cdots \;\; \beta \;\; \alpha \;\; \beta \;\; \alpha \;\; \beta \;\; \alpha \mid \alpha \;\; \beta \;\; \alpha \;\; \alpha \;\; \beta \;\; \alpha \;\; \cdots$$

is biperiodic with $s = -3$. The vertical bar $|$ indicates the position 0 (according to the labelling introduced above), the letter to the right of $|$ is l_1. The letters l_{-3} and l_1 are emphasized.

Proof of the Main Theorem. We have shown that the modules $C(x)$ are linearly compact as modules over the shift ring, thus they are linearly compact as modules over the endomorphism ring, and therefore algebraically compact as A-modules. In case x is contracting, we have seen that $C(x)$ as a module over its shift ring is artinian, therefore $C(x)$ as a module over its endomorphism ring is artinian. Thus, in this case, $C(x)$ is (as an A-module) even Σ-algebraically compact.

Remark (H.Krause). In case x is an expanding $\mathbb{N}$-word or an expanding $\mathbb{Z}$-word, there is a more direct proof in order to show that $C(x)$ is algebraically compact. Namely, in this case $C(x) = \overline{M}(x)$, and we claim that $\overline{M}(x)$ is just the k-dual of a corresponding A^{op}-module $M(x_*)$. Recall that the quiver of A^{op} is $Q^* = (Q_0, Q_1^*)$, where Q_1^* is the set of inverse letters for Q; in order to avoid confusion, given an arrow $\alpha \in Q_1$, we write α^* (instead of α^{-1}) for the corresponding element of Q_1^*; similarly, we write $(\alpha^{-1})^*$ instead of α. Given a word $w = l_1 \cdots l_n$, the dual word is $w_* = (l_1^{-1})^* \cdots (l_n^{-1})^*$ (clearly the conditions (W1), (W2) are satisfied). Similarly, the dual $\mathbb{N}$-word of the $\mathbb{N}$-word $x = l_1 l_2 \cdots$ is $x_* = (l_1^{-1})^*(l_2^{-1})^* \cdots$, and the dual $\mathbb{Z}$-word of the $\mathbb{Z}$-word $x = \cdots l_{-1} l_0 l_1 \cdots$ is $x_* = \cdots (l_1^{-1})^*(l_0^{-1})^*(l_1^{-1})^* \cdots$. There are the corresponding A^{op}-modules $M(w_*)$ and $M(x_*)$, and it is easy to check that

$$\overline{M}(x) = \mathrm{Hom}_k(M(x_*), k),$$

for any $\mathbb{N}$-word or $\mathbb{Z}$-word x. It is well-known that the k-dual of any module is algebraically compact (see e.g. Exercise 7.10 in [JL]).

10. Some Other Algebraically Compact Modules

For the convenience of the reader, we want to exhibit also some other algebraically compact modules, related to a primitive cyclic word w. This should be well-known, but there seems to be no reference.

First, consider the polynomial ring $k[T]$ in one variable. A $k[T]$-module will be written in the form (V, ϕ), where V is a vector space and ϕ is the endomorphism of V given by the multiplication with T (thus ϕ is just an arbitrary endomorphism of V). Some special $k[T]$-modules will be of importance. First of all, the field of rational functions $k(T)$, considered as a $k[T]$ module, will be called the *generic* $k[T]$-module G; it is of importance that the endomorphism ring of G (as a $k[T]$-module) is just the field $k(T)$; of course, G is one-dimensional as a module over its endomorphism ring (thus it is a generic $k[T]$-module in the sense of [CB], and the only one).

If L is a simple $k[T]$-module, let us denote by $L[\infty]$ its injective hull; these modules $L[\infty]$ are usually called *Prüfer modules.* Note that the submodules of $L[\infty]$ form a chain: there is just one submodule of length n, we will denote it by $L[n]$ (thus $L[0] = 0,\ L[1] = L$):

$$L[1] \subset L[2] \subset L[3] \subset \cdots \subset \bigcup_{n \in \mathbb{N}} L[n] = L[\infty].$$

Of course, the simple $k[T]$-modules are the modules of the form $L = L_{\mathbf{p}} = k[T]/\mathbf{p}$, where $\mathbf{p}$ is a maximal ideal of $k[T]$, and we denote by $J_{\mathbf{p}}$ the $\mathbf{p}$-adic completion of $k[T]$; there is a chain of epimorphisms

$$\cdots \longrightarrow L_{\mathbf{p}}[3] \longrightarrow L_{\mathbf{p}}[2] \longrightarrow L_{\mathbf{p}}[1],$$

and $J_{\mathbf{p}}$ is just the corresponding inverse limit. The module $J_{\mathbf{p}}$ is said to be the $\mathbf{p}$-*adic* $k[T]$-*module.* Also, it is easy to see that we have

$$J_{\mathbf{p}} = \mathrm{Hom}_k(L_{\mathbf{p}}[\infty], k).$$

Example: Consider the special case of $\mathbf{p} = \langle T\rangle$. We have

$$L_{\langle T\rangle}[\infty] = \Big(\bigoplus_{i\in\mathbb{N}} k, \sigma\Big) \qquad \text{where} \qquad \sigma(\lambda_1,\lambda_2,\dots) = (\lambda_2,\lambda_3,\dots),$$
$$J_{\langle T\rangle} = \Big(\prod_{i\in\mathbb{N}} k, \varepsilon\Big), \qquad \text{where} \qquad \varepsilon(\lambda_1,\lambda_2,\dots) = (0,\lambda_1,\lambda_2,\dots).$$

These two modules may be considered as rather typical. In case the field k is algebraically closed, any maximal ideal $\mathbf{p}$ of $k[T]$ is of the form $\langle T-c\rangle$ with $c\in k$, thus it is the image of $\langle T\rangle$ under the automorphism of $k[T]$ which sends T to $T-c$. It follows that the Prüfer module $L_{\mathbf{p}}[\infty]$ and the $\mathbf{p}$-adic module $J_{\mathbf{p}}$ are obtained from $L_{\langle T\rangle}[\infty]$ or $J_{\langle T\rangle}$, respectively, by the use of this automorphism.

The generic module and the Prüfer modules are Σ-algebraic compact; the $\mathbf{p}$-adic modules are algebraically compact. For a proof, one only has to observe that the generic module and the Prüfer modules are artinian when considered as modules over the endomorphism ring; the algebraic compactness of the $\mathbf{p}$-adic modules follows from the fact that these are k-duals of other modules.

Now, we return to the quiver Q and to words using the elements of $\overline{Q}_1$ as letters. Let w be a primitive cyclic word. We define a functor F_w from the category of $k[T]$-modules to the category of A-modules, as follows: Let (V,ϕ) be a $k[T]$-module. We want to construct $M = F_w(V,\phi)$. As before, we write $w = l_1l_2\cdots l_n$, and $c(i) = s(l_i)$, and $c(0) = t(l_1)$; of course, we have $c(0) = c(n)$. *We only consider the special case where l_n is an inverse letter.*

Given two vertices a, b, let $V_{ab} = V$ in case $a = b$, and zero otherwise. Let

$$M_a = \bigoplus_{i=1}^{n} V_{ac(i)},$$

an element in M_a is of the form $(\lambda_i)_{1\le i\le n}$ with $\lambda_i \in V_{ac(i)}$.

For any arrow $\alpha\colon a\to b$, the map $M_\alpha\colon M_a \to M_b$ is defined by

$$M_\alpha\big((\lambda_i)_i\big) = (\mu_i)_i$$

where

$$\mu_i = \delta(\alpha, l_i^{-1})\lambda_{i-1} + \delta(\alpha, l_{i+1})\lambda_{i+1}$$

for $1\le i<n$ and

$$\mu_n = \delta(\alpha, l_n^{-1})\phi(\lambda_{n-1}) + \delta(\alpha, l_1)\lambda_1;$$

again we note that always at most one of two summands can be non-zero.

Lemma. *The functor F_w commutes with the formation of direct sums and direct products. Thus, it maps algebraically compact $k[T]$-modules to algebraically compact A-modules, and Σ-algebraically compact $k[T]$-modules to Σ-algebraically compact A-modules.*

Proof: The first assertion follows directly from the construction. This implies the second assertion, since there is the following characterization: A module M is algebraically compact if and only if for any index set I the summation map $\bigoplus_I M \to M$ can be extended to a map $\prod_I M \to M$ (see [JL], Theorem 7.1)

Given a cyclic word $w = l_1 l_2 \cdots l_n$, the words $w_t = l_{t+1} \cdots l_n l_1 \cdots l_t$ are said to be obtained from w by *rotation.* The words obtained from a cyclic word w using rotation and inversion are said to be *equivalent* to w.

Remark. Consider the special case of $\mathbf{p} = \langle T \rangle$. The modules $L_{\langle T \rangle}[\infty]$ and $J_{\langle T \rangle}$ are sent under F_w to modules which we have worked with before: Let $w = l_1 \cdots l_n$, and denote by v the word $v = l_{n-1}^{-1} \cdots l_1^{-1} l_n^{-1}$; note that v and w are equivalent cyclic words. The $\mathbb{N}$-word w^∞ is expanding, whereas v^∞ is contracting (since we assume that l_n is an inverse letter). We have

$$F_w\left(L_{\langle T \rangle}[\infty]\right) = M(v^\infty) \qquad \text{and} \qquad F_w\left(J_{\langle T \rangle}\right) = \overline{M}(w^\infty).$$

This can easily be verified, using the explicit description of the modules $L_{\langle T \rangle}[\infty]$ and $J_{\langle T \rangle}$. One should not be puzzled about the appearance of the two rather differently looking words v and w: Note that the simple module $L_{\langle T \rangle}$ is characterized by the fact that the multiplication by T is the zero map, thus the multiplication by T is singular on $L_{\langle T \rangle}[n]$ for all $n \geq 1$, on $L_{\langle T \rangle}[\infty]$ and on $J_{\langle T \rangle}$. Correspondingly, we see that

$$F_w(L_{\langle T \rangle}) = M(l_1 \cdots l_{n-1}) \qquad \text{and} \qquad F_w(L_{\langle T \rangle}[2]) = M(l_1 \cdots l_{n-1} l_n l_1 \cdots l_{n-1}).$$

Note that the letters $l_1, \ldots, l_{n-1}$ on the one hand and the letter l_n on the other play a different role: The word $l_1 \cdots l_{n-1}$ describes the module $X = F_w(L_{\langle T \rangle})$, whereas the letter l_n yields a particular selfextension of this module X; this selfextension of X is the one which comes from the selfextension of $L_{\langle T \rangle}$ (using the multiplication by T) under our functor F_w.

Note that the generic module G, as well as all the modules $L_{\mathbf{p}\langle T \rangle}[\infty]$ and $J_{\mathbf{p}}$ with $\mathbf{p} \neq \langle T \rangle$ are actually $k[T, T^{-1}]$-modules: they are of the form (V, ϕ), where ϕ is an automorphism of V. Thus, if we want to avoid overlaps between lists of modules constructed in different ways, it may be convenient to consider only the restriction of F_w to the category of $k[T, T^{-1}]$-modules (as it was done in [GP] and elsewhere).

11. String Algebras

We recall that the algebra A is said to be a *string algebra* (or a *special biserial algebra*), provided $A = kQ/\mathcal{I}$ where $Q = (Q_0, Q_1)$ is a quiver and $\mathcal{I}$ an admissible ideal generated by monomials, with the following properties:

- (B1) Every vertex of Q is endpoint of at most two arrows and starting point of at most two arrows.
- (B2) For any arrow β, there is at most one arrow α such that $\alpha\beta$ does not belong to $\mathcal{I}$, and at most one arrow γ such that $\beta\gamma$ does not belong to $\mathcal{I}$.

Proposition 1. *Let A be a string algebra. Then: Any almost periodic $\mathbb{N}$-word is expanding or contracting. If x is a biperiodic $\mathbb{Z}$-word, then x is expanding, or mixed, or contracting, or else x^{-1} is mixed.*

Proof. We show the following: Let y be a contracting periodic $\mathbb{N}$-word. If l_0 is a letter such that $x = l_0 y$ is a non-periodic $\mathbb{N}$-word, then x is also contracting.

For the contrary, assume that $x = l_0 y$ is non-periodic, and not contracting. Let $w = l_1 \cdots l_n$ be a cyclic word such that $y = w^\infty$. Since w^∞ is contracting, we know that the letter l_n is direct. By assumption, x is not contracting, thus l_0 is also a direct letter. We have to distinguish two cases.

First, let as assume that the letter l_1 is inverse. Then the arrows l_0 and l_1^{-1} are (different) arrows starting in $c(0) = t(w)$. Since w is cyclic, we have $s(l_n) = s(w) = c(0)$. Thus also the arrow l_n starts in $c(0)$, and, in addition, the two letters l_n and l_1^{-1} are different. Condition (B1) implies that $l_0 = l_n$, thus x is a periodic word, a contradiction.

Second, consider the case of l_1 being direct. Now all three letters l_0, l_1, l_n are direct. Since w^2 is a word, we know that $l_n l_1$ does not belong to the ideal $\mathcal{I}$. Similarly, since x is a $\mathbb{N}$-word, also $l_0 l_1$ does not belong to $\mathcal{I}$. By condition (B2), we conclude that $l_0 = l_n$, thus again x has to be periodic.

By duality, we also have: Let y be a expanding periodic $\mathbb{N}$-word. If l_0 is a letter such that $x = l_0 y$ is a non-periodic $\mathbb{N}$-word, then x is also expanding.

It is easy to see that both statements together yield a proof of the assertions of Proposition 1.

One may ask under what conditions there do exist $\mathbb{N}$-words which are not almost periodic, or $\mathbb{Z}$-words which are neither periodic nor biperiodic. Call a cyclic word *normal* provided the first letter is direct, the last letter is inverse. If α is an arrow, let $\mathcal{N}(\alpha)$ be the set of normal cyclic words with first letter α.

Proposition 2. *Let A be a string algebra. The following conditions are equivalent:*

(i) *There are only finitely many primitive cyclic words.*
(ii) *For any arrow α, there is at most one primitive cyclic word in $\mathcal{N}(\alpha)$.*
(iii) *Any $\mathbb{N}$-word is almost periodic.*
(iv) *Any $\mathbb{Z}$-word is periodic or biperiodic.*

Proof. We note the following: If w belongs to $\mathcal{N}(\alpha)$, then the last letter of w is of the form β^{-1}, where α, β are different arrows ending in one vertex, say in the vertex c, thus they are the only arrows ending in c. This shows that all words in $\mathcal{N}(\alpha)$ have β^{-1} as last letter. As a consequence, we see: If v, w belong to $\mathcal{N}(\alpha)$, then the concatenation vw is defined, thus $\mathcal{N}(\alpha)$ is a semigroup. It is obvious that the subset $\mathcal{N}_1(\alpha)$ of all elements which do not contain $\beta^{-1}\alpha$ as a subword form a free generating set of $\mathcal{N}(\alpha)$; in particular, $\mathcal{N}(\alpha)$ is a free semigroup. and $\mathcal{N}_1(\alpha) = \mathcal{N}(\alpha) \setminus \mathcal{N}(\alpha)^2$. If $\mathcal{N}_1(\alpha)$ contains a unique element, say w, then w is the only primitive cyclic word in $\mathcal{N}(\alpha)$. If $\mathcal{N}_1(\alpha)$ contains two different elements v, w, then all the words vw^n with $n \in \mathbb{N}$ are primitive cyclic words.

(i) $\implies$ (ii). Assume that there are two primitive cyclic words v, w in $\mathcal{N}(\alpha)$. This implies that $\mathcal{N}_1(\alpha)$ contains at least two elements, therefore there are infinitely many primitive cyclic words in $\mathcal{N}(\alpha)$, a contradiction to (i).

(ii) $\implies$ (i): Assume that there are infinitely many primitive cyclic words. Since a cyclic word w contains both direct and inverse letters, there always exists a normal word obtained from w by rotation. This shows that there are infinitely many primitive cyclic words which are normal. Since there are only finitely many arrows in Q, there exists an arrow α such that $\mathcal{N}(\alpha)$ contains infinitely many primitive cyclic words, in contrast to (ii).

(ii) $\implies$ (iii): Let $x = l_1 l_2 \cdots$ be an $\mathbb{N}$-word. Let $c(i) = s(l_i)$, and $c(0) = t(l_1)$. Call an index i a *sink* provided l_i is an inverse letter, whereas l_{i+1} is a direct letter. Since $\mathcal{I}$ is an admissable ideal, there is some natural number m such that all paths of length m belong to $\mathcal{I}$, thus there must exist infinitely many sinks. Since Q has only finitely many arrows, there exists an infinite set J of sinks i such that $l_{i+1} = \alpha$ for some fixed arrow α and all $i \in J$. Denote the elements of J by $n_1 < n_2 < \dots$, and let

$$w_i = l_{n_i+1} l_{n_i+2} \cdots l_{n_{i+1}}.$$

This is a word in $\mathcal{N}(\alpha)$. Since $\mathcal{N}(\alpha)$ is generated by a unique element w, all the words w_i are powers of w. Note that $x = l_1 \cdots l_{n_1} w_1 w_2 \cdots$, thus x is almost periodic.

(iii) $\implies$ (iv): This is trivial.

(iv) $\implies$ (ii): Assume that some $\mathcal{N}(\alpha)$ contains at least two primitive cyclic words. Then $\mathcal{N}_1(\alpha)$ contains two different elements, say v, w. We can form the $\mathbb{Z}$-word

$$\cdots v w^3 v w^2 v w v w^2 v w^3 v \cdots ;$$

it is neither periodic nor biperiodic, in contrast to (iv).

Corollary 1. *Assume that A is a string algebra with only finitely many primitive cyclic words. If w and w' are primitive cyclic words, and $(V, \phi), (V', \phi')$ are $k[T]$-modules such that the A-modules $F_w(V, \phi)$ and $F_{w'}(V', \phi')$ have an isomorphic simple submodule, then w and w' are equivalent.*

Proof. Given a vertex c of the quiver Q of A, we denote by L_c the corresponding simple A-module.

Let $w = l_1 l_2 \cdots l_n$ with letters l_i, and let $w_i = l_{i+1} \cdots l_n l_1 \cdots l_i$; in particular, $w_0 = w_n = w$. Let J be the set of all indices $1 \le i \le n$ such that w_i is normal. Let (V, ϕ) be a $k[T]$-module, let $M = F_w(V, \phi) = \bigoplus_{i=1}^t V$. Then $\bigoplus_{i \in J} V$ is the socle of M, and the copy of V with index i is a direct sum of simple A-modules of the form $L_{c(i)}$.

Now, assume that L_c is a submodule of $F_w(V, \phi)$. Then $c = c(i)$ for some $i \in J$. If L_c is also a submodule of $F'_w(V', \phi')$, then $c = c(j)$ for some $j \in J$. The words w_i, w'_j are normal primitive cyclic words and they start in c. Thus w_i and w'_j are either equal or inverse words. This shows that w and w' are equivalent.

Corollary 2. *Assume that A is a string algebra with only finitely many primitive cyclic words. Then the number of equivalence classes of primitive cyclic words is at most the number of simple A-modules.*

12. Examples

An example of a string algebra is the Kronecker algebra, it is the path algebra of the following *Kronecker quiver*

$$a \;\overset{\alpha}{\underset{\beta}{\rightrightarrows}}\; b\,,$$

the representations $M = (M_a, M_b; M_\alpha, M_\beta)$ of the Kronecker quiver are called *Kronecker modules.*

Let $w = \alpha\beta^{-1}$, this is a primitive cyclic word (and, up to equivalence, the only one). Here is the list of the infinite dimensional indecomposable algebraically compact Kronecker modules:

(i) The finite dimensional indecomposables.
(ii) The *generic* module $F_w(G)$.
(iii) The *Prüfer modules*: the modules $F_w\big(L_{\mathbf{p}}[\infty]\big)$, where $\mathbf{p}$ is a maximal ideal of $k[T]$, and, in addition $M\big((\beta^{-1}\alpha)^\infty\big)$.
(iv) The $\mathbf{p}$-*adic modules*: the modules $F_w\big(J_{\mathbf{p}}\big)$, where $\mathbf{p}$ is a maximal ideal of $k[T]$, and, in addition, $\overline{M}\big((\beta\alpha^{-1})^\infty\big)$.

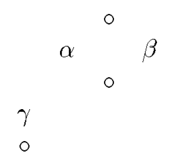
The methods presented in this paper yield all the indecomposable algebraically compact Kronecker modules (of course, this case is well-known, see [JL,P]). One may conjecture that for any string algebra A with only finitely many primitive cyclic words, one obtains in this way all the indecomposable algebraically compact modules.

Let us exhibit a Kronecker module V which is not algebraically compact, with a submodule U such that both U and V/U are algebraically compact. Consider $V = (k[T], k[T]; 1, T\cdot)$; this module is not algebraically compact. The submodule $U = (0, k\cdot 1; 0, 0)$ of V is one-dimensional, thus algebraically compact. The factor module $V/U = M((\beta^{-1}\alpha)^\infty)$ is a Prüfer module, thus also algebraically compact.

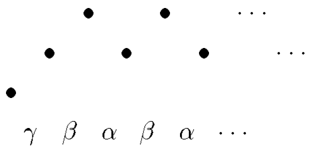
* * *

For the Kronecker quiver, all $\mathbb{N}$-words and all $\mathbb{Z}$-words are periodic. Let us write down typical examples of almost periodic, but non-periodic $\mathbb{N}$-words, and of biperiodic $\mathbb{Z}$-words. All the algebras exhibited below will be string algebras. Always, we present the quiver (and mark the zero relations using dotted lines); in addition, we indicate the shape of some $\mathbb{N}$-word or $\mathbb{Z}$-word.

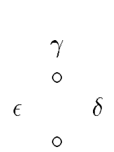
(1) A non-periodic expanding $\mathbb{N}$-word. Consider the algebra A_1 which is the one-point coextension of the Kronecker quiver by a two-dimensional regular module, thus we consider the path algebra of the quiver



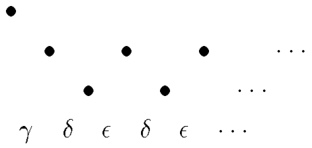
with $\gamma\alpha = 0$. The $\mathbb{N}$-word $x = \gamma(\beta\alpha^{-1})^{\infty}$ is expanding:



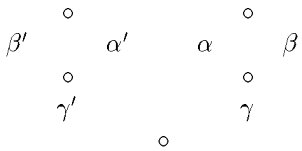
(2) A non-periodic contracting $\mathbb{N}$-word. Here, we consider the one-point extension A_2 of the Kronecker algebra by a two-dimensional regular module, thus



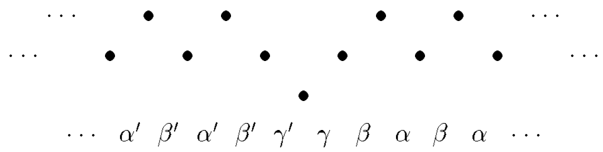
with the relation $\epsilon\gamma = 0$. The $\mathbb{N}$-word $x = \gamma^{-1}(\delta^{-1}\epsilon)^{\infty}$ is contracting:



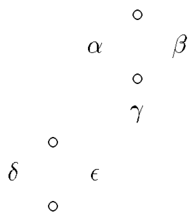
(3) An expanding $\mathbb{Z}$-word. The algebra A_3 is given by the quiver



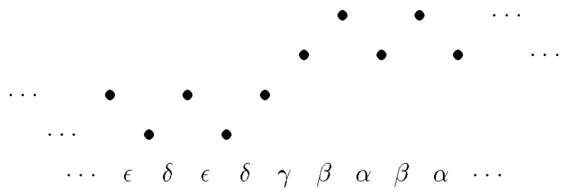
with the relations $\gamma\alpha = 0 = \gamma'\alpha'$. The $\mathbb{Z}$-word $x = z^{-1}y$, where $z = \gamma'(\beta'\alpha'^{-1})^{\infty}$ and $y = \gamma(\beta\alpha^{-1})^{\infty}$, is expanding:



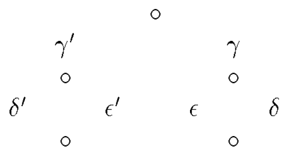
(4) A mixed $\mathbb{Z}$-word. The algebra A_4 is given by the quiver



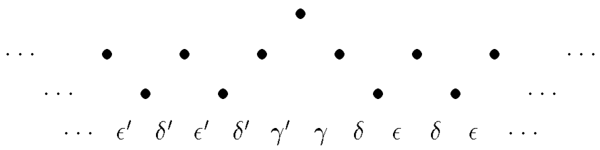
with the relations $\gamma\alpha = 0 = \epsilon\gamma$. The $\mathbb{Z}$-word $x = z^{-1}y$ with $z = (\delta^{-1}\epsilon)^{\infty}$ and $y = \gamma(\beta\alpha^{-1})^{\infty}$ is mixed:



(5) A contracting $\mathbb{Z}$-word. The algebra A_5 is given by the quiver



with the relations $\epsilon\gamma = 0 = \epsilon'\gamma'$. The $\mathbb{N}$-word $x = z^{-1}y$ with $z = \gamma'^{-1}(\delta'^{-1}\epsilon')^{\infty}$ and $y = \gamma^{-1}(\delta^{-1}\epsilon)^{\infty}$ is contracting:



Acknowledgement. The author is endebted to H. Krause for many stimulating remarks concerning the preparation of the paper.

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