Connected Hopf Algebras with Dixmier Basis and Infinite Primary Decomposition

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SUMMARY : *In this paper we study the Hopf algebra actions on commutative rings and modules that admit invariant primary decompositions.*

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Let H be a Hopf algebra over a field $\mathbb{K}$ and R a commutative $\mathbb{K}$-algebra with an H-module structure. Let $I \subseteq R$ be an ideal. Then I is called **invariant** if

$$\mathrm{H}(I) \subseteq I.$$

Assume that I has a (possibly infinite) primary decomposition

$$I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3 \cap \cdots.$$

In this paper we study when an invariant ideal with a primary decomposition has an **invariant primary decomposition**, i.e.,

$$I = \mathfrak{q}_1' \cap \mathfrak{q}_2' \cap \mathfrak{q}_3' \cap \cdots,$$

where the $\mathfrak{q}_i'$'s and $Rad\,(\mathfrak{q}_i')$'s are invariant ideals for all i.

In the special case where $\mathrm{H} = P$ is the mod-p-Steenrod algebra and R is an unstable noetherian $\mathbb{K}$-algebra over P the existence of invariant primary decompositions was established in [6]. This was extended to unstable noetherian $R \odot P$-modules in [4] (see also [2]). This was further generalized to *non*noetherian unstable $R \odot P$-modules (R still noetherian) in the sense that if an unstable module admits a *finite* primary decomposition then it admits an invariant (still finite) primary decomposition, see [5].

In [10] arbitrary pointed Hopf algebras are considered. It is shown that in the categories of commutative noetherian $\mathbb{K}$-algebras R and noetherian (H, R)-modules invariant primary decompositions exist.

Finally, [11] deals with pointed Hopf algebras over a field $\mathbb{K}$ of characteristic zero and *non*-commutative noetherian rings R over it. It is shown that the nilradical of R as well as all minimal primes are invariant.

In this paper we come back to the study of commutative rings R and modules M, but we drop any finiteness assumption. In particular, neither R nor M need to be noetherian. We assume that H is a Hopf algebra of Dixmier type. We determine when an invariant ideal (or a module) that admits a (possibly infinite) primary decomposition, admits an invariant primary decomposition. In Section 1 we define Hopf algebras of Dixmier type, and prove the existence of Dixmier bases in important cases, like, e.g., for the Steenrod algebra of reduced powers. In Section 2 we introduce the J_D-functor that turns arbitrary ideals (or modules) into invariant ones and show that the minimal prime ideals containing an invariant ideal are invariant, see Corollary 2.4. We proceed with the verification of several properties of J_D in Section 3. We prove that the minimal primary ideals belonging to minimal prime ideals over an invariant ideal are invariant, see Proposition 4.1. In Section 4 we obtain the existence of invariant primary decompositions if J_D commutes with taking radicals, see Theorem 4.5. This property is satisfied by, e.g., unstable actions of the Steenrod algebra as shown in Proposition 4.6. Finally in Section 5 we translate these results into the context of modules.

§1. Dixmier Bases for Hopf Algebras

In [11] it is shown that every connected Hopf algebra over a field of characteristic zero is a quotient of a Hopf algebra with Dixmier basis. In this section we extend this result to positive characteristic for several important cases.

We recall the definition of Dixmier basis, see [11].

DEFINITION: *Let* H *be a Hopf algebra over a field* $\mathbb{K}$*. Denote by* Δ *the comultiplication. We say that the subset* $D \subseteq$ H *is a* **Dixmier basis** *for* H*, if*

(1) *D is a $\mathbb{K}$-linear basis for* H.
(2) *D is well ordered by some ordering "<".*
(3) *There exists a multiplication*

$$D \times D \longrightarrow D,\ (d,t) \longmapsto d \odot t$$

such that

$$(⌘) \qquad \Delta(d \odot t) = \lambda\, d \otimes t + \sum_{d'<d} \alpha_{d',t''}\, d' \otimes t'' + \sum_{t'<t} \beta_{d'',t'}\, d'' \otimes t'$$

for some $\lambda \in \mathbb{K}^\times$ and $\alpha_{d',t''},\ \beta_{d'',t'} \in \mathbb{K}$.

We call the property (⌘) *the* **Dixmier Property**.

The following example is taken from [11], Example 4.

EXAMPLE 1.1 : Let H $= \mathbb{K}[t]$ the algebra of polynomial in one variable t over a field $\mathbb{K}$. The comultiplication is given by

$$\Delta(t) = t \otimes 1 + 1 \otimes t.$$

If the field $\mathbb{K}$ has characteristic zero, then we can choose

$$D = \{1, t, t^2, \ldots\}$$

as a Dixmier basis with multiplication

$$t^i \odot t^j = t^{i+j}.$$

The set D is ordered in the obvious way: $t^i < t^j$ if and only if $i < j$. If the characteristic of $\mathbb{K}$ is $p > 0$, then $\mathbb{K}[t]$ admits no Dixmier basis as we see next. Since D is a linear basis it must contain $\mu_i t^i + M_i$, $\mu_i \in \mathbb{K} \setminus 0$ and some $M_i \in \mathbb{K}[t]$, for all $i \in \mathbb{N}_0$. Thus

$$t^i \odot t^{p-i} = \sum_{k=1}^{n} \lambda_k t^{m_k}$$

for some $m_k \in \mathbb{N}_0$ and $\lambda_k \in \mathbb{K}$. Then

$$\Delta(t^i \odot t^{p-i}) = \Delta(\sum_{k=1}^{n} \lambda_k t^{m_k}) = \sum_{k=1}^{n} \lambda_k \sum_{j=0}^{m_k} \binom{m_k}{j} t^j \otimes t^{m_k - j}.$$

Since $t^i \otimes t^{p-i}$ must be a nontrivial summand in the sum on the right, we have that $m_k = p$ for certain k. However this gives

$$\Delta(t^i \odot t^{p-i}) = \Delta(t^p + \sum_{k=1, x_k \neq p}^{n} \lambda_k t^{m_k}) = 1 \otimes t^p + t^p \otimes 1 + \Delta(\sum_{k=1, m_k \neq p}^{n} \lambda_k t^{m_k}).$$

Thus $t^i \otimes t^{p-i}$ does not occur as a nontrivial summand in $\Delta(t^i \otimes t^{p-i})$, and hence in positive characteristic $\mathbb{K}[t]$ does not admit a Dixmier basis.

We can see this also in the following way: Consider the truncated polynomial algebra

$$R = \mathbb{K}1 + \mathbb{K}x + \mathbb{K}x^2 + \cdots + \mathbb{K}x^{p-1}$$

over a field of characteristic p. Then the Hopf algebra $\mathrm{H} = \mathbb{K}[t]$ acts on R via

$$t(x) = 1.$$

The nil radical of R is

$$Nil\,(R) = (x).$$

If H had a Dixmier basis then the nil radical of R would be invariant under the action of H, see Theorem 4.3 in [11]. However $1 \notin Nil\,(R)$, cf. Examples 3 and 4 in [11].

More generally we cite the following result.

THEOREM 1.2: *If* H *is a connected Hopf algebra over a field of characteristic zero, then* H *is a quotient of a Hopf algebra with Dixmier basis.*

PROOF: See Theorem 12 in [11]. ☯

We need a similar result for Hopf algebras over fields of positive characteristic. For this we start with the following construction which is taken from [8].

Let $\mathbb{K}$ be a field of any characteristic. Denote by

$$\mathrm{H}_\infty = \mathbb{K} < h_1,\ h_2,\ h_3,\ \ldots >$$

the free Hopf algebra on the h_i's over $\mathbb{K}$ with comultiplication given by

$$\Delta(h_k) = \sum_{i=0}^{k} h_i \otimes h_{k-i},$$

where $h_0 = 1$. Note that this is a cocommutative Hopf algebra. We want to show that the set D consisting of all monomials in the h_i's is a Dixmier basis for H_∞. Obviously D is a $\mathbb{K}$-linear basis for H. Next we need to define an order on D.

DEFINITION: *Let*

$$d = h_{i_1} \cdots h_{i_k} \in D$$

be a monomial. The **special degree** *of d is defined by*

$$\mathrm{spdeg}(d) = i_1 + \cdots + i_k.$$

We denote the **length** *of d by*

$$l(d) = k.$$

With the help of these two degrees associated to a monomial $d \in D$ we define a well order on D as follows.

DEFINITION: *Let d and d' be elements in D. We say that $d < d'$ if one of the following statements is true:*

(1) $\mathrm{spdeg}(d) < \mathrm{spdeg}(d')$ *or*
(2) $\mathrm{spdeg}(d) = \mathrm{spdeg}(d')$ *and* $l(d) < l(d')$ *or*
(3) $\mathrm{spdeg}(d) = \mathrm{spdeg}(d')$ *and* $l(d) = l(d')$ *and*[1] $d <_{\mathrm{lex}} d'$.

LEMMA 1.3 ([8]): *The set D of all monomials is well ordered by "<".*

[1] "$<_{\mathrm{lex}}$" denotes the lexicographic order.

PROOF: It is obvious that any two elements in D are comparable. To show that any nonempty subset has a least element, pick a chain

$$d_0 > d_1 > d_2 > \cdots$$

with $d_i \in D$ for all i. Since the special degree $\mathrm{spdeg}(d_0)$ is finite there are only finitely many d_i's in the chain of smaller special degree. Thus without loss of generality we can assume that

$$\mathrm{spdeg}(d_i) = \mathrm{spdeg}(d_j) \quad \forall\ i,\ j.$$

Similary, the length of d_0 is finite, and so without loss of generality we assume that

$$l(d_i) = l(d_j) \quad \forall\ i, j.$$

Thus

$$d_i >_{\mathrm{lex}} d_{i+1} \quad \forall\ i.$$

Since the lexicographic order turns the set of monomials into a well ordered set, we are done. ☯

Next we need to define a multiplication on the set D.

DEFINITION: *Let d and t be elements in D with $l(d) \le l(t)$. We assume without loss of generality that*

$$d = h_{i_1} \cdots h_{i_k}$$

and

$$t = h_{m_1} \cdots h_{m_p} h_{j_1} \cdots h_{j_k}.$$

We define a multiplication as follows

$$d \odot t = h_{m_1} \cdots h_{m_p} h_{i_1+j_1} \cdots h_{i_k+j_k}.$$

If $l(d) > l(t)$ then we define

$$d \odot t = t \odot d.$$

Note that $d \odot t \in D$.

PROPOSITION 1.4 ([8]): *With the preceding notation D is a Dixmier basis for* H_∞.

PROOF: By definition D is a $\mathbb{K}$-linear basis for H_∞. By Lemma 1.3 we know that "<" defines a well ordering on D. Thus we need to show that the Dixmier property (⌘) holds.

CASE $l(d) \le l(t)$: We find

$$\Delta(d \odot t) = \sum_{\alpha_1=0}^{m_1} \cdots \sum_{\alpha_p=0}^{m_p} \sum_{\alpha_{p+1}=0}^{i_1+j_1} \cdots \sum_{\alpha_{p+k}=0}^{i_k+j_k} h_{\beta_1} \cdots h_{\beta_{p+k}} \otimes h_{\alpha_1} \cdots h_{\alpha_{p+k}},$$

where $\beta_r = m_r - \alpha_r$ for $r \le p$ and $\beta_{p+r} = i_r + j_r - \alpha_r$. Let

$$a \otimes b = h_{\beta_1} \cdots h_{\beta_{p+k}} \otimes h_{\alpha_1} \cdots h_{\alpha_{p+k}}$$

be a summand of $\Delta(d \odot t)$. We note that

$$(*) \qquad \mathrm{spdeg}(d \odot t) = \mathrm{spdeg}(d) + \mathrm{spdeg}(t) = \mathrm{spdeg}(a) + \mathrm{spdeg}(b).$$

Furthermore, we observe

$$(\bigstar) \qquad l(d \odot t) = l(t) \ge l(b).$$

We need to show that $a \geq d$, $b \geq t$ implies that $a = d$ and $b = t$.

Let $a \geq d$ and $b \geq t$. Thus by $(*)$ and $(\bigstar)$ we obtain that $\operatorname{spdeg}(a) \geq \operatorname{spdeg}(d)$, $\operatorname{spdeg}(b) \geq \operatorname{spdeg}(t)$, and $l(b) = l(t)$. Moreover, since $b \geq t$ we obtain that

$$b = h_{\alpha_1} \cdots h_{\alpha_{p+k}} \geq_{\text{lex}} t = h_{m_1} \cdots h_{m_p} h_{j_1} \cdots h_{j_k}.$$

Hence

$$\beta_r \geq m_r \quad \forall\ r = 1, \ldots, p.$$

Therefore, $\beta_r = m_r$ for $r = 1, \ldots, p$, and thus $\alpha_r = 0$ for $r = 1, \ldots, p$. Therefore d and a have the same length.

We proceed by proof by contradiction. To this end assume that $b > t$. Then there exists an index x such that

$$\beta_{p+x} > j_x \quad \text{and} \quad \beta_y = j_y$$

for $y = 1, \ldots, x-1$. Hence $\alpha_y = i_y$ and $\alpha_x < i_x$, and thus $a < d$. This is a contradicts since $a \geq d$. Therefore $t = b$, and hence $d = a$.

Finally observe that the case $t = b$ occur exactly once in the above sum.

CASE $l(d) > l(t)$: This follows immediately from the first case, because H_∞ is cocommutative.☯

We summarize these results in the following proposition.

PROPOSITION 1.5 ([8]): *Let H_∞ be the free $\mathbb{K}$-algebra on countably many generators $h_0 = 1$, h_1, h_2, ... with an Hopf algebra structure given by*

$$\Delta(h_k) = \sum_{i=0}^{k} h_i \otimes h_{k-i}.$$

Let D be the linear basis containing all monomials in the h_i's. Then D is a Dixmier basis for H_∞.

PROOF: ☯

DEFINITION: *We call an Hopf algebra H that is a quotient of a Hopf algebra H_D with Dixmier basis a* **Hopf algebra of Dixmier type**.

EXAMPLE 1.6 : By the preceding Proposition 1.5 any Hopf algebra H that is a quotient of H_∞ is a Hopf algebra of Dixmier type.

PROPOSITION 1.7 (Dixmier Basis of P): *The* mod *p-Steenrod algebra of reduced powers is a Hopf algebra of Dixmier type.*

PROOF: Denote by P the Steenrod algebra of reduced powers over a finite field $\mathbb{F}_q$ of order q. It is the free associative $\mathbb{F}_q$-algebra generated by the reduced powers $\mathcal{P}^0 = \mathrm{id}$, $\mathcal{P}^1$, $\mathcal{P}^2$, ... modulo the Adem-Wu relations

$$\mathcal{P}^i \mathcal{P}^j = \sum_{k=0}^{[i/q]} (1)^{i+qk} \binom{(q-1)(j-k)-1}{i-qk} \mathcal{P}^{i+j-k} \mathcal{P}^k, \quad \text{whenever } i,\ j > 0 \text{ and } i < qj.$$

The Steenrod algebra has an $\mathbb{F}_q$-linear basis D consisting of admissible monomials

$$\mathcal{P}^I \stackrel{\text{def}}{=} \mathcal{P}^{i_1} \cdots \mathcal{P}^{i_k} \quad \text{with} \quad i_s \geq q\, i_{s+1} \ \ \forall\ s = 1, \ldots, k,$$

see, e.g., Proposition 2.1 in [9]. We define an order on D as we did in the case of H_∞ by replacing the special degree by the moment $m(\mathcal{P}^I)$:

$$m(\mathcal{P}^I) = \sum_{s=1}^{k} s\, i_s.$$

We define a multiplication $\odot$ on D in the following way:

$$\mathcal{P}^I \odot \mathcal{P}^J = \mathcal{P}^{i_1+j_1} \cdots \mathcal{P}^{i_k+j_k}\mathcal{P}^{j_{k+1}} \cdots \mathcal{P}^{j_l},$$

where $\mathcal{P}^J = \mathcal{P}^{j_1} \cdots \mathcal{P}^{j_l}$ and $l \geq k$. If $k \geq l$ we set

$$\mathcal{P}^I \odot \mathcal{P}^J = \mathcal{P}^J \odot \mathcal{P}^I.$$

We find

$$\begin{aligned}
\Delta(\mathcal{P}^I \odot \mathcal{P}^J) &= \Delta(\mathcal{P}^{i_1+j_1} \cdots \mathcal{P}^{i_k+j_k}\mathcal{P}^{j_{k+1}} \cdots \mathcal{P}^{j_l}) \\
&= \sum_{\alpha_1=0}^{i_1+j_1} \cdots \sum_{\alpha_l=0}^{j_l} \mathcal{P}^{\alpha_1} \cdots \mathcal{P}^{\alpha_l} \otimes \mathcal{P}^{i_1+j_1-\alpha_1} \cdots \mathcal{P}^{j_l-\alpha_l} \\
&= \mathcal{P}^I \otimes \mathcal{P}^J + \sum_{\alpha_1=0}^{i_1+j_1} \cdots \sum_{\alpha_k=0}^{i_k+j_k} \sum_{\alpha_{k+1}=1}^{j_{k+1}} \cdots \sum_{\alpha_l=1}^{j_l} \mathcal{P}^{\alpha_1} \cdots \mathcal{P}^{\alpha_l} \otimes \mathcal{P}^{i_1+j_1-\alpha_1} \cdots \mathcal{P}^{j_l-\alpha_l},
\end{aligned}$$

where $(\alpha_1, \ldots, \alpha_l) \neq (i_1, \ldots, i_k, 0, \ldots, 0)$. If the first component of one of the summands has moment

$$m(\mathcal{P}^{\alpha_1} \cdots \mathcal{P}^{\alpha_l}) = \sum_{s=1}^{l} s\alpha_s \leq \sum_{s=1}^{k} s\, i_s = m(\mathcal{P}^I)$$

then it can be written as a sum of admissible monomials of smaller moment, see, e.g., Proposition 2.1 in [9]. If its moment is larger than the moment of $\mathcal{P}^I$, then the moment of the second component

$$m(\mathcal{P}^{i_1+j_1-\alpha_1} \cdots \mathcal{P}^{j_l-\alpha_l}) = \sum_{s=1}^{l} s(i_s + j_s - \alpha_s) \leq \sum_{s=1}^{l} s j_s = m(\mathcal{P}^J).$$

Thus in this case the second component can be written as a sum of admissible monomials of smaller moment. Therefore, our product on D satisfies the Dixmier Property. ☯

§2. Primary Decomposition: Reduction Arguments and Prime Ideals

Let H be a Hopf algebra of Dixmier type. Let R be a commutative $(\mathbb{K}, \mathrm{H})$-module algebra. An ideal $I \subseteq R$ is called **invariant** if

$$\mathrm{H}(I) \subseteq I.$$

Assume that I has a (possibly infinite) primary decomposition

$$I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3 \cap \ldots..$$

Our goal is to determine when I has an invariant primary decomposition

$$I = \mathfrak{q}'_1 \cap \mathfrak{q}'_2 \cap \mathfrak{q}'_3 \cap \ldots.,$$

i.e., all the primary components $\mathfrak{q}'_i$ as well as their prime radicals $\mathfrak{p}'_i$ are invariant.

We start with two reduction arguments.

First, we can assume without loss of generality that the Hopf algebra H is a Hopf algebra with Dixmier basis, H_D. This follows from the fact that the $(\mathbb{K}, \mathrm{H})$-module algebra R can be considered as a $(\mathbb{K}, \mathrm{H}_D)$-module algebra via the canonical projection

$$\varphi : \mathrm{H}_D \longrightarrow \mathrm{H}.$$

Before we come to the second reduction argument we need a the functor J_D that turns arbitrary ideals into invariant ideals. It is defined as follows, cf. Chapter 9 in [7] and Section 4 in [11].

DEFINITION: *Let $I \subseteq R$ be an ideal. Denote by*

$$J_\infty(I) \subseteq I$$

the maximal invariant subideal of I. We define

$$J_D(I) = \{r \in I \mid d(r) \in I \ \forall \ d \in D\},$$

where D forms a Dixmier basis for H*.*

PROPOSITION 2.1: *With the above notation*

$$J_D(I) = J_\infty(I),$$

for any ideal $I \subseteq R$.

PROOF: Let $r \in J_\infty(I)$. Then

$$d(r) \in J_\infty(I) \subseteq I$$

for all $d \in D$. Hence $r \in J_D(I)$.

Conversely, since D is a linear basis for H, the set $J_D(I)$ is invariant. Thus

$$J_D(I) \subseteq J_\infty(I)$$

by maximality of $J_\infty(I)$. ☯

REMARK: Note that the preceding result means in particular that $J_D(I)$ is an ideal.

The following result has been proven in Lemma 1.1 in [5] in the context of modules over the Steenrod algebra.

LEMMA 2.2: *The functor J_D commutes with arbitrary intersections:*

$$J_D(\bigcap_i I_i) = \bigcap_i J_D(I_i)$$

PROOF: By definition

$$J_D(\bigcap_i I_i) \subseteq \bigcap_i I_i$$

is the largest invariant subideal. Since $\bigcap_i J_D(I_i) \subseteq \bigcap_i I_i$ is also invariant we find that

$$\bigcap_i J_D(I_i) \subseteq J_D(\bigcap_i I_i).$$

To prove the reverse inclusion let $r \in J_D(\bigcap_i I_i)$. Then

$$d(r) \in I_i$$

for all i and $d \in D$. Thus
$$r \in \bigcap_i J_D(I_i)$$
as claimed. ☯

This result leads to the second reduction argument: If
$$I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3 \cap \cdots$$
is a primary decomposition of an invariant ideal $I \subseteq R$, then
$$I = J_D(\mathfrak{q}_1) \cap J_D(\mathfrak{q}_2) \cap J_D(\mathfrak{q}_3) \cap \cdots.$$
Thus it is enough to show that $J_D(\mathfrak{q})$ is has an invariant primary decomposition for every primary ideal $\mathfrak{q}$.

We prove the case where $\mathfrak{q} = \mathfrak{p}$ is a prime ideal and some technical corollaries, in the remainder of the section postponing the general case to the next section. In particular, we show that the prime ideals $\mathfrak{p} \supseteq I$ minimal over an invariant ideal I are invariant.

The following three results were proven in [11] in the case of Hopf algebras over a field of characteristic zero, and in [6], resp. [5] in the case of unstable actions of the Steenrod algebra.

PROPOSITION 2.3: *Let* H *be a Hopf algebra of Dixmier type, and let R be a commutative $\mathbb{K}$-algebra over* H. *Let $\mathfrak{p} \subseteq R$ be a prime ideal. Then $J_D(\mathfrak{p}) \subseteq R$ is a prime ideal also.*

PROOF: Let $r,\ s \in R \setminus J_D(\mathfrak{p})$. Choose minimal elements $d, t \in D$ such that
$$d(r) \notin \mathfrak{p} \qquad t(s) \notin \mathfrak{p}.$$
Then, for some $\lambda \in \mathbb{K}^\times$ and $\alpha_{d',t''},\ \beta_{d'',t'} \in \mathbb{K}$ we have
$$(d \odot t)(rs) = \lambda\, d(r)t(s) + \sum_{d'<d} \alpha_{d',t''}\, d'(r)t''(s) + \sum_{t'<t} \beta_{d'',t'}\, d''(r)t'(s).$$
By minimality of d and t the two sums on the right hand side of this equation are in $\mathfrak{p}$. Since $\mathfrak{p}$ is prime, the first summand $d(r)t(s) \notin \mathfrak{p}$. Therefore
$$(d \odot t)(rs) \notin \mathfrak{p}$$
and hence $rs \notin J_D(\mathfrak{p})$ as desired. ☯

COROLLARY 2.4: *Let $I \subseteq R$ be an invariant ideal. Then all minimal prime ideals $I \subseteq \mathfrak{p} \subseteq R$ containing I are invariant.*

PROOF: Consider the canonical projection
$$\varphi : R \longrightarrow R/I.$$
The minimal prime ideals $I \subseteq \mathfrak{p} \subseteq R$ project down to the minimal prime ideals $(0) \subseteq \overline{\mathfrak{p}} \subseteq R/I$. They are invariant by the preceding Proposition 2.3. Thus the ideals $\mathfrak{p}$ are also invariant. ☯

COROLLARY 2.5: *If $I \subseteq R$ is an invariant ideal, then so is its radical.*

PROOF: This is true, because the radical of any ideal is the intersection of the prime ideals containing it. ☯

PROPOSITION 2.6: *Let $I \subseteq R$ be a radical ideal. Then*
$$J_D(I) = Rad\,(J_D(I)).$$

PROOF: The inclusion "$\subset$" is obvious. In order to show the reverse inclusion take an element

$$a \in Rad\,(J_D(I)).$$

Then there exists some power $n \in \mathbb{N}$ such that

$$a^n \in J_D(I).$$

Hence $d(a^n) \in J_D(I)$ for all elements in the Dixmier basis $d \in D$. Assume that $a \notin J_D(I)$. Then there exists a minimal element $d \in D$ such that

$$d(a) \notin I.$$

We observe that

$$d^{\odot n}(a^n) = \lambda\, d(a)^n + \sum \alpha_{i_1,\dots,i_n} d_{i_1}(a) \cdots d_{i_n}(a),$$

for some $i_1, \dots, i_n \in \mathbb{N}_0$, $\lambda \in \mathbb{K}^\times$, and $\alpha_{i_1,\dots,i_n} \in \mathbb{K}$. Note that for every summand of the sum on the right we have that

$$d_{i_j} < d$$

for at least one index i_j. Thus

$$\sum d_{i_1}(a) \cdots d_{i_n}(a) \in I$$

by minimality of d. Since $a^n \in J_D(I)$ we have that

$$d^{\odot n}(a^n) \in J_D(I) \subseteq I.$$

Therefore $d(a)^n \in I$ and thus $d(a) \in I$, because I is radical. This contradicts our assumption, and concludes the proof. ☯

REMARK: It follows from the preceding result that

$$Rad\,(J_D(I)) \subseteq J_D(Rad\,(I))$$

for any ideal $I \subseteq R$. The reverse inclusion is not true in general. We illustrate this with the next example.

EXAMPLE 2.7 ([8]): Let $\mathbb{K}$ be a field of characteristic zero. Let $R = \mathbb{K}[x_1, x_2, \cdots]$ the polynomial ring in infinitely (but countably) many variables over $\mathbb{K}$. Let H be the Hopf algebra over $\mathbb{K}$ generated by derivations $t_1, t_2, \cdots$ acting on R via

$$t_i(x_j) = \begin{cases} x_i & \text{for } j = 1 \\ 0 & \text{for } j > 1. \end{cases}$$

We note that H has a Dixmier basis D consisting of $t_0 = 1_{\mathrm{H}}, t_1, t_2, \cdots$ with multiplication given by

$$t_i \odot t_j = t_{i+j}$$

and order $t_i \le t_j$ if and only if $i \le j$. Let $I = (x_1, x_2^2, x_3^3, \cdots) \subseteq R$. Then

$$J_D(I) = (x_2^2, x_3^3, \cdots)$$

so that

$$Rad\,(J_D(I)) = (x_2, x_3, \cdots).$$

On the other hand

$$J_D(Rad\,(I)) = J_D(x_1, x_2, x_3, \cdots) = (x_1, x_2, x_3, \cdots).$$

§3. Primary Decomposition: More Technical Results

Let $I \subseteq R$ be an ideal. Let $P \subseteq R$ be a prime ideal. Then

$$S_P(I) = \mathrm{Ker}(R \longrightarrow R_P)$$

denotes the saturation of I with respect to P.

PROPOSITION 3.1: *If $I \subseteq R$ is an invariant ideal then so is $S_P(I)$ for any prime ideal $P \subseteq R$.*

PROOF: Let $r \in S_P(I)$. Then there exists an element $s \in R \setminus P$ such that $rs \in I$. Assume that $d'(r) \in S_P(I)$ for all $d' < d$. Then

$$(\bigstar) \qquad d(r)s = \lambda\left(d(rs) - \sum_{d'<d} \alpha_{d',d''}\, d'(r) d''(s)\right)$$

where $\lambda \in \mathbb{K}^{\times}$ and $\alpha_{d',d''} \in \mathbb{K}$. By assumption we have that

$$d'(r)s'_{d'} \in I$$

for some $s'_{d'} \in R \setminus P$. Set $S = \prod s'_{d'} \in R \setminus P$ where the product runs over all d' such that $\alpha_{d' t} \neq 0$. Then

$$d(r)sS = \lambda\left(d(rs)S - \sum_{d'<d} \alpha_{d',d''}(d'(r)S)d''(s)\right) \in I$$

and thus $d(r) \in S_P(I)$. ☯

Let $I \subseteq R$ be an invariant ideal. Denote by $\mathcal{D}(R, I)$ the set of all prime ideals P in R containing I with the following property: There exists an $r \in R$ such that P is a minimal prime containing $(I : r)$.

LEMMA 3.2: *Let $I \subseteq R$ be an ideal. Then the maximal ideals in $\mathcal{D}(R, I)$ have the form $(I : r)$ for some $r \in R$. Furthermore, if I is invariant, then so are the maximal ideals in $\mathcal{D}(R, I)$.*

PROOF: Consider the set of colon ideals $(I : r)$ for some $r \in R \setminus I$. Let $(I : r)$ be maximal in this set. Assume we have

$$st \in (I : r) \quad \text{and} \quad s \notin (I : r).$$

Then

$$t \in (I : rs) = (I : r)$$

by maximality of $(I : r)$. Thus the maximal colon ideals are prime ideals. Hence the maximal elements in $\mathcal{D}(R, I)$ are colon ideals.

We come to the second statement: Let $s \in \mathfrak{p} = (I : r)$. Then

$$d(s)r = \lambda\left(d(sr) - \sum \alpha_{d',d''}\, d'(s)d''(r)\right),$$

where $\lambda \in \mathbb{K}^{\times}$ and $\alpha_{d',d''} \in \mathbb{K}$. By induction the sum on the right hand side is in $\mathfrak{p}$. Since I is invariant, we have $d(sr) \in I \subseteq \mathfrak{p}$. Therefore

$$d(s) \in (I : r) = \mathfrak{p}.$$

☯

LEMMA 3.3: *The set of minimal prime ideals in $\mathcal{D}(R, I)$ is exactly the set of isolated prime ideals of I. Moreover, they are invariant if I is invariant.*

PROOF: The second statement follows from the first by Corollary 2.4.

Let $I \subseteq P_i$ be an isolated prime ideal of I. Then for any $r \notin P_i$ we have

$$I \subseteq (I : r) \subseteq (P_i : r) = P_i.$$

Therefore $P_i \in \mathcal{D}(R, I)$ is minimal. Conversely, if $P \in \mathcal{D}(R, I)$ is a minimal prime ideal, then there exists an element $s \in R$ such that P is minimal over $(I : s)$. If P is not minimal over I then it contains an isolated prime ideal $P_i \subseteq P$. However, then $P \in \mathcal{D}(R, I)$ is not minimal. ☯

§4. Primary Decomposition: Main Results

PROPOSITION 4.1: *Let $I \subseteq R$ be an invariant ideal such that the set $\mathcal{D}(R, I)$ consists of minimal prime ideals. Then*

$$I = \bigcap_{P \in \mathcal{D}(R,I), P \text{ min}} S_P(I)$$

is an invariant primary decomposition of I.

PROOF: By Exercise 10 (iv), Page 55 in [1] we have

$$I = \bigcap_{P \in \mathcal{D}(R,I)} S_P(I), \tag{$*$}$$

We write this intersection as

$$I = \bigcap_{P \in \mathcal{D}(R,I), P \text{ min}} S_P(I) \cap \bigcap_{Q \in \mathcal{D}(R,I), Q \text{ emb}} S_Q(I),$$

where the first intersection runs over all minimal prime ideals $P \in \mathcal{D}(R, I)$, and the second over the embedded ones. Since the minimal prime ideals in $\mathcal{D}(R, I)$ are the minimal prime ideals of R/I, we obtain that $P \subseteq R$ is invariant by Corollary 2.4. By Exercise 11, Page 56 in [1] the ideals $S_P(I)$ are the minimal P-primary ideals in R containing I. Therefore,

$$\bigcap_{P \in \mathcal{D}(R,I), P \text{ min}} S_P(I)$$

is an intersection of invariant primary ideals with invariant radicals.☯

We append an immediate corollary.

COROLLARY 4.2: *If I is an invariant ideal, then the minimal primary ideals $\mathfrak{q} \supseteq I$ belonging to minimal prime ideals over I are invariant.*☯

The following lemma was proven in [6] in the context of unstable actions of the Steenrod algebra.

LEMMA 4.3: *Let $I \subseteq R$ be an invariant ideal. Let $r \in R$ such that $(I : r) \subseteq R$ is an invariant ideal. Then*

$$(I : r) \subseteq (I : d(r))$$

for all $d \in D$.

PROOF: Let $s \in (I : r)$. Thus

$$s\,d_0(r) = s\,r \in I.$$

We consider the case $d > d_0$. We find

$$s\,d(r) = \lambda\, d(s\,r) - \sum_{d'<d} \alpha_{d'',d'}\, d''(s) d'(r).$$

Next, $d(s\,r) \in I$ because I is invariant by assumption. Since $(I : r)$ is invariant we have that

$$d''(s) \in (I : r) \subseteq (I : d'(r)),$$

where the inclusion is true by induction. Thus $s\,d(r) \in I$ as desired.☯

THEOREM 4.4: *Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. If J_D commutes with taking radicals. then $J_d(\mathfrak{q})$ is a $J_D(\mathfrak{p})$-primary ideal.*

PROOF: We assume that $Rad\,(J_D(\mathfrak{q})) = J_D(\mathfrak{p})$. By the preceding result a maximal element P in $\mathcal{D}(R, J_D(\mathfrak{q}))$ is an invariant prime ideal of the form

$$P = (J_D(\mathfrak{q}) : r) \subseteq (\mathfrak{q} : r) \subseteq \mathfrak{p}.$$

Thus $J_D(\mathfrak{p}) \supset P$. Since

$$J_D(\mathfrak{p}) = Rad\,(J_D(\mathfrak{q})) = \bigcap_{P \in \mathcal{D}(R, J_D(\mathfrak{q}))} P \subseteq P \subseteq J_D(\mathfrak{p})$$

we obtain equality and therefore $\mathcal{D}(R, J_D(\mathfrak{q}))$ consists of one element. Thus by Proposition 4.1 $J_D(\mathfrak{q})$ is $J_D(\mathfrak{p})$-primary.☯

THEOREM 4.5: *Let* H *be a Hopf algebra of Dixmier type over a field $\mathbb{K}$, let R be a commutative $\mathbb{K}$-algebra with an* H*-module structure. Let $I \subseteq R$ be an invariant ideal with a primary decomposition*

$$I = \bigcap \mathfrak{q}.$$

If the functor J_D commutes with taking radicals then I has an invariant primary decomposition.

PROOF: If J_D commutes with taking radicals then we obtain an invariant primary decomposition

$$I = \bigcap J_D(\mathfrak{q})$$

by Theorem 4.4.☯

REMARK: We can refine the preceding result by applying Proposition 4.1: If $\mathcal{D}(R, J_d(\mathfrak{q}))$ consists of isolated prime ideal for every primary ideal $\mathfrak{q} \subseteq R$, then every invariant ideal I with a primary decomposition has an invariant primary decomposition.

REMARK: We note that the property of J_D commuting with taking radicals seems not to be a property of the Hopf algebra but rather of its action.

PROPOSITION 4.6 (Steenrod Algebra): *Let R be a graded connected commutative algebra over a finite field $\mathbb{F}$. Let R be an unstable algebra over the Steenrod algebra. Then every ideal invariant under the Steenrod algebra action which has a primary decomposition admits an invariant primary decomposition*

PROOF: By the preceding result we need to show that J_D commutes with taking radicals. By construction we have that $Rad\,(J_D(I)) \subseteq J_D(Rad\,(I))$.

For the reverse inclusion, take an element $r \in J_D(Rad\,(I))$. Then

$$\mathcal{P}^i(r) \in Rad\,(I)$$

for all $i \in \mathbb{N}_0$, where the $\mathcal{P}^i$'s are the reduced powers. Thus for every i there exists a $k_i \in \mathbb{N}_0$ such that

$$(\mathcal{P}^i(r))^{k_i} \in I.$$

Since the action is unstable $\mathcal{P}^i(r) = 0$ for all $i > \deg(r)$. Let

$$p^s \geq \max\{k_0, \dots, k_{\deg(r)}\}.$$

Then

$$\mathcal{P}^j(r^{p^s}) = \begin{cases} \mathcal{P}^{j/p^s}(r)^{p^s} \in I & \text{if } p^s \mid j \\ 0 \in I & \text{otherwise.} \end{cases}$$

Thus $r^{p^s} \in J_D(I)$ and hence $r \in Rad\,(J_D(I))$ as claimed.☯

REMARK: Note that the preceding proof shows that J_D commutes with taking radicals for unstable actions of the Steenrod algebra, i.e., $J_D(\mathfrak{q})$ is $J_D(\mathfrak{p})$-primary whenever $\mathfrak{q}$ is $\mathfrak{p}$-primary.

§5. Primary Decomposition of Modules

In this section we translate the preceding results to R-modules M, where R as well as M admit an action of an Hopf algebra H of Dixmier type, and the two actions are compatible.

Let $N \subseteq M$ be an R-submodule of M. Assume that N admits a (possibly infinite) primary decomposition

$$N = Q_1 \cap Q_2 \cap Q_3 \cap \dots.$$

We define the functor J_D on the category of modules exactly as we did for ideals, see Section 2. By Lemma 2.2 we obtain

$$(\star) \qquad J_D(N) = J_D(Q_1) \cap J_D(Q_2) \cap J_D(Q_3) \cap \dots.$$

By definition $J_D(N) \subseteq M$ is an invariant H-module. We claim that the $J_D(Q_i)$'s admit invariant primary decompositions.

Since $Q_i \subseteq M$ is a primary module, the ideal

$$\mathfrak{q}_i = (Q_i : M) \subseteq R$$

is primary. Assume that

$$J_D(\mathfrak{q}_i) = J_D(Q_i : M)$$

admits an invariant primary decomposition

$$J_D(\mathfrak{q}_i) = \cap_j \mathfrak{q}_{ij}.$$

We note that

$$J_D(Q_i : M) = (J_D(Q_i) : M)$$

by Lemma 1.4. in [4].[2]

[2] Since this reference deals with the special case of unstable modules over the Steenrod algebra we add the proof:

We have that

$$J_D(Q_i) = \cap_j \mathfrak{q}_{ij} M$$

by definition of $\mathfrak{q}_{ij}$. By construction, the submodules $\mathfrak{q}_{ij} M \subseteq M$ are primary. Since $\mathfrak{q}_{ij} \subseteq R$ is an invariant ideal, we have that $\mathfrak{q}_{ij} M \subseteq M$ is an invariant submodule. Thus we have proven the following result:

THEOREM 5.1: *Let* H *be a Hopf algebra of Dixmier type over a field* $\mathbb{K}$*, let* R *be a commutative* $\mathbb{K}$*-algebra and* M *be an* R*-module. Assume that* R *admits an action of* H*, and* M *is an* (H, R)*-module. Let* $N \subseteq M$ *be an* (H, R)*-submodule of* M*. Assume that* N *admits a (possibly infinite) primary decomposition*

$$N = Q_1 \cap Q_2 \cap Q_3 \cap \dots.$$

Then N *admits an invariant primary decomposition if* $J_D(Q_i : M)$ *does.*

PROOF: ☯

EXAMPLE 5.2 : Let P be the mod-p-Steenrod algebra and M an unstable $P \odot R$-module. If N is an invariant submodule with a primary decomposition, then N has an invariant primary decomposition with invariant associated prime ideals.

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Since $J_D(Q) \subseteq Q$ we have that

$$(J_D(Q) : M) \subseteq (Q : M).$$

Next we show that the ideal $(J_D(Q) : M) \subseteq R$ is invariant. To this end let $r \in (J_D(Q) : M)$. Then

$$d(rM) \subseteq J_D(Q) \quad \forall \ d \in D.$$

We obtain by the Dixmier property

$$d(r)M = \lambda\big((d \odot 1)(rM) - \sum_{d'<d} \alpha_{d',d''} d'(r) d''(M)\big),$$

for $\lambda \in \mathbb{K}^{\times}$ and $\alpha_{d',d''} \in \mathbb{K}$. By induction we have that the right hand side of this equation is in $J_D(Q)$, hence so is the left hand side, i.e.,

$$d(r) \in (J_D(Q) : M) \quad \forall \ d \in D.$$

Thus it follows that

$$(J_D(Q) : M) \subseteq J_D(Q : M).$$

To show the reverse inclusion, take an element $r \in J_D(Q : M)$. Then by the Dixmier property

$$d(rM) = \lambda\, d(r)M + \sum_{d'<d} \alpha_{d',d''} d'(r) d''(M),$$

References

[1] M. F. Atiyah and I.G. Macdonald: *Introduction to Commutative Algebra*, Addison-Wesley Publishing Company, Reading MA 1969.

[2] Dagmar M. Meyer and Larry Smith: The Lasker-Noether Theorem for Unstable Modules over the Steenrod Algebra, *Communications in Algebra* 31 (2003), 5841-5845.

[3] Mara D. Neusel: Integral Extensions of Unstable Algebras over the Steenrod Algebra, *Forum Mathematicum* 12 (2000), 155-166.

[4] Mara D. Neusel: The Lasker-Noether Theorem in the Category $U(H^*)$, *Journal of Pure and Applied Algebra* 163 (2003), 221-233.

[5] Mara D. Neusel: The Existence of Thom Classes, *Journal of Pure and Applied Algebra* 191 (2004), 265-283.

[6] Mara D. Neusel and Larry Smith: The Lasker–Noether Theorem for $\mathcal{P}^*$–Invariant Ideals, *Forum Mathematicum* 10 (1998), 1–18.

[7] Mara D. Neusel and Larry Smith: *Invariant Theory of Finite Groups*, Mathematical Surveys and Monographs 94, AMS, Providence RI 2002.

[8] Mara D. Neusel and Piotr Wiśniewski: *private communication*, November 2004 - September 2005.

[9] N. E. Steenrod: *Cohomology Operations*, written and revised by D. B. A. Epstein, Annals of Mathematical Studies 50, Princeton University Press, Princeton NJ 1962.

[10] Andrzej Tyc and Piotr Wiśniewski: The Lasker-Noether Theorem for Commutative and Noetherian Modules Algebras over a pointed Hopf Algebra, *Journal of Algebra* 267 (2003), 58-95.

[11] Piotr Wiśniewski: Coalgebras with Dixmier Basis and Invariance of Minimal Primes of H-Module Algebras over Connected Hopf Algebras over a Field of Characteristic 0, preprint, Torun 2004.

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for $\lambda \in \mathbb{K}^\times$ and $\alpha_{d',d''} \in \mathbb{K}$. By induction we can assume that the right hand side is in Q, thus so is the left hand side, and we are done.