AG-Invariantentheorie

The Lasker-Noether Theorem in the Category $\mathcal{U}(\mathrm{H}^*)$

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Journal of Pure and Applied Algebra
– to appear –

SUMMARY: *We prove the Lasker-Noether Theorem in the category $\mathcal{U}(\mathrm{H}^*)$ of unstable $\mathrm{H}^* \odot \mathcal{P}^*$-modules. Along the way, we generalize Lam's $\mathcal{J}$-functor to the context of modules.*

AMS CODE: 55S10 Steenrod Algebra, 13A50 Invariant Theory, 13XX Commutative Rings and Algebras, 55XX Algebraic Topology
KEYWORDS: Lasker-Noether Theorem, Unstable Modules, Steenrod Algebra, Dickson Algebra, Polynomial Invariants of Finite Groups

Typeset by $\mathcal{LS}$TEX

Let $\mathbb{F}$ be a Galois field of characteristic p with q elements. Consider a faithful representation of degree n

$$\rho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$$

of a finite group G. Then G acts via ρ on the vector space $V = \mathbb{F}^n$, and hence on the ring of polynomial functions

$$\mathbb{F}[V] = \mathbb{F}[x_1, \dots, x_n]$$

in n variables via

$$gf(v) = f(\rho(g)^{-1}v) \quad \forall f \in \mathbb{F}[V],\ v \in V,\ g \in G.$$

The ring of polynomials invariant under this action is denoted by $\mathbb{F}[V]^G$. By a classical theorem of Emmy Noether, any ring of invariants $\mathbb{F}[V]^G$ is Noetherian, [11]. Since the groundfield $\mathbb{F}$ is finite, the full general linear group $\mathrm{GL}(n,\ \mathbb{F})$ is finite, and is moreover present in every ring of invariants $\mathbb{F}[V]^G$. In 1911 Leonard E. Dickson, [3], proved that the ring of invariants of $\mathrm{GL}(n,\ \mathbb{F})$

$$\mathbb{F}[V]^{\mathrm{GL}(n,\ \mathbb{F})} = \mathbb{F}[\mathbf{d}_{n,0}, \dots, \mathbf{d}_{n,n-1}] = \mathcal{D}^*(n),$$

is a polynomial ring in $\mathbf{d}_{n,0}, \dots, \mathbf{d}_{n,n-1}$, which are now called the **Dickson classes**. The algebra $\mathcal{D}^*(n)$ is called the **Dickson algebra**.

As described in Chapters 10 and 11 in [12], or in the introduction of [8], $\mathbb{F}[V]^G$ inherits from $\mathbb{F}[V]$ an unstable action of the Steenrod algebra $\mathcal{P}^*$, i.e., $\mathbb{F}[V]^G$ is an objects in $\mathcal{K}_{fg}$, the category of finitely generated unstable (graded connected commutative $\mathbb{F}$-) algebras over $\mathcal{P}^*$. By the Imbedding Theorem 8.1.5 in [8] *every* object H^* in $\mathcal{K}_{fg}$ contains a fractal[1] of the Dickson algebra such that

$$\mathcal{D}^*(n)^{q^t} \hookrightarrow \mathrm{H}^*$$

is an integral extension, where t can be choosen to be zero, if H^* is $\mathcal{P}^*$-inseparably closed. Therefore every finitely generated unstable $\mathbb{F}$-algebra over the Steenrod algebra can be considered as a module over $\mathcal{D}^*(n)^{q^t}$, i.e., as a finitely generated module over a Noetherian ring. In classical theory every such module has a primary decomposition. In this paper we prove the $\mathcal{P}^*$-invariant version of this statement in its most general form: Let H^* be an unstable Noetherian algebra over the

[1] Recall that a **fractal** of the Dickson algebra is

$$\mathcal{D}^*(n)^{q^t} = \mathbb{F}[\mathbf{d}_{n,0}^{q^t}, \dots, \mathbf{d}_{n,n-1}^{q^t}] = \mathbb{F}[x_1^{q^t}, \dots, x_n^{q^t}]^{\mathrm{GL}(n,\mathbb{F})}.$$

Steenrod algebra. Then every finitely generated unstable H^*-module M has a primary decomposition

$$M = Q_1 \cap \ldots \cap Q_m ,$$

consisting of unstable primary components $Q_1, \ldots, Q_m$. Moreover, the associated prime ideals

$$\mathcal{R}ad(Q_i : M) \subseteq \mathrm{H}^* \quad \forall\, i = 1, \ldots, m ,$$

are $\mathcal{P}^*$-invariant ideals, i.e., ideals that are closed under the action of the Steenrod algebra.

This solves a long open problem, see [10] and Section 6 in [13], that has some surprising immediate applications, see [9].

§1. Lam's $\mathcal{J}$ for Modules

Let H^* be an unstable algebra over the Steenrod algebra. Note that, in this first section, we do not need to assume that the ground ring H^* is Noetherian. We follow the notation introduced in [14] and denote by $\mathcal{U}(\mathrm{H}^*)$ the category of unstable $\mathrm{H}^* \odot \mathcal{P}^*$-modules. The semitensor product[2] was introduction by W.S. Massey and F.P. Peterson in [5], Definition 2.5. It summarizes that we are looking at objects M that are

(1) left H^*-modules, as well as,
(2) unstable left modules over the Steenrod algebra $\mathcal{P}^*$, and

both structures are compatible in the sense that

$$\mathcal{P}^l(h\,m) = \sum_{i+k=l} \mathcal{P}^i(h)\mathcal{P}^k(m)$$

for every element $\mathcal{P}^l \in \mathcal{P}^*$, $h \in \mathrm{H}^*$ and $m \in M$. In other words, the map

$$\mathrm{H}^* \otimes M \longrightarrow M ,$$

defining the H^*-module structure on M, is a homomorphism of left $\mathcal{P}^*$-modules.

Recall that the category $\mathcal{U}(\mathrm{H}^*)$ is abelian. In this section we want to generalize Lam's $\mathcal{J}$-functor to the context of modules, see [4] or Section 11.2 in [12]. Since this functor played a significant role in the proof of

[2] The multiplication in the semitensor product is defined as follows

$$(h \otimes \mathcal{P}^l)(h' \otimes \mathcal{P}^k) := \sum_{i+j=l} h\mathcal{P}^i(h') \otimes \mathcal{P}^j\mathcal{P}^k ,$$

see (2.3) or (2.4) in [5].

the Lasker-Noether Theorem for ideals, [10], it should not surprise that we need it here also.

Let M be an object in $\mathcal{U}(\mathrm{H}^*)$. Denote by $\mathcal{M}od_{\mathrm{H}^*}$ the category of H^*-modules, and by $\mathcal{M}od_{\mathrm{H}^*}(M)$ its full subcategory of H^*-submodules of M. Let N be an object in $\mathcal{M}od_{\mathrm{H}^*}(M)$. By restriction, we can define the images of the Steenrod powers on the elements of N. However, the module N might not be closed under this action. This motivates the following definition.

DEFINITION AND LEMMA 1.1 : Let M be an unstable $\mathrm{H}^* \odot \mathcal{P}^*$-module. Let N be an object in $\mathcal{M}od_{\mathrm{H}^*}(M)$. We define

$$\mathcal{J}(N) := \{n \in N \mid \mathcal{P}^i(n) \in N \quad \forall\ i \geq 0\},$$

and iteratively

$$\mathcal{J}_j(N) = \mathcal{J}(\mathcal{J}_{j-1}(N)) \quad \text{for } j \geq 2.$$

This leads to a descending chain

$$\mathcal{J}_0(N) := N \supseteq \mathcal{J}_1(N) \supseteq \mathcal{J}_2(N) \supseteq \ldots$$

of H^*-modules in $\mathcal{M}od_{\mathrm{H}^*}(M)$. We denote the intersection of this chain by

$$\mathcal{J}_\infty(N) = \bigcap_{j \geq 0} \mathcal{J}_j(N).$$

Then $\mathcal{J}_\infty(N)$ is an unstable $\mathrm{H}^* \odot \mathcal{P}^*$-module, and moreover, the maximal H^*-submodule in N that is closed under the action of the Steenrod algebra.

PROOF: For $n_1,\ n_2 \in \mathcal{J}_j(N)$ and $h_1,\ h_2 \in \mathrm{H}^*$, we have

$$\begin{aligned}\mathcal{P}^l(h_1 n_1 + h_2 n_2) &= \mathcal{P}^l(h_1 n_1) + \mathcal{P}^l(h_2 n_2) \\ &= \sum_{i+k=l} \left(\mathcal{P}^i(h_1)\mathcal{P}^k(n_1) + \mathcal{P}^i(h_2)\mathcal{P}^k(n_2)\right),\end{aligned}$$

where we made use of the Cartan formulae. Since

$$\mathcal{P}^i(h_1),\ \mathcal{P}^i(h_2) \in \mathrm{H}^* \quad \forall\ i \geq 0,$$

and

$$\mathcal{P}^k(n_1),\ \mathcal{P}^k(n_2) \in \mathcal{J}_{j-1}(N) \quad \forall\ k \geq 0,$$

by definition of $\mathcal{J}_j(N)$, we see that

$$\mathcal{P}^l(h_1 n_1 + h_2 n_2) \in \mathcal{J}_{j-1}(N) \quad \forall\ l \geq 0.$$

This in turn means that

$$h_1 n_1 + h_2 n_2 \in \mathcal{J}_j(N),$$

making $\mathcal{J}_j(N)$ into a H^*-module. Therefore we have a chain of H^*-modules in $\mathcal{Mod}_{\mathrm{H}^*}(M)$:

$$N =: \mathcal{J}_0(N) \supseteq \mathcal{J}_1(N) \supseteq \mathcal{J}_2(N) \supseteq \ldots \supseteq \mathcal{J}_j(N) \supseteq \ldots \supseteq \mathcal{J}_\infty(N).$$

Finally we need to show that $\mathcal{J}_\infty(N)$ is closed under the action of the Steenrod algebra. To this end, let $n \in \mathcal{J}_\infty(N)$, then

$$n \in \mathcal{J}_j(N) \quad \forall\, j \geq 0.$$

Hence

$$\mathcal{P}^l(n) \in \mathcal{J}_{j-1}(N) \quad \forall\, j \geq 1,\ \forall\, l \geq 0,$$

i.e.,

$$\mathcal{P}^l(n) \in \mathcal{J}_\infty(N) \quad \forall\, l \geq 0,$$

as claimed. The maximality of $\mathcal{J}_\infty(N)$ is by construction clear. ☯

Let $I = (i_1, \ldots, i_k)$ be a multi index, and set

$$\mathcal{P}^I = \mathcal{P}^{i_1} \cdots \mathcal{P}^{i_k}.$$

Then an element $n \in N$ is in $\mathcal{J}_\infty(N)$ if, and only if,

$$\mathcal{P}^I(n) \in N \quad \forall \text{ multi index } I.$$

In the following series of technical lemmata we show that the category $\mathcal{U}(\mathrm{H}^*)$ of unstable $\mathrm{H}^* \odot \mathcal{P}^*$-modules is closed under certain standard module-theoretic operations. Moreover, we investigate the behaviour of such operations under the $\mathcal{J}_\infty$-functor. Needless to say, we do this, because we will use these results later on.

LEMMA 1.2: *Let N, N' be objects in $\mathcal{Mod}_{\mathrm{H}^*}(M)$, and M in $\mathcal{U}(\mathrm{H}^*)$. Then*

$$\mathcal{J}_\infty(N \cap N') = \mathcal{J}_\infty(N) \cap \mathcal{J}_\infty(N').$$

PROOF: Take an element $n \in \mathcal{J}_\infty(N \cap N')$ then

$$n \in N \cap N' \quad \text{and} \quad \mathcal{P}^I(n) \in N \cap N' \quad \forall \text{ multi index } I.$$

This means

$$n \in \mathcal{J}_\infty(N) \cap \mathcal{J}_\infty(N'),$$

establishing the inclusion "$\subseteq$". The converse inclusion is proved by using this argument backward. ☯

LEMMA 1.3: *Let M and M' be objects in $\mathcal{U}(\mathrm{H}^*)$. Then so is their quotient*

$$\left(M' : M\right).$$

PROOF: The quotient $(M' : M) \subseteq \mathrm{H}^*$ is an ideal in H^*. So we need to show that it is $\mathcal{P}^*$-invariant. So, take an element $h \in (M' : M)$, i.e.,

$$h \cdot M \subseteq M'.$$

We claim that $\mathcal{P}^i(h) \in (M' : M)$, i.e., we claim that

$$\mathcal{P}^i(h) \cdot M \subseteq M'$$

for every $i \geq 0$. We proceed by induction on i. Since $\mathcal{P}^0$ is the identity map, the case $i = 0$ is trivial. However, we need to start our induction with $i = 1$. Let $m \in M$. Then we have by the Cartan formulae

$$\mathcal{P}^1(h)m = \mathcal{P}^1(hm) - h\mathcal{P}^1(m).$$

Now, the first summand $\mathcal{P}^1(hm) \in M'$, because $hm \in M'$ and M' is closed under the action of the Steenrod algebra. The second summand, $h\mathcal{P}^1(m)$, is equally in M', because $m \in M$, therefore $\mathcal{P}^1(m) \in M$, and $h \in (M' : M)$. Hence

$$\mathcal{P}^1(h)m \in M'$$

for every $m \in M$. This means that

$$\mathcal{P}^1(h) \in (M' : M).$$

Let $i > 1$. Then the Cartan formulae tell us that

$$\mathcal{P}^i(h)m = \mathcal{P}^i(hm) - \sum_{k+l=i, k<i} \mathcal{P}^k(h)\mathcal{P}^l(m) \quad \forall\, m \in M.$$

The sum on the right hand side is, by induction, in M'. Since $\mathcal{P}^i(hm) \in M'$, because M' is in $\mathcal{U}(\mathrm{H}^*)$, we conclude that also $\mathcal{P}^i(h)m \in M'$, in other words

$$\mathcal{P}^i(h) \in (M' : M) \quad \forall\, i \geq 0,$$

as claimed. ☯

LEMMA 1.4: *Let M be an object in $\mathcal{U}(\mathrm{H}^*)$, N in $\mathcal{Mod}_{\mathrm{H}^*}(M)$. Then*

$$\left(\mathcal{J}_\infty(N) : M\right) = \mathcal{J}_\infty\left(N : M\right).$$

PROOF: Since $\mathcal{J}_\infty(N) \subseteq N$, we have

$$\begin{aligned}\left(\mathcal{J}_\infty(N) : M\right) &:= \{h \in \mathrm{H}^* \mid hM \subseteq \mathcal{J}_\infty(N)\}\\ &\subseteq \{h \in \mathrm{H}^* \mid hM \subseteq N\} =: (N : M).\end{aligned}$$

Since $\mathcal{J}_\infty(N)$ and M are $\mathcal{P}^*$-modules, so is their quotient, by the preceding Lemma 1.3. Therefore

$$\left(\mathcal{J}_\infty(N) : M\right) = \mathcal{J}_\infty\left(\mathcal{J}_\infty(N) : M\right) \subseteq \mathcal{J}_\infty\left(N : M\right),$$

by maximality of $\mathcal{J}_\infty(N : M)$ in $(N : M)$. This establishes the one inclusion. To show the reverse inclusion, we take an element $h \in \mathcal{J}_\infty\big(N : M\big)$. Then

$$hM \subseteq N \quad \text{and} \quad \mathcal{P}^I(h)M \subseteq N \quad \forall \text{ multi index } I.$$

We need to show that $h \in \big(\mathcal{J}_\infty(N) : M\big)$, i.e.,

$$hM \subseteq \mathcal{J}_\infty(N).$$

Since $hM \subseteq N$, this means for every multi index I and every element $m \in M$ we have to verify that

$$\mathcal{P}^I(hm) \in N.$$

We employ the Cartan formulae, set $I = (i_1, \ldots, i_k)$ and get

$$\mathcal{P}^I(hm) = \mathcal{P}^{I'}\left(\sum_{j_k+l_k=i_k} \mathcal{P}^{j_k}(h)\mathcal{P}^{l_k}(m)\right),$$

where $I' = (i_1, \ldots, i_{k-1})$. Hence, setting $I'' = (i_1, \ldots, i_{k-2})$ and iterating, we arrive at

$$\begin{aligned}
\mathcal{P}^I(hm) = \mathcal{P}^{I'}&\left(\sum_{j_k+l_k=i_k} \mathcal{P}^{j_k}(h)\mathcal{P}^{l_k}(m)\right)\\
&= \mathcal{P}^{I''}\left(\sum_{j_{k-1}+l_{k-1}=i_{k-1}}\ \sum_{j_k+l_k=i_k} \mathcal{P}^{j_{k-1}}\mathcal{P}^{j_k}(h)\mathcal{P}^{l_{k-1}}\mathcal{P}^{l_k}(m)\right)\\
&= \sum_{j_1+l_1=i_1} \cdots \sum_{j_k+l_k=i_k} \mathcal{P}^{j_1}\cdots\mathcal{P}^{j_k}(h)\mathcal{P}^{l_1}\cdots\mathcal{P}^{l_k}(m)\\
&= \sum_{j_1+l_1=i_1} \cdots \sum_{j_k+l_k=i_k} \mathcal{P}^{J}(h)\mathcal{P}^{L}(m),
\end{aligned}$$

for multi indices $J = (j_1, \ldots, j_k)$ and $L = (l_1, \ldots, l_k)$. Since M is an object in $\mathcal{U}(\mathrm{H}^*)$, we have $\mathcal{P}^L(m) \in M$ for all L. Now, $h \in \mathcal{J}_\infty\big(N : M\big)$, i.e.,

$$hm \in N \quad \forall\, m \in M \quad \text{and}\ \mathcal{P}^J(h)m \in N \quad \forall\, m \in M,\ \forall\, J.$$

This means that the right hand side is an element of N, hence so is the left, as desired. ☯

We collect these results, and extend them to

PROPOSITION 1.5: *Let M, M' be unstable $\mathrm{H}^* \odot \mathcal{P}^*$-modules, let N be an object in $\mathcal{M}od_{\mathrm{H}^*}(M)$. Then*

(1) *$(M' : M) \subseteq \mathrm{H}^*$ is $\mathcal{P}^*$-invariant.*
(2) *If $(N : M) \subseteq \mathrm{H}^*$ is $\mathcal{P}^*$-invariant, then there exists an unstable $\mathrm{H}^* \odot \mathcal{P}^*$-module $M_N \subseteq M$ such that*

$$\left(M_N : M\right) = (N : M).$$

PROOF: Statement (1) is the contents of Lemma 1.3. To prove the second statement, recall from Lemma 1.4 that

$$(N : M) = \mathcal{J}_\infty\left(N : M\right) = \left(\mathcal{J}_\infty(N) : M\right).$$

Choose $M_N = \mathcal{J}_\infty(N)$ and finish the proof with Lemma 1.1. ☯

Denote by $\mathcal{R}ad(-)$ the radical of $(-)$. We need the following result correlating the functor $\mathcal{J}_\infty$ with $\mathcal{R}ad$.

LEMMA 1.6: *Let M be an object in $\mathcal{U}(\mathrm{H}^*)$, and let N an object in $\mathcal{M}od_{\mathrm{H}^*}(M)$. Then*

$$\mathcal{R}ad\left(\mathcal{J}_\infty(N) : M\right) = \mathcal{J}_\infty\left(\mathcal{R}ad(N : M)\right).$$

PROOF: Let $h \in \mathcal{R}ad\left(\mathcal{J}_\infty(N) : M\right)$. Then

$$h^r M \subseteq \mathcal{J}_\infty(N)$$

for some large $r \in \mathbb{N}$. Hence, a fortiori,

$$h^r M \subseteq N,$$

or,

$$h \in \mathcal{R}ad(N : M).$$

This means

$$\mathcal{R}ad\left(\mathcal{J}_\infty(N) : M\right) \subseteq \mathcal{R}ad(N : M).$$

By Lemma 1.3 $\left(\mathcal{J}_\infty(N) : M\right)$ is $\mathcal{P}^*$-invariant. Therefore so is its radical, by Lemma 1.4 in [10]. By maximality this implies that

$$\mathcal{R}ad\left(\mathcal{J}_\infty(N) : M\right) \subseteq \mathcal{J}_\infty\left(\mathcal{R}ad(N : M)\right).$$

Conversely, by Lemma 1.3 in [10] we have

$$\mathcal{J}_\infty\left(\mathcal{R}ad(N : M)\right) = \mathcal{R}ad\left(\mathcal{J}_\infty(N : M)\right).$$

We want to show that $\mathcal{R}ad\left(\mathcal{J}_\infty(N : M)\right) \subseteq \mathcal{R}ad\left(\mathcal{J}_\infty(N) : M\right)$. So, it is enough to show that

$$\mathcal{J}_\infty(N : M) \subseteq \left(\mathcal{J}_\infty(N) : M\right),$$

For that take an element $h \in \mathcal{J}_\infty(N : M)$. Then, by definition, we have that

$$h \cdot M \subseteq N \quad \text{and} \quad \mathcal{P}^I(h) \cdot M \subseteq N,$$

for every multi index $I = (i_1, \dots, i_k)$. Let $m \in M$. We need to show that $h \cdot m \in \mathcal{J}_\infty(N)$, i.e., we need to show that

$$\mathcal{P}^I(h \cdot m) \in N \quad \forall \text{ multi index } I.$$

We induct on the length $k = |I|$, where $I = (i_1, \dots, i_k)$.

Case $|I| = 1$: Then $I = (i_1) = (i)$ and

$$\mathcal{P}^i(hm) = \sum_{k+l=i} \mathcal{P}^k(h)\mathcal{P}^l(m)$$

Since $h \in \mathcal{J}_\infty(N : M)$ we have that $\mathcal{P}^k(h) \in \mathcal{J}_\infty(N : M) \subseteq (N : M)$ for all k. Hence, because $\mathcal{P}^l(m) \in M$ for all l

$$\mathcal{P}^k(h) \cdot \mathcal{P}^l(m) \in N \quad \forall\, k,\ \forall\, l.$$

Therefore

$$\mathcal{P}^i(hm) = \sum_{k+l=i} \mathcal{P}^k(h)\mathcal{P}^l(m) \in N.$$

Case $|I| > 1$: We rewrite $I = (I', i)$, where $I' = (i_1, \dots, i_{k-1})$, $I = (i_1, \dots, i_k)$ and $i = i_k$. We have

$$\begin{aligned}
\mathcal{P}^I(h \cdot m) &= \mathcal{P}^{I'}\mathcal{P}^i(h \cdot m) \\
&= \mathcal{P}^{I'}\left(\sum_{k+l=i} \mathcal{P}^k(h)\mathcal{P}^l(m)\right) \\
&= \sum_{k+l=i} \mathcal{P}^{I'}\left(\mathcal{P}^k(h)\mathcal{P}^l(m)\right)
\end{aligned}$$

As in the preceding case we conclude that

$$\begin{aligned}
&\mathcal{P}^l(m) \in M \quad \forall\, l \geq 0, \text{ and} \\
&\quad \mathcal{P}^k(h) \in \mathcal{J}_\infty(N : M) \quad \forall\, k \geq 0,
\end{aligned}$$

hence, by induction we have

$$\mathcal{P}^{I'}\left(\mathcal{P}^k(h)\mathcal{P}^l(m)\right) \in N$$

and we are done. ☯

§2. Primary Unstable $\mathrm{H}^* \odot \mathcal{P}^*$-Modules

We want to show that an unstable Noetherian module over an unstable Noetherian algebra, H^*, has a primary decomposition consisting of unstable components. For that we follow the classical route as described in the Appendix to Chapter IV on page 252f of [15]. We start with recollecting some terminology. Let H^* be an unstable Noetherian algebra over the Steenrod algebra. Let M be an unstable $\mathrm{H}^* \odot \mathcal{P}^*$-module, and $Q \subseteq M$ a submodule. Q is said to be **primary**, if whenever

$$h \cdot m \in Q \quad \text{for } h \in \mathrm{H}^*,\ m \in M$$

then

$$\text{either} \quad m \in Q \quad \text{or} \quad h \in \mathcal{Rad}(Q) := \mathcal{Rad}(Q : M).$$

If a module $Q \subseteq M$ is primary, then the ideal

$$(Q : M) := \{h \in \mathrm{H}^* \mid hM \subseteq Q\} \subseteq \mathrm{H}^*$$

is primary (but not conversely!)[3]. Let M be Noetherian, and let $M' \subseteq M$ be unstable $\mathrm{H}^* \odot \mathcal{P}^*$-modules. As a H^*-submodule of M, the module M' has a primary decomposition

$$M' = Q_1 \cap \ldots \cap Q_m,$$

where $Q_1, \ldots, Q_m \subseteq M$ are primary H^*-submodules of M. The prime ideals

$$\mathfrak{p}_i = \mathcal{Rad}(Q_i : M) \subset \mathrm{H}^* \quad \forall\, i = 1, \ldots, m$$

are called **associated prime ideals** of M'. Moreover, the decomposition is called **irredundant** if

$$\bigcap_{i \neq j} Q_i \not\subseteq Q_j \quad \forall\, j = 1, \ldots, m.$$

It is called **minimal** if

$$\mathcal{Rad}(Q_i : M) \neq \mathcal{Rad}(Q_j : M) \quad \text{whenever } i \neq j.$$

We want to show that the primary modules

$$Q_1, \ldots, Q_m$$

as well as the associated prime ideals can be choosen to be $\mathcal{P}^*$-invariant. We start by showing that the associated prime ideals are $\mathcal{P}^*$-invariant.

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[3] See page 252 in [15] for a counterexample.

LEMMA 2.1: *Let M be Noetherian. Let $M' \subseteq M$ be objects in $\mathcal{U}(\mathrm{H}^*)$, let*

$$M' = Q_1 \cap \ldots \cap Q_m$$

be a primary decomposition of M' as a H^*-module. Then the associated prime ideals of M', i.e., the prime ideals

$$\mathfrak{p}_i := \mathcal{R}ad\left(Q_i : M\right) \subseteq \mathrm{H}^* \quad \forall\ i = 1, \ldots, m$$

are $\mathcal{P}^*$-invariant.

PROOF: Since $M' \subseteq M$ are objects in $\mathcal{U}(\mathrm{H}^*)$, the ideal $(M' : M)$ in H^* is $\mathcal{P}^*$-invariant, by Lemma 1.3. As such, the ideal $(M' : M)$ has a $\mathcal{P}^*$-invariant minimal irredundant primary decomposition by Theorem 3.5 in [10],

$$(M' : M) = \widehat{\mathfrak{q}_1} \cap \ldots \cap \widehat{\mathfrak{q}_k},$$

where the associated prime components are

$$\widehat{\mathfrak{p}_j} = \mathcal{R}ad\left(\widehat{\mathfrak{q}_j}\right) \subseteq \mathrm{H}^*,$$

for $j = 1, \ldots, k$. Hence

$$\begin{aligned}\widehat{\mathfrak{p}_1} \cap \ldots \cap \widehat{\mathfrak{p}_k} &= \mathcal{R}ad(M' : M)\\ &= \mathcal{R}ad(Q_1 : M) \cap \ldots \cap \mathcal{R}ad(Q_m : M)\\ &= \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_m.\end{aligned}$$

Hence, since both decompositions are minimal and irredundant, we have

$$k = m,$$

and, after possibly reordering,

$$\widehat{\mathfrak{p}_i} = \mathcal{R}ad(Q_i : M) \quad \forall\ i = 1, \ldots, k,$$

cf Lemma 2.4.10 in [1]. ☯

The following result extends Lemma 1.5 in [10] to the context of modules.

PROPOSITION 2.2: *Let M be an unstable Noetherian $\mathrm{H}^* \odot \mathcal{P}^*$-module. Let Q in $\mathcal{M}od_{\mathrm{H}^*}(M)$ be a primary module, such that its radical*

$$\mathfrak{p} = \mathcal{R}ad(Q : M) \subseteq \mathrm{H}^*$$

is $\mathcal{P}^*$-invariant. Then $\mathcal{I}_\infty(Q)$ is primary in $\mathcal{U}(\mathrm{H}^*)$ with radical $\mathfrak{p}$.

PROOF: Let Q be primary as a H^*-submodule of M. Then

$$\mathfrak{q} = (Q : M) \subseteq \mathrm{H}^*$$

is a primary ideal with radical, $\mathfrak{p} \subseteq \mathrm{H}^*$. By Theorem 3.3 in [7] we know that

$$\mathcal{J}_\infty(\mathfrak{q})$$

is a primary ideal with radical $\mathcal{J}_\infty(\mathfrak{p}) = \mathfrak{p}$. Hence, by Proposition 1.5,

$$\big(\mathcal{J}_\infty(Q) : M\big) = \mathcal{J}_\infty(Q : M)$$

is a primary ideal with radical $\mathfrak{p}$. We need to show that

$$\mathcal{J}_\infty(Q) \subseteq M$$

is a primary module. We have

$$\mathfrak{p} = \mathcal{J}_\infty(\mathfrak{p}) = \mathcal{J}_\infty\left(\mathcal{Rad}(Q : M)\right) = \mathcal{Rad}\left(\mathcal{J}_\infty(Q) : M\right) \supseteq \left(\mathcal{J}_\infty(Q) : M\right).$$

Let $h \in \mathrm{H}^*$, $m \in M$ and $hm \in \mathcal{J}_\infty(Q)$. We assume that $m \notin \mathcal{J}_\infty(Q)$, and have to show that

$$h \in \mathcal{Rad}\left(\mathcal{J}_\infty(Q) : M\right) = \mathfrak{p}.$$

Note that we have a chain of modules

$$\mathcal{J}_\infty(Q) \subseteq \mathcal{J}_1(Q) \subseteq Q$$

with the same ($\mathcal{P}^*$-invariant) prime radical

$$\mathfrak{p} = \mathcal{Rad}(Q) = \mathcal{Rad}(\mathcal{J}_1(Q)) = \mathcal{J}_\infty(\mathcal{Rad}(Q)).$$

Therefore, by iteration, it is enough to show that $\mathcal{J}_1(Q)$ is primary. To this end let $h \in \mathrm{H}^*$ and $m \in M$ such that $hm \in \mathcal{J}_1(Q)$, and assume that $m \notin \mathcal{J}_1(Q)$. We need to show that

$$h \in \mathcal{Rad}(\mathcal{J}_1(Q) : M) = \mathfrak{p}.$$

CASE $m \in Q \setminus \mathcal{J}_1(Q)$: Then there exists an $i \in \mathbb{N}_0$ such that

$$\mathcal{P}^i(m) \notin Q.$$

Let i be minimal with this property. Then

$$\mathcal{P}^i(hm) = \sum_{k+l=i,\, l<i} \mathcal{P}^k(h)\mathcal{P}^l(m) + h\mathcal{P}^i(m).$$

Since $hm \in \mathcal{J}_1(Q)$ we have that also $\mathcal{P}^i(hm) \in Q$. By minimality of i we know that $\mathcal{P}^l(m) \in Q$ for all $l < i$. Therefore

$$h\mathcal{P}^i(m) = \mathcal{P}^i(hm) - \sum_{k+l=i,\, l<i} \mathcal{P}^k(h)\mathcal{P}^l(m) \in Q.$$

Since $\mathcal{P}^i(m) \notin Q$ by assumption we conclude, because Q is primary, that

$$\begin{aligned} h \in \mathcal{Rad}\left(Q : M\right) &= \mathcal{J}_\infty\left(\mathcal{Rad}(Q : M)\right) \\ &= \mathcal{Rad}\left(\mathcal{J}_\infty(Q) : M\right) \\ &= \mathfrak{p} \\ &= \mathcal{Rad}\left(\mathcal{J}_1(Q) : M\right), \end{aligned}$$

where we used Lemma 2.1 and Lemma 1.6.

CASE $m \notin Q$: By assumption, $hm \in \mathcal{J}_1(Q) \subseteq Q$, so, because Q is primary,

$$h \in \mathcal{Rad}\left(Q : M\right) = \mathfrak{p}.$$

This shows that

$$\mathcal{J}_1(Q) \subseteq M$$

is a primary H^*-module. Hence, iteratively, we get that $\mathcal{J}_\infty(Q) \subseteq M$ is primary in the category $\mathcal{U}(\mathrm{H}^*)$. ☯

THEOREM 2.3 (Lasker-Noether Theorem): *Let* H^* *be an unstable Noetherian algebra over the Steenrod algebra. Let* M *be Noetherian, and let* $M' \subseteq M$ *be unstable* $\mathrm{H}^* \odot \mathcal{P}^*$*-modules. Then* M' *admits a minimal irredundant primary decomposition in* $\mathcal{U}(\mathrm{H}^*)$*, i.e., all primary components, as well as the associated prime ideals are unstable* $\mathrm{H}^* \odot \mathcal{P}^*$*-modules.*

PROOF: Choose a primary decomposition of M' as a H^*-module

$$M' = Q_1 \cap \ldots \cap Q_m.$$

By Lemma 2.1 we know that the associated prime ideals are $\mathcal{P}^*$-invariant. Hence by Proposition 2.2 we have that $\mathcal{J}_\infty(Q_i)$ is a primary module with radical

$$\mathcal{Rad}(Q_i) := \mathcal{Rad}(Q_i : M) = \mathcal{J}_\infty\left(\mathcal{Rad}(Q_i)\right).$$

By Lemma 1.1 these are a modules in $\mathcal{U}(\mathrm{H}^*)$. So

$$M' = \mathcal{J}_\infty(M') = \mathcal{J}_\infty(Q_1 \cap \cdots \cap Q_m) = \mathcal{J}_\infty(Q_1) \cap \cdots \cap \mathcal{J}_\infty(Q_m),$$

using Lemma 1.2, is a primary decomposition in the category $\mathcal{U}(\mathrm{H}^*)$. We make it irredundant by throwing away superfluous modules, and minimal by combining these modules which have the same radical. ☯

COROLLARY 2.4: *Let* H^* *be an unstable Noetherian algebra over the Steenrod algebra. Let* M *be an unstable Noetherian* $\mathrm{H}^* \odot \mathcal{P}^*$*-module. Let* Q *in* $\mathcal{Mod}_{\mathrm{H}^*}(M)$ *be a primary module, then* $\mathcal{J}_\infty(Q)$ *is primary in* $\mathcal{U}(\mathrm{H}^*)$ *with radical* $\mathcal{J}_\infty\left(\mathcal{Rad}(Q : M)\right)$.

PROOF: Let Q be a primary module in $\mathcal{M}od_{\mathrm{H}^*}(M)$. Then, by definition, the ideal

$$(Q : M) \subseteq \mathrm{H}^*$$

is primary. From Theorem 3.3 in [7], we know that

$$\mathcal{J}_\infty(Q : M) \subseteq \mathrm{H}^*$$

is $\mathcal{P}^*$-invariant and primary with $\mathcal{P}^*$-invariant radical

$$\mathcal{J}_\infty\left(\mathcal{R}ad(Q : M)\right) = \mathcal{R}ad\left(\mathcal{J}_\infty(Q) : M\right),$$

where the last equality follows from Lemma 1.6. We need to show that $\mathcal{J}_\infty(Q)$ is a primary module. To this end let

$$\mathcal{J}_\infty(Q) = Q_1 \cap \ldots \cap Q_m$$

be an irredundant and minimal primary decomposition of $\mathcal{J}_\infty(Q)$. By Theorem 2.3 we can assume that the $Q_1, \ldots, Q_m$ are unstable modules. Hence the $\mathcal{P}^*$-invariant primary ideal $(\mathcal{J}_\infty(Q) : M)$ can be written as an intersection of $\mathcal{P}^*$-invariant primary ideals

$$(\mathcal{J}_\infty(Q) : M) = \bigcap_{i=1}^{m}(Q_i : M).$$

By irredundancy and minimality we have that $m = 1$, as desired. ☯

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