The connection between a conjecture of Carlisle and Kropholler, now a theorem of Benson and Crawley–Boevey, and Grothendieck's Riemann–Roch and duality theorems

Amnon Neeman*

December 22, 1994

Contents

0 Introduction

Recently Benson and Crawley–Boevey succeeded in proving a conjecture of Carlisle and Kropholler. The proof breaks into two parts; an easy local computation, and a global argument to reduce to the local computation. The global argument proceeds by two main lemmas. In this article, they are Lemma 2.5(=Corollary 2.5 of [1]), and Lemma 2.6(=Section 3 of [1]). The key point of this note is that Lemma 2.5 is a special case of the

[1]The research was partly supported by NSF grant DMS–9204940

Grothendieck Riemann–Roch Theorem, while Lemma 2.6 follows immediately from Grothendieck's Duality Theorem.

This in no way detracts from the beautiful idea of Benson and Crawley–Boevey. It should be noted that both the Riemann–Roch and the duality theorem are deep results; it is not so surprising that by using them the argument simplifies. The simplified argument allows one to highlight the central idea of the proof.

1 The Statement

Let V be a finite-dimensional vector space over a field k. Let G be a finite group acting on V linearly. Then the symmetric algebra $k[V]$ is a graded ring, as is the invariant subring $k[V]^G$. Let the corresponding Hilbert functions be denoted $P(k[V],n)$ and $P(k[V]^G,n)$. That is

$$P(k[V],n) = dim_k\left(k[V]_n\right) \qquad \text{and} \qquad P(k[V]^G,n) = dim_k\left(k[V]_n^G\right).$$

Of course, $P(k[V],n)$ is completely known; after all, $k[V]$ is just a polynomial ring. The theorem of Benson and Crawley-Boevey computes the top two coefficients of $P(k[V]^G,n)$. Precisely, form the Poincaré series

$$p(k[V]^G,t) = \sum_{i\geq 0} P(k[V]^G,n)t^i.$$

It is known that $p(k[V]^G,t)$ is a rational function in t, whose poles occur only at roots of unity. In other words, there is an expression

$$p(k[V]^G,t) = R(t) + \sum_{i,j} \frac{\alpha_{ij}}{(1-\omega_i t)^j}$$

where $R(t)$ is a polynomial in t, α_{ij} are complex numbers, and ω_i are roots of unity. The sum is assumed finite.

It is known that the numbers j satisfy the inequality $0 \leq j \leq r$, where $r+1 = \dim_k(V)$. It is also known that the order of the pole at $t=1$ is exactly r; in other words, the coefficient α_{ij} where $\omega_i = 1$ and $j=r$ is non zero. Put still differently, the term $\frac{\alpha}{(1-t)^r}$ in the expansion above has $\alpha \neq 0$. The conjecture of Carlisle and Kropholler, now a theorem of Benson and Crawley–Boevey, states

Theorem (Benson, Carlisle, Crawley–Boevey, Kropholler) *Suppose $k = \mathbb{F}_p$ is a prime field. In the expansion of $p(k[V]^G, t)$ around $t = 1$, that is*

$$p(k[V]^G, t) = \frac{\alpha_0}{(1-t)^r} + \frac{\alpha_1}{(1-t)^{r-1}} + O\left(\frac{1}{(1-t)^{r-2}}\right)$$

one has $\alpha_0 = 1/|G|$, and

$$\alpha_1 = \frac{1}{2|G|} \sum_W ((p-1)a_W + h_W - 1)$$

where the sum runs over hyperplanes $W \subset V$, and the integers a_W and h_W are defined by $|G_W| = h_W p^{a_W}$, where $G_W \subset G$ is the subgroup stabilising W pointwise, and h_W is prime to the characteristic p.

Note now that both sides of the equation in the theorem remain unchanged if we replace V by $V \oplus k \oplus k$, with the action of G being trivial on $k \oplus k$. It is clear that the right hand side in the formula remains the same, since the hyperplanes of $V \oplus k \oplus k$ fixed by a non–trivial subgroup of G are just $W \oplus k \oplus k$, where W is a hyperplane of V fixed by a non–trivial subgroup. To see that the coefficients α_0 and α_1 are also unchanged on replacing V by $V \oplus k \oplus k$, observe that

$$k[V \oplus k \oplus k]^G = k[V]^G \otimes k[x, y],$$

and hence

$$p(k[V \oplus k \oplus k]^G, t) = \frac{1}{(1-t)^2}\, p(k[V]^G, t)$$

and it is therefore immediate that α_0 and α_1 do not change. The advantage of replacing V by $V \oplus k \oplus k$ is twofold. First, the group G does not meet the center of $GL(V \oplus k \oplus k)$, and hence acts faithfully on the associated projective space. Secondly, one has an estimate

$$p(k[V \oplus k \oplus k]^G, t) - R(t) = \frac{\alpha_0}{(1-t)^{r+2}} + \frac{\alpha_1}{(1-t)^{r+1}} + \sum_{i,j \leq r} \frac{\alpha_{ij}}{(1-\omega_i t)^j}$$

where $R(t)$ is a polynomial. This leads to the estimate

$$P(k[V \oplus k \oplus k]^G, n) = a_0 n^{r+2} + a_1 n^{r+1} + O(n^r),$$

where $a_0 = \alpha_0/(r+2)!$, and a_1 can be similarly expressed as a linear function in α_0 and α_1. The point is that in the Taylor series for

$$\frac{1}{(1-t)^m} = \sum_{n=0}^{\infty} \left(\begin{array}{c} m+n-1 \\ m \end{array} \right) t^n$$

the coefficients are of degree $\leq m$ in the number n. It follows that in the sum

$$\sum_{i,j \leq r} \frac{\alpha_{ij}}{(1-\omega_i t)^j}$$

the coefficients of t^n is of degree $\leq r$ in n. The only terms contributing coefficients of degree $r+1$ or $r+2$ to the sum

$$p(k[V \oplus k \oplus k]^G, t) - R(t) = \frac{\alpha_0}{(1-t)^{r+2}} + \frac{\alpha_1}{(1-t)^{r+1}} + \sum_{i,j \leq r} \frac{\alpha_{ij}}{(1-\omega_i t)^j}$$

are the ones involving α_0 and α_1. The Taylor series allows one to compute a_0 and a_1 in terms of α_0 and α_1.

From now on we will assume that V is a vector space on which G acts, we will assume that G contains no scalar matrices, and furthermore that there is an estimate

$$P(k[V]^G, n) = a_0 n^r + a_1 n^{r-1} + O(n^{r-2}),$$

where $r+1 = dim_k(V)$. The idea of the proof of Benson and Crawley–Boevey is to replace $k[V]^G$ by some other module M, so that there is also an estimate

$$P(M, n) = a_0 n^r + a_1 n^{r-1} + O(n^{r-2}),$$

for the same a_0 and a_1. It follows that then the α_0 and α_1 also agree for the modules $k[V]^G$ and M, and we can then use M to compute. Since the computation of α_0 and α_1 is admirably explained in [1], I will only deal with the finding of M.

2 The Proof

Let V be a finite dimensional representation of the finite group G, satisfying the assumptions at the end of Section 1. Let $X = \mathrm{Proj}(k[V])$ and $Y =$

$\mathrm{Proj}(k[V]^G)$. Let $\pi : X \to Y$ be the natural projection. Then $X = \mathbb{P}^r$ is projective space, and on it there is a line bundle $\mathcal{O}(1)$, the hyperplane bundle. The map $\pi : X \to Y$ is given as $\mathrm{Proj}(i)$, where $i : k[V]^G \hookrightarrow k[V]$ is the inclusion.

It is slightly non–trivial but well–known that the rational map $\pi : X \to Y$ is everywhere regular. In other words, one needs to show that the base locus is trivial. In this case, this is known because $K[V]$ is a finite module over $K[V]^G$. By the going–up theorem, we then know that the only prime ideal of $K[V]$ lying over the zero ideal of $K[V]^G$ is the zero ideal; thus the base locus is trivial.

The scheme $Y = Proj(k[V]^G)$ is projective, simply by virtue of being $Proj$ of something finitely generated over k. Let us encode it as a remark.

Remark 2.1 Soppose k is a field, R a finitely (positively) generated graded k–algebra. Let $\{r_1, r_2, \ldots, r_l\}$ be generators of degrees $\{b_1, b_2, \ldots, b_l\}$ respectively. Let B be the least common multiple of the integers $\{b_1, b_2, \ldots, b_l\}$. Then the terms in R of degree lB give an embedding of $Y = Proj(R)$ into projective space.

The proof is very simple. Given any monomial m of R of degree $d > lB$, then it has the form $m = r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_l^{\alpha_l}$. Because the total degree is $b_1\alpha_1 + b_2\alpha_2 + \cdots + b_l\alpha_l$ and exceeds lB, it follows that some $b_i\alpha_i > B$. But then $\alpha_i > \frac{B}{b_i}$, and $\beta_i = \frac{B}{b_i}$ is an integer because B was chosen divisible by all b_i. Thus,

$$m = r_i^{\beta_i} m'$$

where $m' = r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_i^{\alpha_i - \beta_i} \cdots r_l^{\alpha_l}$ is an element of R, and the degree of m' is exactly B less than the degree of m. We deduce that any monomial of R of degree $> lB$ can be expanded as a product of elements of degree B and an element of degree $\leq lB$. It immediately follows that if m is a monomial of degree hlB, that is a multiple of lB, then m can be written as a product of elements of degree precisely lB.

The fact that this can be done is very classical; it goes back to the Italian school. It is classically known as the fact that "weighted projective space" can be embedded in projective space. Since the argument is very simple, we included it.

Thus, if we choose t to be an integer sufficiently divisible by the degrees of the generators of $k[V]^G$ as a k–algebra, then the degree t part of $k[V]^G$

gives an embedding of $Y = Proj\left(k[V]^G\right)$ into a projective space of some dimension. Let $\mathcal{L}_t$ be the corresponding very ample line bundle on Y. Then it is obvious that the map $\pi : X \to Y$ satisfies $\pi^*\mathcal{L}_t = \mathcal{O}(t)$. This amounts to no more than considering the degree t part of the homomorphism $i : k[V]^G \hookrightarrow k[V]$ defining π.

More generally, let $\mathcal{L}_n$ be the coherent sheaf $\pi_*\mathcal{O}(n)^G$ on Y. We should perhaps remind the reader of the notation. The sheaf $\pi_*\mathcal{O}(n)$ on Y is clearly well–defined, being the ordinary direct image. The group G acts on this sheaf, since G acts on $\mathcal{O}(n)$, and the map π is G–equivariant. But then the sheaf $\pi_*\mathcal{O}(n)^G$ is defined as the equaliser of all the maps

$$g : \pi_*\mathcal{O}(n) \longrightarrow \pi_*\mathcal{O}(n),$$

where $g \in G$. Practically by the definition of Y we know that for some integer t, (same t as above), the sheaf $\mathcal{L}_t = \pi_*\mathcal{O}(t)^G$ is a line bundle on Y, and in fact for any integer n, there is an equality $\mathcal{L}_{n+t} = \mathcal{L}_t \otimes \mathcal{L}_n$. The point is that the question is local in Y, and locally in Y $\mathcal{L}_t$ is isomorphic to $\mathcal{O}_Y$.

Remark 2.2 The preceding few paragraphs are all fairly standard consequences of geometric invariant theory. In the last few paragraphs I tried to give a self–contained exposition, largely because to give references would be awkward. Traditionally, geometric invariant theory is always exposed over fields of charcteristic 0. It is known, as a consequence of Haboush's proof of the Mumford conjecture and of related work by Nagata, that the theory is only slightly different in finite characteristic. However, there is no convenient general reference I could appeal to. In any case, the preceding results are so easy to deduce directly in the case of an action by a finite group, that it seemed unkind to the non–expert to make him chase this down in the literature. I included this remark to make it clear that I claim no originality for the preceding discussion.

We are assuming that there is an estimate

$$\begin{aligned} dim_k\left(k[V]_n^G\right) &= dim_k\left\{H^0(Y, \mathcal{L}_n)\right\} \\ &= a_0 n^r + a_1 n^{r-1} + O(n^{r-2}). \end{aligned}$$

Let $\omega_{X/Y}$ be the sheaf $Hom_{\mathcal{O}_X}\left(\pi^*\left\{\wedge^r \Omega_Y\right\}, \wedge^r \Omega_X\right)$. Let $\omega_{X/Y}(n) = \omega_{X/Y} \otimes \mathcal{O}(n)$ be the Serre twist. There is, of course, an estimate

$$dim_k \left\{ H^0\left(X, \mathcal{O}(n) \oplus \omega_{X/Y}(n)\right)\right\} \;\; = \;\; b_0 n^r + b_1 n^{r-1} + O(n^{r-2})$$

given by the Hilbert polynomial. In this section we will prove that $b_0 = 2|G|a_0$ and $b_1 = 2|G|a_1$. The calculation of $b_0/2|G|$ and $b_1/2|G|$ is then the same as in [1], Section 4.

It suffices of course to show that in the expansions

$$dim_k \left\{ H^0(Y, \mathcal{L}_{nt})\right\} \;\; = \;\; a_0 n^r + a_1 n^{r-1} + O(n^{r-2}).$$

and

$$dim_k \left\{ H^0\left(X, \mathcal{O}(nt) \oplus \omega_{X/Y}(nt)\right)\right\} \;\; = \;\; b_0 n^r + b_1 n^{r-1} + O(n^{r-2}).$$

the coefficients satisfy $b_0 = 2|G|a_0$ and $b_1 = 2|G|a_1$. After all, replacing n by nt only replaces a_0 by $a_0 t^r$, b_0 by $b_0 t^r$, a_1 by $a_1 t^{r-1}$ and b_1 by $b_1 t^{r-1}$. But of course $\mathcal{O}(t) = \pi^* \mathcal{L}_t$, and hence $H^0\left(X, \mathcal{O}(nt) \oplus \omega_{X/Y}(nt)\right)$ can be rewritten as $H^0\left(X, \left\{\mathcal{O} \oplus \omega_{X/Y}\right\} \otimes \pi^* \mathcal{L}_t^n\right)$. But the map $\pi : X \longrightarrow Y$ is finite, and hence

$$H^0(X, \mathcal{S}) = H^0(Y, \pi_* \mathcal{S})$$

for any coherent sheaf $\mathcal{S}$. In particular, letting $\mathcal{S} = \left\{\mathcal{O} \oplus \omega_{X/Y}\right\} \otimes \pi^* \mathcal{L}_t^n$ and recalling that the projection formula gives

$$\pi_* \left[\left\{\mathcal{O} \oplus \omega_{X/Y}\right\} \otimes_{\mathcal{O}_X} \pi^* \mathcal{L}_t^n\right] = \pi_* \left\{\mathcal{O} \oplus \omega_{X/Y}\right\} \otimes_{\mathcal{O}_Y} \mathcal{L}_t^n$$

we have an identity

$$H^0\left(X, \mathcal{O}(nt) \oplus \omega_{X/Y}(nt)\right) = H^0\left(Y, \pi_* \left\{\mathcal{O} \oplus \omega_{X/Y}\right\} \otimes_{\mathcal{O}_Y} \mathcal{L}_t^n\right).$$

Thus, we will try to establish that the estimates

$$(1) \qquad h^0\left(Y, \pi_* \left\{\mathcal{O} \oplus \omega_{X/Y}\right\} \otimes_{\mathcal{O}_Y} \mathcal{L}_t^n\right) \;\; = \;\; a_0 n^r + a_1 n^{r-1} + \mathcal{O}(n^{r-2})$$

$$(2) \qquad h^0\left(Y, \bigoplus_{2|G| \text{ times}} \mathcal{L}_t^n\right) \;\; = \;\; a_0 n^r + a_1 n^{r-1} + \mathcal{O}(n^{r-2})$$

both hold, with the same a_0 and a_1. We will prove a more general sequence of lemmas.

Lemma 2.3 *Let $\mathcal{V}$ be a coherent sheaf on a normal variety Y. Let $\mathcal{L}$ be an ample line bundle on Y. Let $U \subset Y$ be the smooth locus of Y. Let the Hilbert polynomial of $\mathcal{V}$ be*

$$h^0\left(Y, \mathcal{V} \otimes_{\mathcal{O}_Y} \mathcal{L}^n\right) = a_0 n^r + a_1 n^{r-1} + \mathcal{O}(n^{r-2}).$$

Finally, let $ch(\mathcal{V})$ be the Chern character of $\mathcal{V}$, and let $ch_0(\mathcal{V})$ and $ch_1(\mathcal{V})$ be the 0^{th} and 1^{st} classes of $ch(\mathcal{V})$ on the smooth variety U. Then the coefficients a_0 and a_1 depend only on $ch_0(\mathcal{V})$ and $ch_1(\mathcal{V})$. That is, there is a formula for a_0 and a_1 in terms of $ch_0(\mathcal{V})$ and $ch_1(\mathcal{V})$.

Remark 2.4 Perhaps we owe the reader an explanation of Chern classes. Let $\mathcal{V}$ be a vector bundle on a manifold. Classically, the Chern classes were some cohomology classes on the manifold. It is an observation of Grothendieck [4] that in fact, very little of the structure of ordinary cohomology is needed to define Chern classes. The key property needed is that the cohomology of projective bundles is a truncated polynomial ring. What this means is the following. Let H^* be a contravariant functor taking spaces X to graded rings $H^*(X)$. Let $f : X \longrightarrow Y$ be a map of spaces. Then $H^*(f) : H^*(Y) \longrightarrow H^*(X)$ expresses $H^*(X)$ as a $H^*(Y)$–algebra. Suppose the map $f : X \longrightarrow Y$ gives X the structure of a $\mathbb{P}^n$–bundle over Y. The key to being able to define Chern classes is that, as an $H^*(Y)$–algebra, $H^*(X)$ should be naturally isomorphic to a truncated polynomial ring; that is

$$H^*(X) \simeq \frac{H^*(Y)[t]}{(P(t))}$$

where $P(t)$ is a monic polynomial of degree $n+1$. In fact, the Chern class c_i is, up to sign, the coefficient of t^{n+1-i} in this polynomial P.

Many cohomology theories other than ordinary cohomology have this property. Topologists refer to a cohomology theory like this as a "complex oriented theory". Complex K–theory and complex cobordism are examples, among many others. See for instance [6].

In algebraic geometry, we often work with varieties defined over fields of characteristic p (for instance right now), and none of the topological work seems relevant. But by Grothendieck's general construction, Chern classes may take their value in any theory satisfying the complex orientability criterion. At least two come to mind: the étale cohomology, and the Chow ring. To fix ideas, in this article the Chern classes of the coherent sheaf $\mathcal{V}$ are understood to take their values in $A^*(U)$, the Chow ring of $U \subset Y$. It is of course possible to take them in any motivic cohomology theory, and these days there are plenty of candidates. This is not the appropriate place for a thorough discussion of motivic cohomology. The Chow ring has the advantage of being well documented in the literature. The bare essentials may be found in Grothendieck's [4]. A far more extensive discussion appears for instance in Seminaire Chevalley, second volume, entitled *Anneaux de Chow et Applications*.

Proof of Lemma 2.3. This is essentially immediate from the Riemann-Roch theorem. The slightly delicate point is that Y may be singular. Here is how we get around this technical difficulty. First, the question is geometric, so we may assume that the ground field k is algebraically closed.

Replacing $\mathcal{L}$ by $\mathcal{L}^m$, we may assume $\mathcal{L}$ very ample. Use $\mathcal{L}$ to embed Y in projective space $\mathbb{P}^N$ for some large N. By projecting from a general linear space of dimension $N-r-1$, we deduce a finite surjective map $p : Y \to \mathbb{P}^r$. The idea will be to study p.

Let $Z \subset \mathbb{P}^r$ be the image of the singular locus of Y. Let $V \subset \mathbb{P}^r$ be the complement, $V = \mathbb{P}^r - Z$. Put $U' = p^{-1}V$. Then $U' \subset U$ is an open set in Y, and the complement $Y - U'$ is of codimension 2 at least. It clearly suffices to show that there is a formula for a_0 and a_1 in terms of $c_0(\mathcal{V}|_{U'})$ and $c_1(\mathcal{V}|_{U'})$. We henceforth replace U by U'.

We want to compute the leading coefficients of $h^0\left(Y, \mathcal{V} \otimes_{\mathcal{O}_Y} \mathcal{L}^n\right)$. But $\mathcal{L} = p^*\mathcal{O}(1)$, since the map $Y \to \mathbb{P}^r$ was obtained from the complete linear system $|\mathcal{L}|$ by projection from general position. Thus, by the projection formula

$$p_*\left(\mathcal{V} \otimes_{\mathcal{O}_Y} p^*\mathcal{O}(n)\right) = (p_*\mathcal{V}) \otimes_{\mathcal{O}_{\mathbb{P}^r}} \mathcal{O}(n),$$

and

$$h^0\left(Y, \mathcal{V} \otimes_{\mathcal{O}_Y} \mathcal{L}^n\right) = h^0\left(\mathbb{P}^r, (p_*\mathcal{V}) \otimes_{\mathcal{O}_{\mathbb{P}^r}} \mathcal{O}(n)\right).$$

Because $\mathcal{O}(1)$ is ample, for n large enough and $i > 0$, $h^i\left(\mathbb{P}^r, (p_*\mathcal{V}) \otimes \mathcal{O}(n)\right) = 0$. Thus for n large

$$h^0\left(\mathbb{P}^r, p_*\mathcal{V} \otimes \mathcal{O}(n)\right) = \chi\left(p_*\mathcal{V} \otimes \mathcal{O}(n)\right)$$

is the Euler characteristic. But the Riemann-Roch theorem says that

$$\chi(p_*\mathcal{V} \otimes \mathcal{O}(n)) = \int ch\,[p_*\mathcal{V} \otimes \mathcal{O}(n)] \cdot \tau(\mathbb{P}^n)$$

where the integral stands for the top degree part of the class inside it, and the "integrand" is the Chern character of $p_*\mathcal{V} \otimes \mathcal{O}(n)$ times the Todd class of $\mathbb{P}^n$. The Chern character is multiplicative. Hence we deduce

$$\chi(p_*\mathcal{V} \otimes \mathcal{O}(n)) = \int [ch(p_*\mathcal{V}) \cdot \tau(\mathbb{P}^n)] \cdot ch(\mathcal{O}(n)).$$

Of course, the Chern character of the line bundle $\mathcal{O}(n)$ is given by e^{nc_1}, where c_1 is the first Chern class of $\mathcal{O}(1)$. It is

$$ch\,(\mathcal{O}(n)) = 1 + nc_1 + \cdots + \frac{n^{r-1}}{(r-1)!}\,c_1^{r-1} + \frac{n^r}{r!}\,c_1^r.$$

It is clear now that the coefficients of n^r and n^{r-1} in the top degree class of the product

$$[ch(p_*\mathcal{V}) \cdot \tau(\mathbb{P}^n)] \cdot ch(\mathcal{O}(n))$$

can only involve degree 0 or 1 terms from $[ch(p_*\mathcal{V}) \cdot \tau(\mathbb{P}^n)]$, in other words they involve only $ch_0(p_*\mathcal{V})$ and $ch_1(p_*\mathcal{V})$.

Now applying the Riemann-Roch theorem to the map $p : U \to V$, we have that

$$p_*[ch(\mathcal{V}) \cdot \tau(U)] = ch(p_*\mathcal{V}) \cdot \tau(V)$$

which gives a formula

$$ch(p_*\mathcal{V}) = p_*[ch(\mathcal{V} \cdot (\tau(U)] \cdot \tau(V)^{-1},$$

and if we only care about $ch_0(p_*\mathcal{V})$ and $ch_1(p_*\mathcal{V})$, the formula for them involves only $ch_0(\mathcal{V})$ and $ch_1(\mathcal{V})$. $\square$

Lemma 2.5 *Let $\pi : X \to Y$ be a finite surjective map of varieties over k, and suppose X is smooth and Y normal. Suppose the degree of the map π is d. Then the following two sheaves on Y have the same two leading coefficients in their Hilbert polynomials.*

(1) $\pi_*\mathcal{O}_X + \mathbf{Hom}_{\mathcal{O}_Y}(\pi_*\mathcal{O}_X, \mathcal{O}_Y)$;

(2) $\bigoplus_{i=1}^{2d} \mathcal{O}_Y$.

Proof. By Lemma 2.3, it suffices to establish that the first two classes ch_0 and ch_1 agree, and in fact it suffices to check this on the smooth locus $U \subset Y$. The zeroth class ch_0 is just the rank of the bundle, and those are clearly two vector bundles on U of equal rank. But $ch_1 = c_1$ is the first Chern class, and can be computed by taking the highest wedge product of the bundle. In both cases, we get $\mathcal{O}_Y$. $\square$

Lemma 2.6 *The two sheaves on* Y

(1) $\mathbf{Hom}_{\mathcal{O}_Y}(\pi_*\mathcal{O}_X, \mathcal{O}_Y)$;

(2) $\pi_*\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \omega_{X/Y})$;

have Hilbert polynomials with equal top two leading coefficients. Once again, $\omega_{X/Y} = Hom_{\mathcal{O}_X}(\pi^*\{\wedge^r\Omega_Y\}, \wedge^r\Omega_X)$.

Proof. By Lemma 2.3, it suffices to show that the two agree as sheaves on the open set $U \subset Y$. But U is a manifold, and over U the sheaf $\omega_{X/Y}$ is just the dualizing sheaf for the map $\pi: \pi^{-1}U \to U$. By Grothendieck's duality theorem, we have

$$R\ \mathrm{Hom}_{\mathcal{O}_U}(R\pi_*\mathcal{O}_{\pi^{-1}U}, \mathcal{O}_U) = R\pi_* R\ \mathrm{Hom}_{\mathcal{O}_{\pi^{-1}U}}\left(\mathcal{O}_{\pi^{-1}U}, \omega_{X/Y}\right).$$

where RHom means the Hom functor in the derived category. It is a complex of sheaves whose 0^{th} cohomology is the ordinary sheaf Hom, the thing that was denoted **Hom** in Lemma 2.6. Because π is a finite (hence affine) map, $R\pi_* = \pi_*$ (there are no higher derived functors). Because $\mathcal{O}_{\pi^{-1}U}$ is locally free on $\pi^{-1}U \subset X$,

$$R\ \mathrm{Hom}_{\mathcal{O}_{\pi^{-1}U}}\left(\mathcal{O}_{\pi^{-1}U}, \omega_{X/Y}\right) = \mathbf{Hom}_{\mathcal{O}_{\pi^{-1}U}}\left(\mathcal{O}_{\pi^{-1}U}, \omega_{X/Y}\right),$$

and, finally, because $\pi_* \mathcal{O}_{\pi^{-1}U}$ is locally free on U (this is because π is flat over U),

$$R \operatorname{Hom}_{\mathcal{O}_U} (\pi_* \mathcal{O}_{\pi^{-1}U}, , \mathcal{O}_U) = \mathbf{Hom}_{\mathcal{O}_U} (\pi_* \mathcal{O}_{\pi^{-1}U}, \mathcal{O}_U) \, .$$

In other words, we may delete all the higher derived functors in the formula.
□

Corollary 2.7 *It follows that the two sheaves on Y*

(1) $\pi_* \mathcal{O}_X \oplus \pi_* \omega_{X/Y}$;

(2) $\bigoplus_{i=1}^{2d} \mathcal{O}_Y$;

have the same two leading coefficients for their Hilbert polynomials. In other words, $2d\ h^0(Y, \mathcal{L}^n)$ is asymtotically $a_0 n^r + a_1 n^{r-1}$ for the same a_0 and a_1 as the top two coefficients in the Hilbert polynomial of $h^0(X, \mathcal{O}(n)) + h^0(X, \omega_{X/Y}(n))$.

References

[1] D.J. Benson and W.W. Crawley–Boevey. *A ramification formula for Poincaré series, and a hyperplane formula in modular invariant theory.* To appear in Bull. London Math. Soc.

[2] A. Borel and J.–P. Serre. *Le théorème de Riemann–Roch.* Bull. Soc. Math. de France 86 (1958) 97–136.

[3] D. Carlisle and P. Kropholler. *Modular invariants of finite symplectic groups.* To appear in J. London Math. Soc.

[4] A. Grothendieck. *La théorie des classes de Chern.* Bull. Soc. Math. de France 86 (1958) 137–154.

[5] R. Hartshorne. *Residues and Duality.* Lecture Notes in Math. 20 (1966) Springer–Verlag, Heidelberg.

[6] D. Quillen. *Elementary proofs of some results of cobordism theory using Steenrod operations.* Adv. in Math. 7 (1971) 29–56.