The Noether Map I

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SUMMARY : *Let* $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ *be a faithful representation of a finite group* G. *In this paper we study the image of the associated Noether map*

$$\eta_G^G : \mathbb{F}[V(G)]^G \longrightarrow \mathbb{F}[V]^G.$$

It turns out that the image of the Noether map characterizes the ring of invariants in the sense that its integral closure $\overline{\mathrm{Im}(\eta_G^G)} = \mathbb{F}[V]^G$. *This is true without any restrictions on the group, representation, or ground field. Moreover, we show that the extension* $\mathrm{Im}(\eta_G^G) \subseteq \mathbb{F}[V]^G$ *is a finite* p*-root extension. Furthermore, we show that the Noether map is surjective, i.e., its image integrally closed, if* $V = \mathbb{F}^n$ *is a projective* $\mathbb{F}G$*-module. We apply these results and obtain upper bounds on the degrees of a minimal generating set of* $\mathbb{F}[V]^G$ *and the Cohen-Macaulay defect of* $\mathbb{F}[V]^G$. *We illustrate our results with several examples.*

Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a faithful representation of a finite group G over a field $\mathbb{F}$. The representation ϱ induces naturally an action of G on the vector space $V = \mathbb{F}^n$ of dimension n and hence on the ring of polynomial functions $\mathbb{F}[V] = \mathbb{F}[x_1, \dots, x_n]$. Our interest is focused on the subring of invariants

$$\mathbb{F}[V]^G = \{f \in \mathbb{F}[V]^G \mid gf = f \ \forall\, g \in G\},$$

which is a graded connected Noetherian commutative algebra. In the first section of this paper we introduce the Noether map and show that the integral closure of its image is the ring of invariants. In Section 2 we show that $\mathrm{Im}(\eta_G^G) \subseteq \mathbb{F}[V]^G$ is a finite p-root extension. In Section 3 we prove that the Noether map is surjective if V is a projective $\mathbb{F}G$-module. In Section 4 we derive some results about degree bounds and the Cohen-Macaulay defect of $\mathbb{F}[V]^G$. Furthermore we present some examples.

§1. The Noether Map

Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a group G of order d. Let $\mathbb{F}[V]$ be the symmetric algebra on V^*. Denote by $\mathbb{F}G$ the group algebra. Let

$$V(G) = \mathbb{F}G \otimes V$$

be the induced module $\mathrm{coind}_1^G(V)$. The group G acts on $V(G)$ by left multiplication on the first component. We obtain a G-equivariant surjection

$$(\bigstar) \qquad V(G) \longrightarrow V,\ (g, v) \longmapsto gv.$$

Let us choose a basis $e_1, \dots, e_n$ for V. Let $x_1, \dots, x_n$ be the standard dual basis for V^*, and set $G = \{g_1, \dots, g_d\}$. Then $V(G)$ can be written as

$$V(G) = \mathrm{span}_{\mathbb{F}}\{e_{ij} \mid i = 1, \dots, n,\ j = 1, \dots, d\},$$

and the map $(\bigstar)$ translates into

$$V(G) \longrightarrow V,\ \ e_{ij} \longmapsto g_j e_i.$$

Similarly, we have

$$V(G)^* = \mathrm{span}_{\mathbb{F}}\{x_{ij} \mid i = 1, \dots, n,\ j = 1, \dots, d\}$$

with

$$V(G)^* \longrightarrow V^*,\ \ x_{ij} \longmapsto g_j x_i.$$

We obtain a surjective G-equivariant map between the rings of polynomial functions

$$\eta_G : \mathbb{F}[V(G)] \longrightarrow \mathbb{F}[V].$$

The group G acts on $\mathbb{F}[V(G)]$ by permuting the basis elements x_{ij}. By restriction to the induced ring of invariants, we obtain the classical

Noether map, cf. Section 4.2 in [11],

$$\eta_G^G : \mathbb{F}[V(G)]^G \longrightarrow \mathbb{F}[V]^G.$$

We note that $V(G)$ is the n-fold regular representation of G. Thus $\mathbb{F}[V(G)]^G$ are the n-fold vector invariants of the regular representation of G.

In the classical nonmodular case, where $p \nmid d$, the map η_G^G is surjective, see Proposition 4.2.2 in [11]. This does not remain true in the modular case as we illustrate in the next example.

EXAMPLE 1: Let $\varrho : \mathbb{Z}/2 \hookrightarrow \mathrm{GL}(3, \mathbb{F}_2)$ be the 3-dimensional representation of $\mathbb{Z}/2$ over the field with two elements afforded by the matrix

$$\varrho(g) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\mathbb{F}[x_1,\ x_2,\ x_3]^{\mathbb{Z}/2} = \mathbb{F}[x_1 + x_2, x_1x_2, x_3]$$

and

$$\begin{aligned} &\mathbb{F}[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}]^{\mathbb{Z}/2} \\ &\quad = \mathbb{F}[x_{i1} + x_{i2}, x_{i1}x_{i2}, x_{i1}x_{i+1,2} + x_{i2}x_{i+1,1}, x_{11}x_{21}x_{31} + x_{12}x_{22}x_{32}], \end{aligned}$$

where $i \in \mathbb{Z}/3$, cf. Example 2 in Section 2.3, [11] or Example 1 in Section 3.2, loc.cit. We obtain

$$\mathrm{Im}(\eta_{\mathbb{Z}/2}^{\mathbb{Z}/2}) = \mathbb{F}[x_1 + x_2, x_1x_2, x_3^2, (x_1 + x_2)x_3].$$

Thus the Noether map is no longer surjective, because the invariant x_3 is not in its image. However, note that the integral closure of the image of the Noether map is the ring of invariants $\mathbb{F}[V]^G$. This is always true as we see in this section.

Recall the **transfer map**

$$\mathrm{Tr}^G : \mathbb{F}[V] \longrightarrow \mathbb{F}[V]^G;\ f \longmapsto \sum_{g \in G} gf,$$

see, e.g., Section 2.2. in [11]. By construction the transfer is an $\mathbb{F}[V]^G$-module homomorphism. We denote by

$$\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)] \subseteq \mathbb{F}[V]^G$$

the subalgeba generated by the image of the transfer.

We observe that any element $\frac{f_1}{f_2} \in \mathbb{F}(V)$ can be written as the quotient of some polynomial by an invariant polynomial in the following way

$$\frac{f_1}{f_2} = \frac{f_1 \frac{\mathbf{N}(f_2)}{f_2}}{\mathbf{N}(f_2)},$$

where $\mathbf{N}(f) = \prod_{g \in G} gf$ denotes the **Norm** of f. This allows us to extend the transfer to a map of $\mathbb{F}(V)^G$-modules between the respective fields of fractions

$$\mathrm{Tr}^G : \mathbb{F}(V) \longrightarrow \mathbb{F}(V)^G; \ \frac{f_1}{f_2} \longmapsto \frac{\sum_{g \in G} gf_1}{f_2},$$

where we assume that $f_2 \in \mathbb{F}[V]^G$.

PROPOSITION 1.1: *We have that*

$$\mathbb{F}(\mathrm{Tr}^G(\mathbb{F}(V))) = \mathbb{F}\!\mathbb{F}(\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)]) = \mathbb{F}(V)^G,$$

where $\mathbb{F}\!\mathbb{F}(-)$ *denotes the field of fractions functor.*

PROOF: Let $\frac{\mathrm{Tr}^G(f_1)}{\mathrm{Tr}^G(f_2)} \in \mathbb{F}\!\mathbb{F}(\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)])$. Then

$$\frac{\mathrm{Tr}^G(f_1)}{\mathrm{Tr}^G(f_2)} = \mathrm{Tr}^G\left(\frac{f_1}{\mathrm{Tr}^G(f_2)}\right) \in \mathrm{Tr}^G(\mathbb{F}(V)).$$

To prove the reverse inclusion take an element

$$\mathrm{Tr}^G(\frac{f_1}{f_2}) \in \mathrm{Tr}^G(\mathbb{F}(V)),$$

where $f_2 \in \mathbb{F}[V]^G$. Choose a polynomial $f \in \mathbb{F}[V]$ such that $\mathrm{Tr}^G(f) \neq 0$. (Recall that the transfer map is never zero by Propositon 2.2.4 in [11].) Then we have

$$\mathrm{Tr}^G(\frac{f_1}{f_2}) = \frac{\mathrm{Tr}^G(f_1)}{f_2} = \frac{\mathrm{Tr}^G(f_1)\mathrm{Tr}^G(f)}{f_2\mathrm{Tr}^G(f)} = \frac{\mathrm{Tr}^G(f_1)\mathrm{Tr}^G(f)}{\mathrm{Tr}^G(ff_2)} \in \mathbb{F}\!\mathbb{F}(\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)]).$$

We come to the second equality. Since $\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)] \subseteq \mathbb{F}[V]^G$ we have that

$$\mathbb{F}\!\mathbb{F}(\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)]) \subseteq \mathbb{F}(V)^G.$$

To prove the reverse inclusion, let $\frac{f_1}{f_2} \in \mathbb{F}(V)^G$ where without loss of generality $f_1, \ f_2 \in \mathbb{F}[V]^G$. Let $\mathrm{Tr}^G(f) \neq 0$ for some suitable $f \in \mathbb{F}[V]$. Thus

$$\frac{f_1}{f_2} = \frac{\mathrm{Tr}^G(f)f_1}{\mathrm{Tr}^G(f)f_2} = \frac{\mathrm{Tr}^G(ff_1)}{\mathrm{Tr}^G(ff_2)} \in \mathbb{F}\!\mathbb{F}(\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)])$$

as desired. □

PROPOSITION 1.2: *The integral closure of the image of the Noether map is the ring of invariants*[1]

$$\overline{\mathrm{Im}(\eta_G^G)} = \mathbb{F}[V]^G.$$

PROOF: By Proposition 1.1 and Lemma 4.2.1 in [11] we have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)] & \subseteq & \mathrm{Im}(\eta_G^G) & \subseteq & \mathbb{F}[V]^G & \subseteq & \mathbb{F}[V] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{FF}(\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)]) & = & \mathbb{FF}(\mathrm{Im}(\eta_G^G)) & = & \mathbb{F}(V)^G & \subseteq & \mathbb{F}(V).
\end{array}$$

Let $x_1, \dots, x_n \in V^*$ be a basis. Then the coefficients of the polynomials

$$F_i(X) = \prod_{g \in G} (X - gx_i),$$

are the orbit chern classes of x_i counted with multiplicities

$$\sigma_1(x_i) = \mathrm{Tr}^G(x_i), \cdots, \sigma_d(x_i) = \mathbf{N}(x_i).$$

Thus they are in the image of η_G^G. Denote by A the $\mathbb{F}$-algebra generated by these coefficients. By construction A is finitely generated, thus noetherian. Furthermore $\mathbb{F}[V]$ is finitely generated as an A-module, thus as an $\mathrm{Im}(\eta_G^G)$-module since $A \subseteq \mathrm{Im}(\eta_G^G)$. Therefore the extension

$$\mathrm{Im}(\eta_G^G) \subseteq \mathbb{F}[V]$$

is finite, and $\overline{\mathrm{Im}(\eta_G^G)} = \mathbb{F}[V]^G$ as desired. □

We close this section with an immediate corollary of the preceding result:

COROLLARY 1.3: *The Krull dimension of the image of the Noether map coincides with the Krull dimension of the ring of invariants, which in turn is equal to* $n = \dim_{\mathbb{F}} V$. □

ADDENDUM: Define a map $E : \mathbb{F}[V] \longrightarrow \mathbb{F}[V(G)]^G$, $x_i \longmapsto \sum_{j=1}^{d} x_{ij}$. Then we obtain a commutative triangle as follows:

$$\begin{array}{ccc}
\mathbb{F}[V(G)]^G & \xrightarrow{\eta_G^G} & \mathbb{F}[V]^G \\
\uparrow E & \nearrow \mathrm{Tr}^G & \\
\mathbb{F}[V] & &
\end{array}$$

If $p \not| d$, then the preceding diagram proves that the Noether map is surjective, since the transfer is surjective, see Lemma 4.2.1 in [11]. We want to add the following observation:

[1] This result has an obvious generalization: Let G be a finite group acting by automorphisms on normal domains A and B. Let $\eta : A \longrightarrow B$ be an G-equivariant homomorphism. Then the integral closure of $\eta(A^G)$ is equal to B^G. The proof remains up to notation the same.

PROPOSITION 1.4: *The algebra generated by the image of the transfer map is equal to the image of the Noether map if and only if V is a nonmodular $\mathbb{F}G$-module.*

PROOF: By Lemma 4.2.1 in [11] the image of the transfer is always contained in the image of the Noether map. Thus if $p \not| |G|$, then the transfer is surjective, and hence the Noether map. If $p \mid |G|$, then the transfer is no longer surjective. Indeed, the height of the image of the transfer is at most $n-1$, see Theorem 6.4.7 in [11]. Thus the Krull dimension of $\mathbb{F}[\mathrm{Im}(\mathrm{Tr}^G)]$ is strictly less than n. On the other hand the Krull dimension of the image of the Noether map is n by Proposition 1.2. Thus they cannot be equal. □

§2. p-root Extensions

In this section we prove that the extension $\mathrm{Im}(\eta_G^G) \hookrightarrow \mathbb{F}[V]^G$ is a p-root extension, i.e., for any $f \in \mathbb{F}[V]^G$ there exists an integer $l \in \mathbb{N}$ such that

$$f^{p^l} \in \mathrm{Im}(\eta_G^G),$$

cf. Section 3 in [4]. We need a relative version of the Noether map, which we obtain in the following way. Let $H \leq G$ be an arbitrary subgroup. Then we have a commutative diagram as follows:

$$\begin{array}{ccc} V(G)^* & \xrightarrow{\eta_G} & V^* \\ \uparrow & & || \\ V(H)^* & \xrightarrow{\eta_H} & V^* \end{array}$$

Since $V(G)^* = \oplus_{|G:H|} V(H)^*$ the vertical map admits a splitting as $\mathbb{F}H$-modules. This induces the following diagram:

$$\begin{array}{ccc} \mathbb{F}[V(G)]^G & \xrightarrow{\eta_G^G} & \mathbb{F}[V]^G \\ \downarrow & & \downarrow \\ \mathbb{F}[V(G)]^H & \xrightarrow{\eta_G^H} & \mathbb{F}[V]^H \\ \uparrow & & || \\ \mathbb{F}[V(H)]^H & \xrightarrow{\eta_H^H} & \mathbb{F}[V]^H \end{array}$$

PROPOSITION 2.1: *Denot by P a p-Sylow subgroup of G. Let H be a subgroup in G containing P.*

(1) *The map η_G^P is surjective if and only if η_P^P is surjective.*

(2) *If $\eta_G^H|_{(d)}$ is surjective, then so is $\eta_G^G|_{(d)}$, where $\eta|_{(d)}$ denotes the restriction of the map η to degree d.*

PROOF: We have that

$$\eta_P^P = \eta_G^P|_{\mathbb{F}[V(P)]^P}.$$

Thus if η_P^P is surjective, so is η_G^P. Conversely, let η_G^P be surjective. By

Theorem 1.3 in [10] it is enough to show that $\eta_P^P|_{(1)}$ is surjective. By assumption $\eta_G^P|_{(1)}$ is surjective. Therefore[2] $\mathbb{F}[V]^P_{(1)}$ is generated as a vector space by images $\eta_G^P(o_P(g_j \otimes x_i))$ for some basis elements $g_j \otimes x_i$ of $V(G)^*$. Note that the stabilizer of $g_j \otimes x_i$ in P is trivial, so it follows that

$$\eta_G^P(o_P(g_j \otimes x_i) = \mathrm{Tr}^P(g_j x_i).$$

Hence, $\mathbb{F}[V]^P_{(1)}$ is generated as a vector space by invariants of the form $\mathrm{Tr}^P(g_j x_i)$. Set

$$g_j x_i = \sum_{k=1}^{n} \alpha_k x_k$$

for suitable $a_k \in \mathbb{F}$. We obtain

$$\mathrm{Tr}^P(g_j x_i) = \mathrm{Tr}^P(\sum_{k=1}^{n} \alpha_k x_k) = \sum_{k=1}^{n} \alpha_k \mathrm{Tr}^P(x_k) = \sum_{k=1}^{n} \alpha_k \eta_P^P(o(g_1 \otimes x_k)) \in \mathrm{Im}(\eta_P^P),$$

where $g_1 \in P$ is the identity.

We come to the second statement. Consider the following diagram

$$\begin{array}{ccc} \mathbb{F}[V(G)]^G & \xrightarrow{\eta_G^G} & \mathbb{F}[V]^G \\ \downarrow & & \downarrow \\ \mathbb{F}[V(G)]^H & \xrightarrow{\eta_G^H} & \mathbb{F}[V]^H \end{array}$$

Let $f \in \mathbb{F}[V]^G \subseteq \mathbb{F}[V]^H$. Since η_G^H is surjective, there exists an element $F \in \mathbb{F}[V(G)]^H$ such that

$$\eta_G^H(F) = f.$$

Then

$$\begin{aligned} \eta_G^G\left(\mathrm{Tr}_H^G(\frac{1}{|G:H|}F)\right) &= \eta_G^G\left(\frac{1}{|G:H|}\mathrm{Tr}_H^G(F)\right) \\ &= \frac{1}{|G:H|}\sum_{gH} g\eta_G^H(F) \\ &= \frac{1}{|G:H|}|G:H|f = f. \end{aligned}$$

Since $\mathrm{Tr}_H^G(F) \in \mathbb{F}[V(G)]^G$ we have that η_G^G is surjective. □

REMARK: We note that the proof of the first statement actually shows more: If $\eta_P^P|_{(d)}$ is surjective then so is $\eta_G^P|_{(d)}$. If $\eta_G^P|_{(1)}$ is surjective then so is $\eta_P^P|_{(1)}$.

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[2] For a graded object A we denote the homogeneous degree i-part by $A_{(i)}$.

PROPOSITION 2.2: *The extension*

$$\mathrm{Im}(\eta_G^G) \hookrightarrow \mathbb{F}[V]^G$$

is a p-root extension.

PROOF: First we consider the case of p-groups, $G = P$. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{F}[V(P)]^P & \xrightarrow{\eta_P^P} & \mathbb{F}[V]^P \\ \downarrow & & \downarrow \\ \mathbb{F}[V(P)] & \xrightarrow{\eta_P} & \mathbb{F}[V] \end{array},$$

and note that the lower map η_P is by construction surjective. Therefore, for any invariant polynomial $f \in \mathbb{F}[V]^P \subset \mathbb{F}[V]$ there exists a polynomial $F \in \mathbb{F}[V(P)]$ such that $\eta_P(F) = f$. The norm of this polynomial

$$\mathsf{N}_P(F) = \prod_{g \in P} gF \in \mathbb{F}[V(P)]^P$$

is invariant under the P-action. Moreover,

$$f^{|P|} = \prod_{g \in P} g\eta_P(F) = \prod_{g \in P} \eta_P(gF) = \eta_P^P(\mathsf{N}_P(F)) \in \mathrm{Im}(\eta_P^P) \subseteq \mathbb{F}[V]^P.$$

Since P is a p-group, its order $|P|$ is a pth power, and we are done.

Next we turn to arbitrary groups G, and we denote a p-Sylow subgroup by P. We obtain a commutative diagram as follows.

$$\begin{array}{ccccc} \mathbb{F}[V(G)]^G & \xrightarrow{\eta_G^G} & \mathrm{Im}(\eta_G^G) & \hookrightarrow & \mathbb{F}[V]^G \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{F}[V(G)]^P & \xrightarrow{\eta_G^P} & \mathrm{Im}(\eta_G^P) & \hookrightarrow & \mathbb{F}[V]^P \\ \uparrow & & \uparrow & & || \\ \mathbb{F}[V(P)]^P & \xrightarrow{\eta_P^P} & \mathrm{Im}(\eta_P^P) & \hookrightarrow & \mathbb{F}[V]^P. \end{array}$$

If $f \in \mathbb{F}[V]^P$, then $f^{p^l} \in \mathrm{Im}(\eta_P^P) \subseteq \mathrm{Im}(\eta_G^P)$ for some $l \in \mathbb{N}_0$. Hence

$$\mathrm{Im}(\eta_G^P) \hookrightarrow \mathbb{F}[V]^P$$

is a p-root extension. Since the index $|G:P|$ is coprime to the characteristic, the relative transfer homomorphism splits the two outer inclusions of the above diagram:

$$\begin{array}{ccccc} \mathbb{F}[V(G)]^G & \xrightarrow{\eta_G^G} & \mathrm{Im}(\eta_G^G) & \subseteq & \mathbb{F}[V]^G \\ \Big\uparrow \mathrm{Tr}_P^G & \text{⌘} & \Big\uparrow \mathrm{Tr}_P^G \,| & & \Big\uparrow \mathrm{Tr}_P^G \\ \mathbb{F}[V(G)]^P & \xrightarrow{\eta_G^P} & \mathrm{Im}(\eta_G^P) & \subseteq & \mathbb{F}[V]^P, \end{array}$$

where the restriction

$$\mathrm{Tr}_P^G \Big| : \mathrm{Im}(\eta_G^P) \longrightarrow \mathrm{Im}(\eta_G^G)$$

is surjective, since the first square ⌘ in the diagram commutes. Since the transfer commutes with taking pth powers, we obtain that

$$\mathrm{Im}(\eta_G^G) \hookrightarrow \mathbb{F}[V]^G$$

is a p-root extension as claimed. □

REMARK: We note that in the case of finite ground fields $\mathbb{F}$, rings of invariants are unstable algebras over the Steenrod algebra. In this category the above result means that the $\mathscr{P}^*$-inseparable closure of the image of the Noether map is the ring of invariants

$$\sqrt[\mathscr{P}^*]{\eta_G^G} = \mathbb{F}[V]^G,$$

see [7] Chapter 4 for detailed information on inseparable closures.

§3. Projective Modules

In this section we want to study the question of when the Noether map is surjective.

We note that the $\mathbb{F}G$ module V is projective if and only if its dual vector space V^* is injective which in turn is equivalent to projective because G is a finite group. We will make frequently use of this fact in what follows.

PROPOSITION 3.1: *If V is a projective $\mathbb{F}G$-module, then the Noether map is surjective.*

PROOF: By construction we have a short exact sequence of $\mathbb{F}G$-modules as follows

$$0 \longrightarrow W^* \longrightarrow V(G)^* \longrightarrow V^* \longrightarrow 0.$$

Since V^* is projective, this sequence splits and

$$V(G)^* \overset{\varphi}{\cong} V^* \oplus W^* \xrightarrow{\mathrm{pr}} V^*$$

as $\mathbb{F}G$-modules. Taking invariants we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{F}[V(G)]^G & \xrightarrow{\varphi^*} & \mathbb{F}[V \oplus W]^G \\ & \searrow \eta_G^G & \big\downarrow \mathrm{pr}^* \\ & & \mathbb{F}[V]^G \end{array}$$

Thus η_G^G is surjective because φ^* as well as pr^* are. □

REMARK: Since nonmodular $\mathbb{F}G$-modules are always projective we recover the classical result that η_G^G is surjective for every nonmodular representation of G.

COROLLARY 3.2: *Let $\varrho : G \hookrightarrow \mathrm{GL}(p, \mathbb{F})$ be a permutation representation of the finite group G over a field $\mathbb{F}$ of characteristic p. Then η_G^G is surjective.*

PROOF: Let $\psi : \Sigma_p \hookrightarrow \mathrm{GL}(p, \mathbb{F})$ be the defining representation of the symmetric group in p letters. Since ϱ is a permutation representation we have that

$$\varrho(G) \leq \psi(\Sigma_p) \leq \mathrm{GL}(p, \mathbb{F}).$$

Since $V = \mathbb{F}^p$ is a projective $\mathbb{F}\,\mathrm{Syl}_p(\Sigma_p)$-module it is projective as a $\mathbb{F}\Sigma_p$-module (see, e.g., Corollary 3 on page 66 in [1]). Thus it is projective as a $\mathbb{F}G$-module by Theorem 6 on page 33 loc.cit. Thus by Proposition 3.1 the Noether map η_G^G is surjective. □

EXAMPLE 1: If $\psi : \Sigma_n \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is the defining representation of the symmetric group in n letter over a field of characteristic p, where $p < n$, then V is not projective as a module over Σ_n nor is $\eta_{\Sigma_n}^{\Sigma_n}$ surjective. The latter is true because in degree one we have

$$\mathbb{F}[V(\Sigma_n)]_{(1)}^{\Sigma_n} = \mathrm{span}_{\mathbb{F}}\{\sum_{j=1}^{n!} x_{ij} \mid i = 1, \ldots, n\}$$

and thus

$$\eta_{\Sigma_n}^{\Sigma_n}(\sum_{j=1}^{n!} x_{ij}) = (n-1)! \sum_{i=1}^{n} x_i \equiv 0 \bmod p.$$

Therefore the first elementary symmetric function $e_1 = x_1 + \cdots + x_n \in \mathbb{F}[V]^{\Sigma_n}$ is not hit. Therefore, V is not $\mathbb{F}\Sigma_n$-projective. This is not a new result: For the defining representation $\psi : \Sigma_n \hookrightarrow \mathrm{GL}(n, \mathbb{F})$, $V = \mathbb{F}^n$ is a projective $\mathbb{F}\Sigma_n$-module if and only if $p \geq n$. This follows from Corollary 7 on Page 33 of [1].

EXAMPLE 2: Let $\psi : A_n \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$ be the defining representation of the alternating group in n letters over a field of characteristic p. By Corollary 3.2 the Noether map $\eta_{A_n}^{A_n}$ is surjective if $n \le p$. We want to check what happens if $n > p$.

We start by considering the Noether map

$$\eta_{A_n}^{A_n} : \mathbb{F}[V(A_n)]^{A_n} \longrightarrow \mathbb{F}[V]^{A_n}$$

in degree one. We have

$$\mathbb{F}[V(A_n)]^{A_n}|_{(1)} = \mathrm{span}_{\mathbb{F}}\{\sum_{j=1}^{|A_n|} x_{ij} \,|\, i = 1, \ldots,\ n\}$$

and

$$\mathbb{F}[V]^{A_n}|_{(1)} = \mathrm{span}_{\mathbb{F}}\{e_1 = x_1 + \cdots + x_n\}.$$

Thus we have

$$\eta_{A_n}^{A_n}(\sum_{j=1}^{|A_n|} x_{ij}) = |\mathrm{Stab}_{A_n}(x_i)|e_1 = |A_{n-1}|e_1 = \frac{(n-1)!}{2}e_1.$$

Thus the elementary symmetric function e_1 is in the image of the Noether map if and only if

$$\frac{(n-1)!}{2} \in \mathbb{F}^{\times}.$$

This in turn happens exactly when

(1) p is odd and $p \ge n$,
(2) $p = 2$ and $n \le 4$.

We know already that the Noether map is surjective in the first case. If p is even and $n \le 3$ we are in the nonmodular case, so the Noether map is again surjective. Thus the only case that we have to check by hand is the defining representation of A_4 over a field of characteristic 2.

We note that the 2-Sylow subgroup of A_4 is the Klein-Four-Group $\mathbb{Z}/2 \times \mathbb{Z}/2$. When we restrict $\psi|_{\mathbb{Z}/2\times\mathbb{Z}/2}$ we obtain the regular representation of $\mathbb{Z}/2\times\mathbb{Z}/2$. Thus V is $\mathbb{F}(\mathbb{Z}/2\times\mathbb{Z}/2)$-projective. Therefore, V is $\mathbb{F}A_4$-projective. Hence the Noether map is surjective. Indeed, a short calculation shows that

$$\eta_{A_4}^{A_4}(o(x_{11})) = 3e_1 = e_1,$$
$$\eta_{A_4}^{A_4}(o(x_{11}x_{12})) = e_2,$$
$$\eta_{A_4}^{A_4}(o(x_{11}x_{21}x_{31})) = 3e_3 = e_3,$$
$$\eta_{A_4}^{A_4}(o(x_{11}x_{12}x_{13}x_{14})) = 3e_4 = e_4,$$

$$\eta_{A_4}^{A_4}(o(x_{11}^3x_{21}^2x_{31})) = o(x_1^3x_2^2x_3),$$

where $o(-)$ denotes the orbit sum of $-$, and $g_1 = (1)$, $g_2 = (12)(34)$, $g_3 = (13)(24)$, and $g_4 = (14)(23)$.

§4. Applications and Examples

Let $\rho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$ be a faithful representation of a finite group of order d. Set $V = \mathbb{F}^n$. Recall that $\beta(\mathbb{F}[V]^G)$ is the maximal degree of an $\mathbb{F}$-algebra generator of $\mathbb{F}[V]^G$ in a minimal generating set.

PROPOSITION 4.1: *If V is a projective $\mathbb{F}G$-module then*

$$\beta(\mathbb{F}[V]^G) \leq \max\{d,\ n\binom{d}{2}\}.$$

PROOF: If V is $\mathbb{F}G$-projective then the Noether map η_G^G is surjective by Proposition 3.1. Thus, since η_G^G is an $\mathbb{F}$-algebra map, a set of generators of $\mathbb{F}[V(G)]^G$ is mapped onto a set of generators of $\mathbb{F}[V]^G$. Since $V(G)$ is a permutation module with n transitive components each of which has degree d, it is generated by elements of degree at most $\max\{d,\ n\binom{d}{2}\}$, by Corollary 3.10.9 in [3] and the result follows. □

REMARK: Let $\rho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$ be a representation of a finite group G of order d. Assume that the characteristic of $\mathbb{F}$ is zero or strictly larger than d. (This is the strongly nonmodular case.) Then

$$\beta(\mathbb{F}[V]^G) \leq \beta(\mathbb{F}[W]^G)$$

where W is the regular $\mathbb{F}G$-module, see Theorem 4.1.4 in [11]. Thus our Proposition 4.1 is a characteristic-free generalization: for *projective* $\mathbb{F}G$-modules V of dimension n, the upper bound for $\beta(\mathbb{F}[V]^G)$ is given by $\beta(\mathbb{F}[W]^G)$ where W is $\oplus_n \mathbb{F}G$.

The degree bound given above is sharp as we illustrate with the following example.

EXAMPLE 1: Let A_3 be the alternating group in three letters. Let $\mathbb{F}$ be a field containing a primitive 3rd root of unity $\omega \in \mathbb{F}$. Then we obtain a faithful representation

$$\rho : A_3 \hookrightarrow \mathrm{GL}(1,\ \mathbb{F}),\ (123) \longmapsto \omega.$$

We have

$$\mathbb{F}[x]^{A_3} = \mathbb{F}[x^3], \quad \text{and} \quad \mathbb{F}[x_{11}, x_{12}, x_{13}]^{A_3} = \mathbb{F}[e_1, e_2, e_3, o(x_{11}^2x_{12})],$$

where the e_i's are the elementary symmetric functions in the x_{1j}'s. Thus

$$\beta(\mathbb{F}[x]^{A_3}) = 3 = \beta(\mathbb{F}[x_{11}, x_{12}, x_{13}]^{A_3}) = \max\{3, \binom{3}{2}\}.$$

Before we proceed we want to compare the degree bound given in Proposition 4.1 with the known general bounds, see [9] for an overview of this topic.

(1) In the nonmodular case, we have that $\beta(\mathbb{F}[V]^G) \le |G|$ by Theorem 2.3.3 in [11]. This bound is better since

$$|G| \le \max\{n|G|, n\binom{|G|}{2}\}.$$

(2) The general degree bound given in Theorem 3.8.11 in [3] is

$$\beta(\mathbb{F}[V]^G) \le n(|G|-1) + |G|^{n2^{n-1}} n^{2^{n-1}+1}.$$

A short calculation shows that

$$\max\{n|G|, n\binom{|G|}{2}\} \le n(|G|-1) + |G|^{n2^{n-1}} n^{2^{n-1}+1}.$$

Thus the bound given in Proposition 4.1 is always better whenever it applies.

(3) If the ground field $\mathbb{F}$ is finite of order q, we have another general degree bound given by:

$$\beta(\mathbb{F}[V]^G) \le \begin{cases} \frac{q^n-1}{q-1}(nq-n-1) & \text{if } n \ge 3, \\ 2q^2-q-2 & \text{if } n = 2, \end{cases}$$

see Theorem 16.4 in [5]. This bound behaves worse than the one of Proposition 4.1 if $q > |G|$ (again, whenever it applies).

(4) Finally in [2] a bound of a completely different flavor is proven. In particular it depends on a choice of a homogeneous system of parameters. In our Example 1 we found that the bound of Proposition 4.1 is sharp. If we apply Theorem 2.3 in [2] to this example we obtain

$$\beta(\mathbb{F}[x]^{A_3}) \le \text{degree}(f),$$

where $f \in \mathbb{F}[x]^{A_3}$ is a system of parameters. If we make the unlucky choice of $f = x^9$ the bound given in [2] is no longer sharp. Of course, in this case it is easy to find a system of parameters, namely x^3, that improves the bound given in [2]. However, even though it is often not hard to construct a system of parameters for any given ring of invariants, in general it is not obvious how to find a "better" system of parameters, i.e., one consisting of polynomials of smaller degree. This in particular applies when the ring of invariants itself is not known.

We denote by CMdefect(–) the Cohen-Macaulay defect. The following result tells us that the Cohen-Macaulay defect of the ring of invariants of n copies of the regular representation of a finite group G is an upper bound for the Cohen-Macaulay defect of the ring of invariants $\mathbb{F}[V]^G$ in the case where V is projective of dimension n.

PROPOSITION 4.2: *If V is $\mathbb{F}G$-projective then*

$$\mathrm{CMdefect}(\mathbb{F}[V]^G) \leq \mathrm{CMdefect}(\mathbb{F}[V(G)]^G).$$

PROOF: Since V is $\mathbb{F}G$-projective, we have the $\mathbb{F}G$-module decomposition

$$V(G) = V \oplus K.$$

Thus the result follows from [8]. □

The inequality in the preceding result is sharp since the Cohen-Macaulay defect of any nonmodular representation is zero. However, we want to illustrate this with a modular example.

EXAMPLE 2: Let $\rho : \mathbb{Z}/2 \hookrightarrow \mathrm{GL}(2, \mathbb{F})$ be the regular representation of the cyclic group of order 2 over a field of characteristic 2 afforded by the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Its ring of invariants is polynomial

$$\mathbb{F}[x, y]^{\mathbb{Z}/2} = \mathbb{F}[x + y, xy]$$

so a fortiori Cohen-Macaulay. The vector space $V(\mathbb{Z}/2)$ is the two-fold regular representation of $\mathbb{Z}/2$. Thus its ring of invariants $\mathbb{F}[V(\mathbb{Z}/2)]^{\mathbb{Z}/2}$ is a complete intersection (see, e.g., [6]). Thus

$$\mathrm{CMdefect}(\mathbb{F}[V]^{\mathbb{Z}/2}) = 0 = \mathrm{CMdefect}(\mathbb{F}[V(\mathbb{Z}/2)]^{\mathbb{Z}/2}).$$

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